SOFIC MEASURES AND
DENSITIES OF LEVEL SETS

Alain Thomas

Dedicated to the memory of Pierre Liardet

ABSTRACT. The Bernoulli convolution associated to the real $\beta > 1$ and the probability vector $(p_0, \ldots, p_{d-1})$ is a probability measure $\eta_{\beta,p}$ on $\mathbb{R}$, solution of the self-similarity relation $\eta = \sum_{k=0}^{d-1} p_k \cdot \eta \circ S_k^{-1}$, where $S_k(x) = \frac{x+k}{\beta}$. If β is an integer or a Pisot algebraic number with finite Rényi expansion, $\eta_{\beta,p}$ is sofic and a Markov chain is naturally associated. If $\beta = b \in \mathbb{N}$ and $p_0 = \cdots = p_{d-1} = \frac{1}{d}$, the study of $\eta_{b,p}$ is close to the study of the order of growth of the number of representations in base b with digits in $\{0, 1, \ldots, d-1\}$. In the case $b = 2$ and $d = 3$ it has also something to do with the metric properties of the continued fractions.

Communicated by Cornelius Kraaikamp

Introduction

The different sections of this paper are relatively independent. In the first two sections we talk about the sofic subshift and sofic measures, also usually called hidden Markov measures, submarkov measures, functions of Markov chains, rational measures. A sofic probability measure on a space $\{0, 1, \ldots, b-1\}^{\mathbb{N}}$, is the image of a Markov probability measure by a shift-commuting continuous map, and can be naturally represented by products of matrices (see for instance [7]). [3] is a collection of papers about hidden Markov processes, involving connections with symbolic dynamics and statistical mechanics. See also [4, 17, 18, 20].

The measures defined by Bernoulli convolution [22], i.e., the measures

$$\eta_{\beta,p} := \mathop{\Large\ast}_{n=1}^{\infty} \left(\sum_{k=0}^{d-1} p_k \delta_{k/\beta^n} \right),$$

where $\beta \in \mathbb{R}$, $p_0, \ldots, p_{d-1} > 0$ and $\sum_k p_k = 1$, are sofic if β is an integer. They are also sofic when β is a Pisot number (i.e., an algebraic number whose conjugates belong to the open unit disk) with finite Rényi expansion (see [24] for the definition). A transducer of normalization [15, Theorem 2.3.39] is naturally associated to β and d. The matrices associated to the measure $\eta_{\beta,p}$ are easy to define when β is an integer, $\beta = b \geq 2$. In the case $p_0 = \cdots = p_{d-1} = \frac{1}{d}$ they are used by different authors (see, for instance [23]) because they are related to the number of representations of the integer n in base b with digits in $\{0, \ldots, d-1\}$, let $\mathcal{N}(n)$. And the Hausdorff dimensions of the level sets of $\eta_{b,p}$ are related with the lower exponential densities of some sets of integers, on which $\mathcal{N}(n)$ has a given order of growth (Theorem 21). There exist many partial results about the level sets of the measures defined by products of matrices (see, for instance [8, 9, 10, 12]).

1. Sofic subshifts

We recall some classical definitions about the subshifts (see for instance [1, 14, 19, 25, 26]). By subshifts we mean closed subsets of $\{0, 1, \ldots, b-1\}^{\mathbb{N}}$ invariant by the shift $\sigma : (\omega_n)_{n \in \mathbb{N}} \mapsto (\omega_{n+1})_{n \in \mathbb{N}}$. Let us give two equivalent definitions of the sofic subshifts.

DEFINITION 1. A sofic subshift is a subshift recognizable by a finite automaton [1].
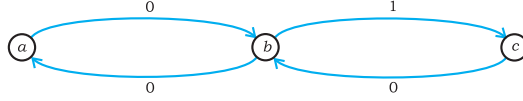
DEFINITION 2. A subshift is a sofic iff it is the image of a topological Markov subshift by a letter-to-letter morphism.

In this definition, "letter-to-letter morphism" can be replaced by "continuous morphism", because any continuous morphism (i.e., any continuous map, for the usual topology, commuting with the shift) has the form

$$\varphi\big((\omega_n)_{n \in \mathbb{N}}\big) = \big(\psi(\omega_{n-s}\omega_{n-s+1} \ldots \omega_{n+s})\big)_{n \in \mathbb{N}}$$

with $\psi : \{0, 1, \ldots, b-1\}^{2s+1} \to \{0, 1, \ldots, b'-1\}$, being understood that $\omega_n = 0$ for $n \leq 0$.
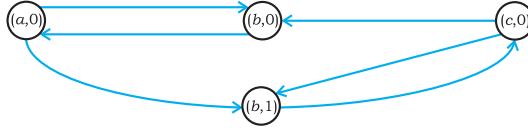
EXAMPLE 3. An example of sofic subshift in the sense of Definition 1, is the set of the labels of the infinite paths in the following automaton:


(this subshift is the set of the sequences $(\omega_n)_{n\in\mathbb{N}} \in \{0,1\}^\mathbb{N}$ without factor $10^{2i}1$ for any $i \in \mathbb{N} \cup \{0\}$, hence it excludes an infinite set of words and it is not of finite type). It is sofic in the sense of Definition 2 because it is the image by the morphism $\pi : (x,y) \mapsto y$ of the Markov subshift of the sequences $(\xi_n)_{n\in\mathbb{N}} \in \{(a,0),(b,0),(b,1),(c,0)\}^\mathbb{N}$ such that, for any $n \in \mathbb{N}$,

$$\xi_n\xi_{n+1} \in \big\{(a,0)(b,0), (a,0)(b,1), (b,0)(a,0), (b,1)(c,0), (c,0)(b,0), (c,0)(b,1)\big\}.$$

The graph of this Markov subshift is:

EXAMPLE 4. An example of sofic subshift in the sense of Definition 2, is the image of the Markov subshift associated to the graph:
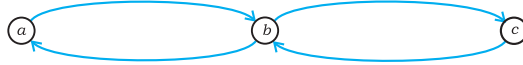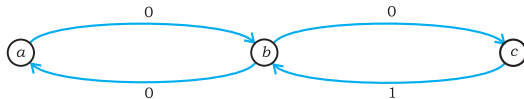

by the letter-to-letter morphism

$$\varphi : \{a,b,c\}^\mathbb{N} \to \{0,1\}^\mathbb{N}$$

associated to

$$\psi : \{a,b,c\} \to \{0,1\}, \ \psi(a) = \psi(b) = 0, \ \psi(c) = 1.$$

As in Example 3, it excludes the words $10^{2i}1$, $i \in \mathbb{N} \cup \{0\}$. It is also a sofic subshift in the sense of Definition 1, because it is recognizable by the following automaton, where each arrow with initial state x has label $\psi(x)$:

Notice that, if we associate to this automaton a Markov subshift and a morphism by the same method as in Example 3, we recover the initial Markov subshift and morphism of Example 4.

2. Markov, sofic and linearly representable measures

We specify now the definitions of the Markov and sofic measures we use in the sequel.

DEFINITION 5. (i) We call a (homogeneous) Markov probability measure (not necessarily shift-invariant), a measure μ on the product set $\{0, 1, \ldots, b - 1\}^{\mathbb{N}}$, defined by setting, for any cylinder set

$$[\omega_1 \ldots \omega_n] = \big\{(\xi_i)_{i \in \mathbb{N}} \ : \ \xi_1 \ldots \xi_n = \omega_1 \ldots \omega_n\big\},$$
$$\mu[\omega_1 \ldots \omega_n] = p_{\omega_1} p_{\omega_1 \omega_2} \ldots p_{\omega_{n-1} \omega_n} , \tag{1}$$

where $p = \big(p_0 \ \ldots \ p_{b-1}\big)$ is a positive probability vector and

$$P = \begin{pmatrix} p_{00} & \cdots & p_{0(b-1)} \\ \vdots & \ddots & \vdots \\ p_{(b-1)0} & \cdots & p_{(b-1)(b-1)} \end{pmatrix}$$

is a nonnegative stochastic matrix.

Clearly, the support of μ is a Markov subshift, and μ is shift-invariant (or stationary) iff p is a left eigenvector of P.

(ii) A probability measure on $\{0, 1, \ldots, b' - 1\}^{\mathbb{N}}$ is called sofic if it is the image of a Markov probability measure by a continuous morphism $\varphi = \{0, 1, \ldots \ldots, b - 1\}^{\mathbb{N}} \to \{0, 1, \ldots, b' - 1\}^{\mathbb{N}}$. This morphism can be chosen letter-to-letter:

$$\varphi\big((\omega_n)_{n \in \mathbb{N}}\big) = \big(\psi(\omega_n)\big)_{n \in \mathbb{N}}.$$

(iii) According to [2] we say that a probability measure η on $\{0, 1, \ldots, b - 1\}^{\mathbb{N}}$ is linearly representable if there exist a set of r-dimensional nonnegative row vectors $\{R_0, \ldots, R_{b-1}\}$, a set of $r \times r$ nonnegative matrices $\mathcal{M} = \{M_0, \ldots, M_{b-1}\}$ and a positive r-dimensional column vector C, satisfying both conditions

$$\bigg(\sum_i R_i\bigg) C = 1 \quad \text{and} \quad \bigg(\sum_i M_i\bigg) C = C, \tag{2}$$

and such that

$$\eta[\omega_1 \ldots \omega_n] = R_{\omega_1} M_{\omega_2} \ldots M_{\omega_n} C. \tag{3}$$

REMARK 6. If $\sum_i M_i$ is irreducible and if $R_i = R M_i$ for any i, where R is the positive left eigenvector of $\sum_i M_i$ such that $RC = 1$, the measure η defined in (3) is σ-invariant and
$$\eta[\omega_1 \ldots \omega_n] = R M_{\omega_1} \ldots M_{\omega_n} C.$$

The next theorem concerns the linear representation of the sofic measures, given for instance in [7]. It uses also the ideas of the proof of [2, Theorem 4.28].

THEOREM 7. *A probability measure on $\{0, 1, \ldots, b-1\}^{\mathbb{N}}$ is sofic if and only if it is linearly representable.*

P r o o f. First we note that any Markov measure is linearly representable because the formula (1) is equivalent to

$$\mu[\omega_1 \ldots \omega_n] = \pi_{\omega_1} P_{\omega_2} \ldots P_{\omega_n} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \tag{4}$$

where

$$\pi_0 = \begin{pmatrix} p_0 & 0 & \ldots & 0 \end{pmatrix}, \qquad \pi_1 = \begin{pmatrix} 0 & p_1 & \ldots & 0 \end{pmatrix}, \qquad \ldots$$

$$P_0 = \begin{pmatrix} p_{00} & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p_{(b-1)0} & 0 & \ldots & 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0 & p_{01} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & p_{(b-1)1} & \ldots & 0 \end{pmatrix}, \quad \ldots$$

Now any sofic measure ν defined from a Markov measure μ and a map

$$\psi : \{0, 1, \ldots, b-1\} \to \{0, 1, \ldots, b'-1\}$$

is linearly representable because (4) implies

$$\begin{aligned}
\nu[\omega_1' \ldots \omega_n'] &= \sum_{\omega_1 \in \psi^{-1}(\omega_1')} \cdots \sum_{\omega_n \in \psi^{-1}(\omega_n')} \mu[\omega_1 \ldots \omega_n] \\
&= \left(\sum_{\omega_1 \in \psi^{-1}(\omega_1')} \pi_{\omega_1} \right) \ldots \left(\sum_{\omega_n \in \psi^{-1}(\omega_n')} P_{\omega_n} \right) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \qquad (5) \\
&= R_{\omega_1} M_{\omega_2} \ldots M_{\omega_n} C,
\end{aligned}$$

where for $0 \le i' \le b'$, the row vector $R_{i'} := \sum_{i \in \psi^{-1}(i')} \pi_i$, the matrix $M_{i'} :=$ $\sum_{i \in \psi^{-1}(i')} P_i$ and the column vector $C := \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ satisfy both conditions in (2).

Conversely, let η be a linearly representable measure, so there exists some r-dimensional row vectors $R_0, \ldots, R_{b-1}$, some $r \times r$ matrices $M_0, \ldots, M_{b-1}$ and r-dimensional column vector C satisfying (2) and (3). We shall modify the linear representation of η in the following way: let Δ be the diagonal matrix whose diagonal entries are the entries of C; setting $R_i' := R_i \Delta$, $M_i' := \Delta^{-1} M_i \Delta$ and $C' := \Delta^{-1} C$, the entries of C' are 1 and

$$\eta[\omega_1 \ldots \omega_n] = R_{\omega_1}' M_{\omega_2}' \ldots M_{\omega_n}' C'.$$

Setting

$$R_0'' := \begin{pmatrix} R_0' & 0 & \dots & 0 \end{pmatrix}, \qquad R_1'' := \begin{pmatrix} 0 & R_1' & \dots & 0 \end{pmatrix}, \qquad \dots,$$

$$M_0'' := \begin{pmatrix} M_0' & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ M_0' & 0 & \dots & 0 \end{pmatrix}, \qquad M_1'' := \begin{pmatrix} 0 & M_1' & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & M_1' & \dots & 0 \end{pmatrix}, \qquad \dots,$$

$$C'' := \begin{pmatrix} C' \\ \vdots \\ C' \end{pmatrix}$$

we have a third linear representation of η:

$$\eta[\omega_1 \dots \omega_n] = R_{\omega_1}'' M_{\omega_2}'' \dots M_{\omega_n}'' C''.$$

From (2), $p := \sum_i R_i''$ is a probability vector and $P := \sum_i M_i''$ is a stochastic matrix, they define a Markov probability measure μ on the product set $\{0, 1, \dots, rb - 1\}^{\mathbb{N}}$, and η is sofic because it is the image of μ by the morphism defined from the map $\psi : i \mapsto \lfloor \frac{i}{r} \rfloor$, $i \in \{0, 1, \dots, rb - 1\}$. $\qquad\square$

3. Level sets and density spectrum associated to a map $f : \mathbb{N} \to \mathbb{R}_+^*$

3.1. Position of the problem

This section is independent of the previous. In Theorem 9 below we do not make any hypothesis on the map $f : \mathbb{N} \to \mathbb{R}_+^*$ but, in the examples we consider later, f has a polynomial rate of growth, meaning that

$$-\infty < \alpha_1 := \liminf_{n \to \infty} \frac{\log f(n)}{\log n} \le \alpha_2 := \limsup_{n \to \infty} \frac{\log f(n)}{\log n} < +\infty. \qquad (6)$$

Our purpose is to associate to any $\alpha \in [\alpha_1, \alpha_2]$, a set of positive integers $\mathcal{E}(\alpha)$ as large as possible such that

$$\lim_{n \in \mathcal{E}(\alpha), \ n \to \infty} \frac{\log f(n)}{\log n} = \alpha \qquad (7)$$

(the notation $\lim_{n \in E, \ n \to \infty} u_n$ stands for $\lim_{k \to \infty} u_{n_k}$, where $E = \{n_1, n_2, \dots\}$ with $n_1 < n_2 < \dots$). Then we call "level sets" the sets $\mathcal{E}(\alpha)$, and "density spectrum" the map which associates to α the "density" (in the following sense) of $\mathcal{E}(\alpha)$: because of the following remark, we do not use the natural densities defined for any $S \subset \mathbb{N}$ by

$$d_-(S) := \liminf_{N \to \infty} \frac{\#S \cap [1, N)}{N},$$

$$\mathrm{d}_+(S) := \limsup_{N \to \infty} \frac{\#S \cap [1, N)}{N}$$

but the exponential densities defined by

$$\mathrm{d}_-^{\exp}(S) := \liminf_{N \to \infty} \frac{\log\left(\#S \cap [1, N)\right)}{\log N},$$

$$\mathrm{d}_+^{\exp}(S) := \limsup_{N \to \infty} \frac{\log\left(\#S \cap [1, N)\right)}{\log N}.$$

REMARK 8. One cannot expect to find for any α in a non trivial interval $[\alpha_1, \alpha_2]$, a set $\mathcal{E}(\alpha)$ with positive natural density $\mathrm{d}_-(\mathcal{E}(\alpha))$ and satisfying the condition (7). Indeed if we have $\mathrm{d}_-(\mathcal{E}(\alpha)) > 0$ for any α in a non countable set A, there exists $\varepsilon > 0$ such that $\mathrm{d}_-(\mathcal{E}(\alpha)) \geq \varepsilon$ for infinitely many values of α. When $\alpha \neq \beta$, the set $\mathcal{E}(\alpha) \cap \mathcal{E}(\beta)$ is obviously finite, so by removing a finite number of elements from one of these sets, $\mathcal{E}(\alpha)$ and $\mathcal{E}(\beta)$ are disjoint. One can construct in this way a sequence of disjoint sets of positive integers $\mathcal{E}(\alpha_n)$ such that $\mathrm{d}_-(\mathcal{E}(\alpha_n)) \geq \varepsilon$, but this is in contradiction with the following inequalities (deduced from the general formula $\liminf_{n \to \infty}(u_n + v_n) \geq \liminf_{n \to \infty}(u_n) + \liminf_{n \to \infty}(v_n)$):

$$1 \geq \mathrm{d}_-\left(\bigcup_{n \in \mathbb{N}} \mathcal{E}(\alpha_n)\right) \geq \sum_{n \in \mathbb{N}} \mathrm{d}_-\left(\mathcal{E}(\alpha_n)\right).$$

THEOREM 9. *We associate to any map $f : \mathbb{N} \to \mathbb{R}_+^*$ and to any $\alpha \geq 0$, $\varepsilon > 0$, the set*

$$\mathcal{E}(\alpha, \varepsilon) := \left\{ n \in \mathbb{N} \ : \ \alpha - \varepsilon \leq \frac{\log f(n)}{\log n} \leq \alpha + \varepsilon \right\}. \tag{8}$$

There exist some integers $1 = N_1 < N_2 < \cdots$ such that the set

$$\mathcal{E}(\alpha) := \bigcup_{k \in \mathbb{N}} \left(\mathcal{E}(\alpha, 1/k) \cap [N_k, N_{k+1},)\right), \tag{9}$$

has densities

$$
\begin{aligned}
\mathrm{d}_-\big(\mathcal{E}(\alpha)\big) &= \lim_{\varepsilon \to 0} \mathrm{d}_-\big(\mathcal{E}(\alpha, \varepsilon)\big) &= \inf_{\varepsilon > 0} \mathrm{d}_-\big(\mathcal{E}(\alpha, \varepsilon)\big), \\
\mathrm{d}_-^{\exp}\big(\mathcal{E}(\alpha)\big) &= \lim_{\varepsilon \to 0} \mathrm{d}_-^{\exp}\big(\mathcal{E}(\alpha, \varepsilon)\big) &= \inf_{\varepsilon > 0} \mathrm{d}_-^{\exp}\big(\mathcal{E}(\alpha, \varepsilon)\big), \\
\mathrm{d}_+\big(\mathcal{E}(\alpha)\big) &= \lim_{\varepsilon \to 0} \mathrm{d}_+\big(\mathcal{E}(\alpha, \varepsilon)\big) &= \inf_{\varepsilon > 0} \mathrm{d}_+\big(\mathcal{E}(\alpha, \varepsilon)\big), \\
\mathrm{d}_+^{\exp}\big(\mathcal{E}(\alpha)\big) &= \lim_{\varepsilon \to 0} \mathrm{d}_+^{\exp}\big(\mathcal{E}(\alpha, \varepsilon)\big) &= \inf_{\varepsilon > 0} \mathrm{d}_+^{\exp}\big(\mathcal{E}(\alpha, \varepsilon)\big).
\end{aligned}
\tag{10}
$$

If $\mathcal{E}(\alpha)$ is not finite, one has $\lim_{n \in \mathcal{E}(\alpha), \ n \to \infty} \frac{\log f(n)}{\log n} = \alpha$ and one says that $\mathcal{E}(\alpha)$ is a level set associated to α. One also says that $\mathrm{d}_-^{\exp}(\mathcal{E}(\cdot))$ and $\mathrm{d}_+^{\exp}(\mathcal{E}(\cdot))$ are <u>the</u> density spectrums of f because they don not depend on the construction of $\mathcal{E}(\alpha)$ in (9), but only depend on the exponential densities of the sets $\mathcal{E}(\alpha, \varepsilon)$.

The second subsection, independent on the first, will be used later to prove the theorem.

3.2. A general lemma about the subsets of $\mathbb{N}$

Given a sequence $(E_k)_k$ of subsets of $\mathbb{N}$, monotonic for the inclusion, we construct a subset of $\mathbb{N}$ whose densities are the limits of the ones of E_k.

LEMMA 10. *Let* $(E_k)_k$ *be a sequence of subsets of* $\mathbb{N}$, *non-increasing or non-decreasing for the inclusion. There exist some integers* $1 = N_1 < N_2 < \cdots$ *such that the set*

$$E = \bigcup_{k \in \mathbb{N}} \big(E_k \cap [N_k, N_{k+1}) \big) \tag{11}$$

has densities

$$\begin{aligned}
\mathrm{d}_-(E) = \ell_- &:= \lim_{k \to \infty} \mathrm{d}_-(E_k), \quad \mathrm{d}_-^{\mathrm{exp}}(E) = \ell_-^{\mathrm{exp}} := \lim_{k \to \infty} \mathrm{d}_-^{\mathrm{exp}}(E_k), \\
\mathrm{d}_+(E) = \ell_+ &:= \lim_{k \to \infty} \mathrm{d}_+(E_k), \quad \mathrm{d}_+^{\mathrm{exp}}(E) = \ell_+^{\mathrm{exp}} := \lim_{k \to \infty} \mathrm{d}_+^{\mathrm{exp}}(E_k).
\end{aligned} \tag{12}$$

P r o o f. Let us define the integers $1 = N_1 < N_2 < \cdots$ by induction: we suppose that we know the value of N_k for some k, and we search N_{k+1} large enough in view to obtain (12). We have $E_k' \subset E_k \subset E_k''$ with

$$E_k' := E_k \cap [N_k, \infty) \quad \text{and} \quad E_k'' := E_k \cup [1, N_k),$$

and E_k', E_k'' have same densities as E_k. So, by definition of "limitinf" and "limitsup", we can chose N_{k+1} such that for all $N \geq N_{k+1}$,

$$\begin{aligned}
N\big(\mathrm{d}_-(E_k) - \tfrac{1}{k}\big) &\leq \#E_k' \cap [1, N) \leq \#E_k'' \cap [1, N) \leq N\big(\mathrm{d}_+(E_k) + \tfrac{1}{k}\big), \\
N^{\mathrm{d}_-^{\mathrm{exp}}(E_k) - \frac{1}{k}} &\leq \#E_k' \cap [1, N) \leq \#E_k'' \cap [1, N) \leq N^{\mathrm{d}_+^{\mathrm{exp}}(E_k) + \frac{1}{k}} \\
N\big(\mathrm{d}_-(E_{k+1}) - \tfrac{1}{k}\big) &\leq \#E_{k+1} \cap [1, N) \leq N\big(\mathrm{d}_+(E_{k+1}) + \tfrac{1}{k}\big), \\
N^{\mathrm{d}_-^{\mathrm{exp}}(E_{k+1}) - \frac{1}{k}} &\leq \#E_{k+1} \cap [1, N) \leq N^{\mathrm{d}_+^{\mathrm{exp}}(E_{k+1}) + \frac{1}{k}}.
\end{aligned} \tag{13}$$

Using again the definition of "limitinf" and "limitsup", we can impose a supplementary conditions to N_{k+1}, according to the value of k mod. 4:

$$\begin{aligned}
\text{if } k \equiv 0 \text{ mod. 4}, \qquad \#E_k'' \cap [1, N_{k+1}] &\leq N_{k+1}\big(\mathrm{d}_-(E_k) + \tfrac{1}{k}\big) \\
\text{if } k \equiv 1 \text{ mod. 4}, \qquad \#E_k'' \cap [1, N_{k+1}] &\leq N_{k+1}^{\mathrm{d}_-^{\mathrm{exp}}(E_k) + \frac{1}{k}} \\
\text{if } k \equiv 2 \text{ mod. 4}, \quad N_{k+1}\big(\mathrm{d}_+(E_k) - \tfrac{1}{k}\big) &\leq \#E_k' \cap [1, N_{k+1}] \\
\text{if } k \equiv 3 \text{ mod. 4}, \qquad N_{k+1}^{\mathrm{d}_+^{\mathrm{exp}}(E_k) - \frac{1}{k}} &\leq \#E_k' \cap [1, N_{k+1}].
\end{aligned} \tag{14}$$

Since

$$E_k' \cap [1, N_{k+1}) \subset E \cap [1, N_{k+1}) \subset E_k'' \cap [1, N_{k+1}),$$

we deduce from (13) (applied to $N = N_{k+1}$) and (14) that, according to the value of $k \bmod. 4$,

$$
\begin{aligned}
\lim_{\substack{k \equiv 0 \\ k \to \infty}} \frac{\#E \cap [1, N_{k+1})}{N_{k+1}} &= \ell_-, \quad \lim_{\substack{k \equiv 1 \\ k \to \infty}} \frac{\log\left(\#E \cap [1, N_{k+1})\right)}{\log N_{k+1}} = \ell_-^{\exp}, \\
\lim_{\substack{k \equiv 2 \\ k \to \infty}} \frac{\#E \cap [1, N_{k+1})}{N_{k+1}} &= \ell_+, \quad \lim_{\substack{k \equiv 3 \\ k \to \infty}} \frac{\log\left(\#E \cap [1, N_{k+1})\right)}{\log N_{k+1}} = \ell_+^{\exp}.
\end{aligned}
\tag{15}
$$

For any $k \in \mathbb{N}$ and $N_{k+1} < N \le N_{k+2}$ we have the following inclusions, if the sequence of sets $(E_k)_k$ is non-increasing:

$$
E_{k+1} \cap [1, N) \subset E \cap [1, N) \subset E_k'' \cap [1, N) \tag{16}
$$

while, if the sequence of sets $(E_k)_k$ is non-decreasing:

$$
E_k' \cap [1, N) \subset E \cap [1, N) \subset E_{k+1} \cap [1, N). \tag{17}
$$

Now, (12) follows from (16), (17), (13) and (15). □

REMARK 11. If one remove the hypothesis that $(E_k)_k$ is monotonic, it may happen that the lower (resp. upper) densities of the set E defined by (11) are smaller (resp. larger) that the limitinf (resp. the limitsup) of the corresponding densities of E_k, even if the sequence $(N_k)_k$ increases quickly, so the method used for the proof of Lemma 10 do not apply. Suppose for instance that, for some positive integer κ, the set $E_1 = \cdots = E_{\kappa-1}$ has lower density $1/2$ and more precisely,
$$
\#\left(E \cap [1, N_\kappa)\right) = \#\left(E_1 \cap [1, N_\kappa)\right) = \lfloor N_\kappa/2 \rfloor.
$$

Suppose that the set E_κ is distinct from E_1 but has also lower density $1/2$ with, for instance, $E_\kappa \cap [1, 2N_\kappa) = [1, N_\kappa)$. Then if $N_{\kappa+1} \ge 2N_\kappa$, the integer $N = 2N_\kappa$ satisfy
$$
\#\left(E \cap [1, N)\right) = \#\left(E_1 \cap [1, N_\kappa)\right) = \lfloor N/4 \rfloor.
$$

Now suppose about the lower exponential density, that $E_1 = \cdots = E_{\kappa-1}$ have lower exponential density $1/2$, more precisely,

$$
\#\left(E \cap [1, N_\kappa)\right) = \#\left(E_1 \cap [1, N_\kappa)\right) = \lfloor N_\kappa^{1/2} \rfloor.
$$

Suppose that E_κ has lower exponential density $1/2$ with, for instance, $E_\kappa \cap [1, N_\kappa^2) = [1, N_\kappa)$. Then if $N_{\kappa+1} \ge N_\kappa^2$, the integer $N = N_\kappa^2$ satisfies

$$
\#\left(E \cap [1, N)\right) = \#\left(E_1 \cap [1, N_\kappa)\right) = \lfloor N^{1/4} \rfloor.
$$

3.3. Proof of Theorem 9

P r o o f. Lemma 10 applies to the non-increasing sequence

$$
E_k = \mathcal{E}(\alpha, 1/k). \qquad \square
$$

EXAMPLE 12. Let $f(n) = n^{1+\sin n}$ then, given $\alpha \in [0, 2]$ and $\varepsilon > 0$, $\mathcal{E}(\alpha, \varepsilon)$ is the set of the integers n such that $1 + \sin n \in [\alpha - \varepsilon, \alpha + \varepsilon]$. It has a positive usual density hence it has exponential density 1, as well as $\mathcal{E}(\alpha)$. The density spectrum of f is:
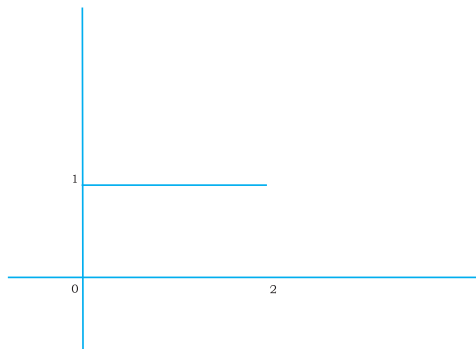

FIGURE 1. Exponential density of $\mathcal{E}(\alpha)$ in function of α in Example 12.

EXAMPLE 13. The number of representations of n in base 2 with digit in $\{0, 1, 2\}$, defined by

$$\mathcal{N}(n) : \# \left\{ (\omega_i)_{i \geq 0} \ : \ n = \sum_{i=0}^{\infty} \omega_i 2^i, \ \omega_i \in \{0, 1, 2\} \right\}, \tag{18}$$

has a polynomial rate of growth: (6) holds with $\alpha_1 = 0$ (because, for any k, $\mathcal{N}(2^k - 1) = 1$) and $\alpha_2 = \log \frac{1+\sqrt{5}}{2} / \log 2$ (see [5, Corollary 6.10]).

By [11, Theorem 19] there exists a subset $S \subset \mathbb{N}$ of natural density 1—and consequently exponential density 1—such that the limit

$$\alpha_0 := \lim_{n \in S, \ n \to \infty} \frac{\log \mathcal{N}(n)}{\log n}$$

exists and is positive. Since $S \cap [N, \infty) \subset \mathcal{E}(\alpha_0, \varepsilon)$ for any $\varepsilon > 0$ and for N large enough, one has also

$$d_{\pm}\big(\mathcal{E}(\alpha_0)\big) = d_{\pm}^{\exp}\big(\mathcal{E}(\alpha_0)\big) = 1.$$

Nevertheless, let us deduce from [11, Corollary 32] that

$$\exists \alpha_3 > \alpha_0, \ d_+^{\exp}\big(\mathcal{E}(\alpha_3)\big) > 0. \tag{19}$$

From [11, Corollary 32], given α_0' and β_0' in the interval $\left(\alpha_0, \frac{\log 3}{\log 2} - 1\right)$ one has for N large enough

$$\#\left\{ n < N \ : \ \frac{\log \mathcal{N}(n)}{\log n} > \alpha_0' \right\} \geq N^{\beta_0'} \tag{20}$$

and [11, Remark 21-3] ensures that $\left(\alpha_0, \frac{\log 3}{\log 2} - 1\right) \neq \emptyset$.

To prove (19) by contraposition, suppose that

$$d_+^{\exp}(\mathcal{E}(\alpha)) < \beta_0' \quad \text{for any} \quad \alpha \geq \alpha_0'.$$

Then by the definition of $\mathcal{E}(\alpha)$, for each $\alpha \geq \alpha_0'$ there exists $\varepsilon_\alpha > 0$ such that $d_+^{\exp}(\mathcal{E}(\alpha, \varepsilon_\alpha)) < \beta_0'$. There exists $\beta_\alpha < \beta_0'$ such that, for N large enough,

$$\#\left\{n < N \; : \; \alpha - \varepsilon_\alpha \leq \frac{\log \mathcal{N}(n)}{\log n} \leq \alpha + \varepsilon_\alpha\right\} \leq N^{\beta_\alpha}.$$

The open intervals $(\alpha - \varepsilon_\alpha, \alpha + \varepsilon_\alpha)$, $\alpha \geq \alpha_0'$, cover the compact set $[\alpha_0', 2]$, so there exists a subcover of $[\alpha_0', 2]$ by n_0 intervals of this form. Let $\beta < \beta_0'$ be the maximum of β_α for α being the center of such an interval. We have

$$\#\left\{n < N \; : \; \alpha_0' \leq \frac{\log \mathcal{N}(n)}{\log n} \leq 2\right\} \leq n_0 N^\beta \quad (N \text{ large enough}). \qquad (21)$$

Since one check easily that $\frac{\log \mathcal{N}(n)}{\log n} \leq 2$ for any $n \in \mathbb{N}$, (21) is in contradiction with (20), hence (19) holds and more precisely there exists

$$\alpha_3 \geq \alpha_0' > \alpha_0 \quad \text{and} \quad \beta_3 \geq \beta_0'$$

such that $d_+^{\exp}(\mathcal{E}(\alpha_3)) = \beta_3$. Since $\frac{\log 3}{\log 2} - 1 \approx 0.585$ the upper density spectrum of $\mathcal{N}$ is:
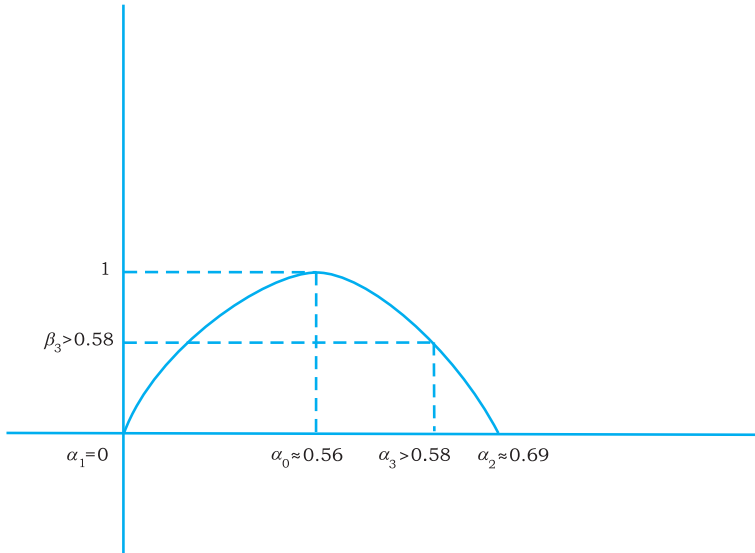

FIGURE 2. Upper exponential density of $\mathcal{E}(\alpha)$ in function of α in Example 13.
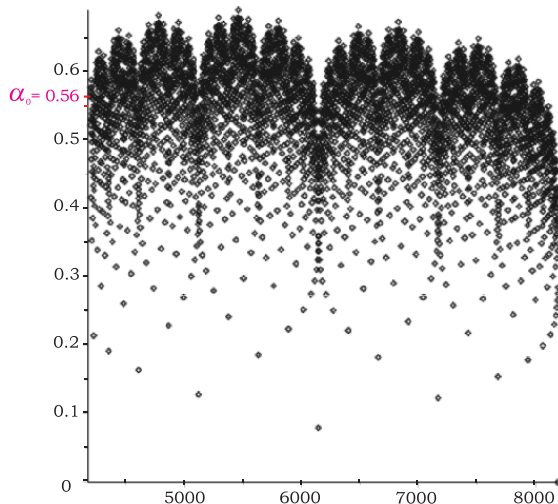
FIGURE 3. The values of $\frac{\log \mathcal{N}(n)}{\log n}$ for $n \in [2^{12}, 2^{13})$ in Example 13. The order of growth of $\mathcal{N}$ can be represented by this graph.

4. Relation between singularity spectrum and density spectrum

Let η be a probability measure on $\{0, 1, \ldots, b-1\}^{\mathbb{N}}$, the level sets $E(\alpha)$ are defined by

$$E(\alpha) := \left\{ (\omega_n)_{n \in \mathbb{N}} \; : \; \lim_{k \to \infty} \frac{\log \eta[\omega_1 \ldots \omega_k]}{\log(1/b^k)} = \alpha \right\} \qquad (22)$$

and one calls the singularity spectrum, the map

$$\alpha \mapsto \text{H-dim}\big(E(\alpha)\big),$$

where H-dim is the Hausdorff dimension, on the understanding that the distance between two sequences $(\omega_n)_{n \in \mathbb{N}}$ and $(\omega'_n)_{n \in \mathbb{N}}$ is $b^{1-\inf\{i \,:\, \omega_i \neq \omega'_i\}}$. Any ball is a cylinder set, and the diameter of a cylinder set is

$$\delta\big([\omega_1 \ldots \omega_k]\big) := b^{-k}.$$

It is natural to associate to η the function $f_\eta : \mathbb{N} \cup \{0\} \to [0, 1]$ defined as follows from the expansion of n in base b:

$$f_\eta(n) := \eta[\omega_1 \ldots \omega_k] \quad \text{for any} \quad n =_b \omega_1 b^{k-1} + \cdots + \omega_k b^0,$$

where the notation $=_b$ means that $\forall i, \; \omega_i \in \{0, 1, \ldots, b-1\}$ and $\omega_1 \neq 0$.

Notice that f_η depends only on the restriction of η to the set of the sequences $(\omega_n)_{n \in \mathbb{N}}$ such that $\omega_1 \neq 0$.

PROPOSITION 14. *Let η be a probability measure on $\{0, 1, \ldots, b-1\}^{\mathbb{N}}$ such that*

$$\lim_{n \to \infty} \frac{\log \eta[0\omega_1 \ldots \omega_n]}{\log \eta[\omega_1 \ldots \omega_n]} = 1 \quad \text{for any} \quad (\omega_n)_{n \in \mathbb{N}}.$$

The level sets $E(\cdot)$ (of the measure η) and $\mathcal{E}(\cdot)$ (of the function f_η) satisfy the inequality

$$\text{H-dim}\big(E(\alpha)\big) \leq \text{d}_{-}^{\exp}\big(\mathcal{E}(-\alpha)\big).$$

P r o o f. From the hypothesis on the probability η, a sequence $(\omega_n)_{n \in \mathbb{N}}$ with the first term $\omega_1 = 0$ belongs to $E(\alpha)$ if and only if $(\omega_{n+1})_{n \in \mathbb{N}}$ do. The set $E'(\alpha)$ of the sequences $(\omega_n)_{n \in \mathbb{N}} \in E(\alpha)$ with the first term $\omega_1 \neq 0$, has same Hausdorff dimension as $E(\alpha)$ because

$$E(\alpha) = \bigcup_{n \geq 0} \{0^n\} \times E'(\alpha) \quad \text{and} \quad \text{H-dim}\big(\{0^n\} \times E'(\alpha)\big) = \text{H-dim}\big(E'(\alpha)\big).$$

According to (10) it is sufficient to prove for any $k \in \mathbb{N}$ the inequality

$$\text{H-dim}(E'(\alpha)) \leq \text{d}_{-}^{\exp}\left(\mathcal{E}\left(-\alpha, \frac{1}{k}\right)\right).$$

The integer k is now fixed and, by definition of the limit inf, there exists an infinite set $E \subset \mathbb{N}$ such that

$$\lim_{N \in E, \ N \to \infty} \frac{\log\left(\#\mathcal{E}(-\alpha, \frac{1}{k}) \cap [1, N)\right)}{\log N} = \text{d}_{-}^{\exp}\left(\mathcal{E}\left(-\alpha, \frac{1}{k}\right)\right). \tag{23}$$

One can assume that the elements of E have the form $N = b^i$ with $i \in \mathbb{N}$: indeed N is in some interval $[b^{i(N)}, b^{i(N)+1})$, the denominator $\log N$ in (23) is equivalent to $\log\big(b^{i(N)}\big)$, and the numerator is greater or equal to

$$\log\left(\#\mathcal{E}\left(-\alpha, \frac{1}{k}\right) \cap \left[1, b^{i(N)}\right)\right).$$

Let $\omega \in E'(\alpha)$. There exists $\kappa \in \mathbb{N}$ such that

$$\alpha - \frac{1}{2k} \leq \frac{\log \eta[\omega_1 \ldots \omega_\kappa]}{\log(1/b^\kappa)} \leq \alpha + \frac{1}{2k} \tag{24}$$

and, since these inequalities are true for any κ large enough, one can chose $\kappa = \kappa(\omega)$, such that $\kappa \geq 2k\alpha + 2$ and $b^\kappa \in E$. Let us prove that the integer

$$n = n(\omega) := \omega_1 b^{\kappa-1} + \cdots + \omega_\kappa b^0 \tag{25}$$

belongs to the level set $\mathcal{E}(-\alpha, \frac{1}{k})$ if k is large enough. The numerator in (24) is $\log(f_\eta(n))$ because $\omega_1 \neq 0$.

One has

$$-\frac{\log(f_\eta(n))}{\log n} = \frac{\log \eta[\omega_1 \dots \omega_\kappa]}{\log(1/b^\kappa)} \frac{\log(b^\kappa)}{\log n}$$

hence, using (24),

$$\left(\alpha - \frac{1}{2k}\right)\frac{\log(b^\kappa)}{\log n} \leq -\frac{\log(f_\eta(n))}{\log n} \leq \left(\alpha + \frac{1}{2k}\right)\frac{\log(b^\kappa)}{\log n}.$$

Now

$$1 \leq \frac{\log(b^\kappa)}{\log n} \leq \frac{\log(b^\kappa)}{\log(b^{\kappa-1})} = \frac{\kappa}{\kappa - 1} \quad \text{and} \quad \left(\alpha + \frac{1}{2k}\right)\frac{\kappa}{\kappa - 1} \leq \alpha + \frac{1}{k}$$

(consequence of the hypothesis $\kappa \geq 2k\alpha + 2$), so $n \in \mathcal{E}(-\alpha, \frac{1}{k})$.

There exists a disjoint cover of $E'(\alpha)$ by a finite or countable family of cylinder sets C_i, each of the C_i having the form $[\omega_1 \dots \omega_\kappa]$, where $\kappa = \kappa(\omega)$ is defined in (24). We consider the cylinder sets C_i such that $\kappa(\omega)$ has a given value κ_0. The corresponding integers $n(\omega)$ are distinct and belong to

$$\mathcal{E}(-\alpha, \frac{1}{k}) \cap [1, b^{\kappa_0}).$$

By the hypotheses on $\kappa(\omega)$ one consider only the integers $\kappa_0 \geq 2k\alpha + 2$ such that $b^{\kappa_0} \in E$; in particular, the larger is k, the larger is κ_0. Let $\varepsilon > 0$ and suppose that k is large enough so that, applying (23) to $N = b^{\kappa_0}$ and setting

$$d_k = \mathrm{d}_-^{\exp}\left(\mathcal{E}\left(-\alpha, \frac{1}{k}\right)\right),$$

$$\frac{\log\left(\#\mathcal{E}(-\alpha, \frac{1}{k}) \cap [1, b^{\kappa_0})\right)}{\log b^{\kappa_0}} \leq d_k + \varepsilon.$$

Consequently, for any $s > d_k + \varepsilon$,

$$\sum_i \delta(C_i)^s \leq \sum_{\kappa_0=0}^{\infty} b^{\kappa_0(d_k+\varepsilon)} b^{-\kappa_0 s} = \frac{1}{1 - b^{d_k+\varepsilon-s}},$$

proving that

$$\text{H-dim}(E'(\alpha)) \leq d_k + \varepsilon.$$

Since it is true for any k large enough and any $\varepsilon > 0$, this implies

$$\text{H-dim}(E'(\alpha)) \leq \mathrm{d}_-^{\exp}(\mathcal{E}(-\alpha)).$$

$\square$

5. Bernoulli convolution and number of representations in integral base

5.1. Bernoulli convolution in integral base and related matrices

The Bernoulli convolution [6, 22] in integral base $b \geq 2$, associated to a positive probability vector $p = (p_0, \ldots, p_{d-1})$ with $d \geq b$, is the probability measure $\eta = \eta_{b,p}$ defined by setting, for any interval $I \subset \mathbb{R}$

$$\eta_{b,p}(I) := P_p\left(\left\{(\omega_k)_{k \in \mathbb{N}} \ : \ 0 \leq \omega_k \leq d-1, \ \sum_k \frac{\omega_k}{b^k} \in I\right\}\right), \qquad (26)$$

where P_p the product probability defined on $\{0, \ldots, d-1\}^{\mathbb{N}}$ from the probability vector p.

We define also, on the symbolic space $\{0, 1, \ldots, b-1\}^{\mathbb{N}}$, the probability measures

$$\eta_{q,symb}[\varepsilon_1 \ldots \varepsilon_k] := \frac{\eta(q + I_{\varepsilon_1 \ldots \varepsilon_k})}{\eta(q + [0, 1))} \quad (q = 0, 1, 2, \ldots),$$

$$\eta_{sum,symb}[\varepsilon_1 \ldots \varepsilon_k] := \sum_{q=0}^{\infty} \eta(q + I_{\varepsilon_1 \ldots \varepsilon_k}). \qquad (27)$$

Let us prove that both are sofic.

REMARK 15. The shift-invariant measure $\eta_{sum,symb}$ is the image of P_p by the shift-commuting map φ which associates to any $(\omega_k)_{k \in \mathbb{N}}$, the b-expansion of the fractional part of $\sum_k \frac{\omega_k}{b^k}$. Unfortunately φ is discontinuous:

$$\lim_{n \to \infty} \varphi((b-1)^n \bar{0}) = \overline{b-1} \quad \text{is different from} \quad \varphi(\lim_{n \to \infty} (b-1)^n \bar{0}) = \bar{0}.$$

So it is not sufficient to use this map for proving that $\eta_{sum,symb}$ is sofic.

Let the bi-infinite matrix

$$M_\infty := \begin{pmatrix} \ddots & \vdots & \vdots & \iddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \iddots \\ \cdots & 0 & p_{d-1} & \cdots & p_{d-b-1} & \cdots & p_0 & \cdots & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \cdots & p_{d-1} & \cdots & p_b & \cdots & p_0 & 0 & \cdots \\ \iddots & \vdots & \vdots & \ddots & \vdots & \iddots & \vdots & \iddots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (28)$$

where each row contains the same probability vector $\begin{pmatrix} p_{d-1} & \cdots & p_0 \end{pmatrix}$, shifted b times to the right at the following row. We define in an unique way some matrices $M_0, \ldots, M_{b-1}$, by setting that $M_0, \ldots, M_{b-1}$ are submatrices of M_∞ of size $a + 1 := \left\lceil \frac{d-1}{b-1} \right\rceil$ and

$$M_0 := \begin{pmatrix} p_0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \ M_1 := \begin{pmatrix} p_1 & p_0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \ \cdots,$$

$$M_{b-1} := \begin{pmatrix} p_{b-1} & p_{b-2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}. \quad (29)$$

Assuming by convention that $p_i = 0$ for $i \notin \{0, \ldots, d-1\}$, we can write

$$M_j = (p_{j+bq-q'})_{\substack{0 \le q \le a \\ 0 \le q' \le a}} = \begin{pmatrix} p_j & p_{j-1} & \cdots & p_{j-a} \\ p_{j+b} & p_{j+b-1} & \cdots & p_{j+b-a} \\ \vdots & \vdots & \ddots & \vdots \\ p_{j+ab} & p_{j+ab-1} & \cdots & p_{j+ab-a} \end{pmatrix}. \quad (30)$$

REMARK 16. About the choice of the size of the matrices, $a = \left\lceil \frac{d-1}{b-1} \right\rceil - 1$ is the largest integer such that the matrix $\sum_j (p_{j+bq-q'})_{\substack{0 \le q \le a \\ 0 \le q' \le a}}$ is irreducible, and the smallest integer such that its transpose is stochastic.

THEOREM 17. *The following formula gives the measure of the translated b--adic interval* $q + I_{\varepsilon_1 \ldots \varepsilon_k}$ *with* $q \in \{0, 1, \ldots, a\}$, $\varepsilon_1, \ldots, \varepsilon_k \in \{0, 1, \ldots, b-1\}$ *and* $I_{\varepsilon_1 \ldots \varepsilon_k} := \left[\sum_{i=1}^{k} \frac{\varepsilon_i}{b^i}, \ \sum_{i=1}^{k} \frac{\varepsilon_i}{b^i} + \frac{1}{b^k} \right)$:

$$\eta(q + I_{\varepsilon_1 \ldots \varepsilon_k}) = E_q M_{\varepsilon_1} \ldots M_{\varepsilon_k} C, \quad (31)$$

where $E_0, E_1, \ldots, E_a$ *are the canonical basis* $(a+1)$*-dimensional row vectors and* C *the unique positive eigenvector of the irreducible matrix* $\sum_i M_i$ *such that* $\sum_i E_i C = 1$. *Consequently the measures* $\eta_{q,symb}$ *and* $\eta_{sum,symb}$ *are linearly representable.*

P r o o f. Let $I = I_{\varepsilon_1 \ldots \varepsilon_n}$ and $I' = I_{\varepsilon_2 \ldots \varepsilon_n}$. With the convention that $p_i = 0$ for any $i \notin \{0, 1, \ldots, d-1\}$, we have

$$\begin{aligned} \eta(q + I) &= \sum_{i=0}^{d-1} P\Big(\{ \omega_1 = i \text{ and } \sum_k \frac{\omega_{k+1}}{b^k} \in qb + \varepsilon_1 - i + I' \} \Big) \\ &= \sum_{i \in \mathbb{Z}} p_i \, \eta \, (qb + \varepsilon_1 - i + I') \\ &= \sum_{q' \in \mathbb{Z}} p_{qb+\varepsilon_1-q'} \, \eta \, (q' + I'). \end{aligned}$$

In fact q' belongs to $\{0, 1, \ldots, a\}$, otherwise $\eta(q' + I')$ is null. Since the coefficients $p_{qb + \varepsilon_1 - q'}$ for $q, q' \in \{0, 1, \ldots, a\}$, are the entries of M_{ε_1},

$$
\begin{pmatrix} \eta(I) \\ \eta(I + 1) \\ \vdots \\ \eta(I + a) \end{pmatrix} = M_{\varepsilon_1} \begin{pmatrix} \eta(I') \\ \eta(I' + 1) \\ \vdots \\ \eta(I' + a) \end{pmatrix}
$$

and, by induction,

$$
\begin{pmatrix} \eta(I) \\ \eta(I + 1) \\ \vdots \\ \eta(I + a) \end{pmatrix} = M_{\varepsilon_1} \ldots M_{\varepsilon_k} C \quad \text{with} \quad C := \begin{pmatrix} \eta([0, 1)) \\ \eta([1, 2)) \\ \vdots \\ \eta([a, a + 1)) \end{pmatrix}. \tag{32}
$$

In the particular case $k = 1$ we have $I = I_{\varepsilon_1}$ and, making the sum in (32) for $\varepsilon_1 = 0, 1, \ldots, b - 1$, we deduce that C is a eigenvector of $\sum_i M_i$. Moreover C is positive because the measure of any nontrivial subinterval of $\left[0, \frac{d-1}{b-1}\right]$ is positive, and $\sum_i E_i C = 1$ because $[0, a + 1] \supset \left[0, \frac{d-1}{b-1}\right]$ (the support of η). $\quad\square$

COROLLARY 18. *The measures $\eta_{q,symb}$ and $\eta_{sum,symb}$ are sofic, as well as the measure η' defined by*

$$
\eta'[\varepsilon_1 \ldots \varepsilon_k] := \eta_{sum,symb}[\varepsilon_k \ldots \varepsilon_1].
$$

Moreover, η' is the continuous image by the map $i \mapsto \left\lfloor \frac{i}{a+1} \right\rfloor$, of the Markov measure on $\{0, 1, \ldots, (a+1)b - 1\}^{\mathbb{N}}$ of transition matrix

$$
P_{\eta'} := \begin{pmatrix} {}^t M_0 & {}^t M_1 & \ldots & {}^t M_{b-1} \\ \vdots & \vdots & \ddots & \vdots \\ {}^t M_0 & {}^t M_1 & \ldots & {}^t M_{b-1} \end{pmatrix}
$$

and initial probability vector

$$
\begin{pmatrix} {}^t C & \ldots & {}^t C \end{pmatrix}.
$$

P r o o f. According to Theorems 17 and 7, $\eta_{q,symb}$ and $\eta_{sum,symb}$ are sofic. Formula (31) gives a linear representation of η':

$$
\eta'[\varepsilon_1 \ldots \varepsilon_k] = {}^t C \; {}^t(M_{\varepsilon_1}) \ldots {}^t(M_{\varepsilon_k}) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.
$$

According to the proof of the converse part of Theorem 7, $P_{\eta'}$ is the transition matrix and $\begin{pmatrix} {}^t C & \ldots & {}^t C \end{pmatrix}$ is the initial probability vector of the Markov chain associated to η'. $\quad\square$

5.2. The matrices M_j are also related to the transducer of normalization [15] and to the number of representations of the integers in base b

DEFINITION 19. (i) The normalization in base b, with $d \geq b$ digits, is the map

$$\mathfrak{n} : \{0, \ldots, d-1\}^{\mathbb{N}} \rightarrow \{0, \ldots, b-1\}^{\mathbb{Z}}$$
$$(\omega_i)_{i \in \mathbb{N}} \mapsto (\varepsilon_i)_{i \in \mathbb{Z}} \quad \text{such that}$$

$$\sum_{i=1}^{\infty} \frac{\omega_i}{b^i} = \sum_{i=-\infty}^{\infty} \frac{\varepsilon_i}{b^i} \left(= \sum_{i=-(h-1)}^{\infty} \frac{\varepsilon_i}{b^i} \text{ with } \varepsilon_{-(h-1)} \neq 0 \text{ if } (\omega_i)_{i \in \mathbb{N}} \neq (0)_{i \in \mathbb{N}} \right)$$

with the additional condition that the ε_i are not eventually $b-1$.

(ii) The normalization in base b also associates to a finite sequence

$$(\omega_{h-1}, \ldots \ldots, \omega_0) \quad \text{with terms in} \quad \{0, \ldots, d-1\}$$

and distinct from 0^h, the sequence

$$(\varepsilon_{k-1}, \ldots, \varepsilon_0) \quad \text{with terms in} \quad \{0, \ldots, b-1\}$$

such that

$$\omega_{h-1}b^{h-1} + \cdots + \omega_0 b^0 = \varepsilon_{k-1}b^{k-1} + \cdots + \varepsilon_0 b^0 \text{ and } \varepsilon_{k-1} \neq 0. \qquad (33)$$

Notice that if we put $\omega_h = \omega_{h+1} = \cdots = \omega_{k-1} = 0$ we have

$$\mathfrak{n}(\omega_{k-1}, \ldots, \omega_0, 0, 0, \ldots) = (\ldots, 0, 0, \varepsilon_{k-1}, \ldots, \varepsilon_0, 0, 0, \ldots).$$
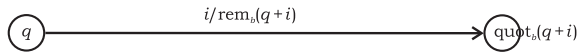
(iii) The number of b-representations of n with digits in $\{0, \ldots, d-1\}$ is

$$\mathcal{N}(n) := \#\left\{ (\omega_i)_{i \geq 0} \ : \ n = \sum_{i=0}^{\infty} \omega_i b^i, \ \omega_i \in \{0, \ldots, d-1\} \right\}.$$

The number of b-representations of length k is

$$\mathcal{N}_k(n) := \#\left\{ (\omega_0, \ldots, \omega_{k-1}) \ : \ n = \sum_{i=0}^{k-1} \omega_i b^i, \ \omega_i \in \{0, \ldots, d-1\} \right\}.$$

Now we define the transducer $\mathcal{T}$ as follows:

where q belongs to the set of states $\{0, \ldots, a\}$ with $a = \left\lceil \frac{d-1}{b-1} \right\rceil - 1$, and $\mathrm{quot}_b(q+i)$ and $\mathrm{rem}_b(q+i)$ are, respectively, the quotient and the remainder of the Euclidean division of $q+i$ by b. It is the transpose of the transducer of normalization defined by Frougny and Sakarovitch in [15, Propositions 2.2.5, 2.2.6 and Theorem 2.2.7], which chose $\{-a, \ldots, a\}$ as set of states.

PROPOSITION 20. *Let us consider the matrices* $M_0, \ldots, M_{b-1}$ *defined by* (29) *or* (30)*, where we suppose that* $p_0 = \cdots = p_{d-1} = \frac{1}{d}$.

(i) *The transpose of* dM_j *is the incidence matrix of the graph of set of vertices* $\{0, \ldots, a\}$*, whose edges are the edges of* $\mathcal{T}$ *with output label* j.

(ii) *Let* $w = \varepsilon_1 \ldots \varepsilon_k \in \{0, \ldots, b-1\}^k$ *and* $M_w = M_{\varepsilon_1} \ldots M_{\varepsilon_k}$*, we have*

$$d^k M_w = \begin{pmatrix} \mathcal{N}_k(n) & \mathcal{N}_k(n-1) & \ldots & \mathcal{N}_k(n-a) \\ \mathcal{N}_k(n+b^k) & \mathcal{N}_k(n+b^k-1) & \ldots & \mathcal{N}_k(n+b^k-a) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{N}_k(n+ab^k) & \mathcal{N}_k(n+ab^k-1) & \ldots & \mathcal{N}_k(n+ab^k-a) \end{pmatrix},$$

where $n = \varepsilon_1 b^{k-1} + \cdots + \varepsilon_k b^0$.

The first row of $d^k M_w$ *is also equal to* $\begin{pmatrix} \mathcal{N}(n) & \mathcal{N}(n-1) & \ldots & \mathcal{N}(n-a) \end{pmatrix}$*, so we have*

$$\eta_{b,p}(I_{\varepsilon_1 \ldots \varepsilon_k}) = \frac{1}{d^k} \begin{pmatrix} \mathcal{N}(n) & \mathcal{N}(n-1) & \ldots & \mathcal{N}(n-a) \end{pmatrix} C. \tag{34}$$

(iii) *To normalize a finite sequence* $(\xi_{h-1}, \ldots, \xi_0) \in \{0, \ldots, d-1\}^h$*, one enters in* $\mathcal{T}$ *the digits* $\xi_0, \ldots, \xi_{h-1}, 0, 0, \ldots$ *from the initial state* 0*, and one obtain in output the digits* $\zeta_0, \ldots, \zeta_{k-1}, 0, 0, \ldots$ *such that*

$$\xi_{h-1} b^{h-1} + \cdots + \xi_0 b^0 = \zeta_{k-1} b^{k-1} + \cdots + \zeta_0 b^0. \tag{35}$$

P r o o f. (i) In (30) we have $p_{j+bq-q'} = \frac{1}{d}$ when $j + bq - q' \in \{0, \ldots, d-1\}$, that is, when a edge of output label j relates q' to q in $\mathcal{T}$.

(ii) Consequently the entry of the $(q+1)^{\text{th}}$ row and $(q'+1)^{\text{th}}$ column of the matrix ${}^t(dM_{\varepsilon_k}) \ldots {}^t(dM_{\varepsilon_1})$ is the number of paths from q to q' whose output label is $\varepsilon_k \ldots \varepsilon_1$. The input label $\omega_k \ldots \omega_1$ and the states $q_k, \ldots, q_0$ of such a path satisfy

$$\begin{aligned} q_k + \omega_k &= bq_{k-1} + \varepsilon_k \text{ (with } q_k = q) \\ q_{k-1} + \omega_{k-1} &= bq_{k-2} + \varepsilon_{k-1} \\ &\vdots \\ q_1 + \omega_1 &= bq_0 + \varepsilon_1 \text{ (with } q_0 = q'). \end{aligned} \tag{36}$$

We deduce

$$\sum_{i=1}^{k} (q_i + \omega_i) b^{k-i} = b \sum_{i=1}^{k} q_{i-1} b^{k-i} + \sum_{i=1}^{k} \varepsilon_i b^{k-i}$$

and, after simplification,

$$\sum_{i=1}^{k} \omega_i b^{k-i} = q' b^k - q + \sum_{i=1}^{k} \varepsilon_i b^{k-i} \tag{37}$$

meaning that $\omega_1 \ldots \omega_k$ is a representation of length k of $n + b^k q' - q$.

Conversely, if (37) holds, then (36) holds with

$$q_i = q' b^i + \sum_{j \le i} (\varepsilon_j - \omega_j) b^{i-j}.$$

It remains to prove that (36) implies $q_i \le a$ by descending induction. Since $q_k = q$ one has $q_k \le a$. If $q_i \le a$, the Euclidean divisions in (36) imply

$$q_{i-1} = \left\lfloor \frac{q_i + \omega_i}{b} \right\rfloor \le \left\lfloor \frac{a + d - 1}{b} \right\rfloor,$$

where the r.h.s. is at most a because

$$\left\lfloor \frac{a + d - 1}{b} \right\rfloor \le \frac{a + d - 1}{b} < \frac{\frac{d-1}{b-1} + d - 1}{b} = \frac{d-1}{b-1} \le \left\lceil \frac{d-1}{b-1} \right\rceil = a + 1.$$

(iii) In particular, if (37) holds with $q = q' = 0$ then $\omega_k \ldots \omega_1$ is the input label and $\varepsilon_k \ldots \varepsilon_1$ the output label of a path from the state 0 to the state 0. Suppose that (35) holds. Equivalently, (37) holds with $q = q' = 0$, with $\omega_1 \ldots \omega_k = 0^{h-k} \xi_{h-1} \ldots \xi_0$ and $\varepsilon_1 \ldots \varepsilon_k = \zeta_{k-1}, \ldots, \zeta_0$. This proves that, if we enter the digits $\xi_0, \ldots, \xi_{h-1}, 0, 0, \ldots$ from the initial state 0, the output digits are $\zeta_0, \ldots, \zeta_{k-1}$. $\qquad\square$

THEOREM 21. *In the case $p_0 = \cdots = p_{d-1} = \frac{1}{d}$ the level sets of the measure $\eta_{b,p}$, that is, the sets*

$$E(\alpha) := \left\{ x \in \mathbb{R} : \lim_{r \to 0} \frac{\log \eta_{b,p}\big((x-r, x+r)\big)}{\log r} = \alpha \right\},$$

are related to the level sets $\mathcal{E}(\cdot)$ of the function $\mathcal{N}$ by the inequality

$$\text{H-dim}\big(E(\alpha)\big) \le \mathrm{d}_-^{\exp}\left(\mathcal{E}\left(\frac{\log d}{\log b} - \alpha \right) \right). \tag{38}$$

P r o o f. We first consider the measure $\eta_{0,symb}$ on the symbolic space $\{0, \ldots \ldots, b-1\}^{\mathbb{N}}$ and we prove that the level sets $E_{0,symb}(\alpha)$ of this measure, defined as in (22), satisfy (38). The condition of Proposition 14 is satisfied: indeed (31) implies

$$\frac{\eta_{0,symb}[0\varepsilon_1 \ldots \varepsilon_k]}{\eta_{0,symb}[\varepsilon_1 \ldots \varepsilon_k]} = \frac{1}{d} \quad \text{and thus} \quad \lim_{k \to \infty} \frac{\log \eta_{0,symb}[0\varepsilon_1 \ldots \varepsilon_k]}{\log \eta_{0,symb}[\varepsilon_1 \ldots \varepsilon_k]} = 1.$$

For any $n =_b \varepsilon_1 b^{k-1} + \cdots + \varepsilon_k b^0$ one has $b^{k-1} \le n < b^k$ and consequently, $k = k(n) = 1 + \left\lfloor \frac{\log n}{\log b} \right\rfloor$. By (34),

$$\eta_{0,symb}[\varepsilon_1 \ldots \varepsilon_{k(n)}] = \frac{1}{d^{k(n)} \eta([0,1))} \big(\mathcal{N}(n) \quad \ldots \quad \mathcal{N}(n-a) \big) C. \tag{39}$$

Let $f(n) = \eta_{0,symb}[\varepsilon_1 \ldots \varepsilon_{k(n)}]$, by Proposition 14 one has

$$\text{H-dim}\big(E_{0,symb}(\alpha)\big) \leq \text{d}_-^{\exp}(\mathcal{E}_f(-\alpha)). \tag{40}$$

We deduce from (39) that f has the same density spectrum as the function $g(n) := \frac{1}{d^{k(n)}}\mathcal{N}(n)$, because, from [11, Lemma 9 (ii)], there exists a positive constant K such that

$$\frac{1}{K \log n} \leq \frac{\mathcal{N}(n)}{\mathcal{N}(n-1)} \leq K \log n. \tag{41}$$

Since $\lim_{n \to \infty} \left(\frac{\log \mathcal{N}(n)}{\log n} - \frac{\log g(n)}{\log n} \right) = \frac{\log d}{\log b}$, the level sets $\mathcal{E}(\cdot)$ of $\mathcal{N}$ are related to the ones of g and f: one has

$$\text{d}_-^{\exp}\left(\mathcal{E}\Big(\frac{\log d}{\log b} - \alpha\Big)\right) = \text{d}_-^{\exp}(\mathcal{E}_g(-\alpha)) = \text{d}_-^{\exp}\left(\mathcal{E}_f(-\alpha)\right).$$

With (40) one deduce

$$\text{H-dim}\big(E_{0,symb}(\alpha)\big) \leq \text{d}_-^{\exp}\left(\mathcal{E}\Big(\frac{\log d}{\log b} - \alpha\Big)\right). \tag{42}$$

Let us now prove by a classical method that the real $x =_b \sum_{k=1}^{\infty} \frac{\varepsilon_k}{b^k}$ belongs to $E(\alpha) \cap (0,1)$ if and only if $(\varepsilon_k)_k$ belongs to $E_{0,symb}(\alpha) \setminus \{\bar{0}\}$. This is due to the fact that $I_{\varepsilon_1 \ldots \varepsilon_k} \subset \left(x - \frac{1}{b^k}, x + \frac{1}{b^k}\right)$, and conversely $(x - r, x + r) \subset I_w \cup I_{w'} \cup I_{w''}$ holds for $\frac{1}{b^{k+1}} \leq r < \frac{1}{b^k}$, the words w' and w'' being respectively the lexicographical predecessor and the lexicographical successor of $w = \varepsilon_1 \ldots \varepsilon_k$. We use of course the relation (34) and the inequality (41). Now let us prove that

$$\text{H-dim}\big(E(\alpha) \cap [0,1)\big) = \text{H-dim}\big(E_{0,symb}(\alpha)\big). \tag{43}$$

To any cover of $E_{0,symb}(\alpha)$ corresponds a cover of $E(\alpha) \cap [0,1)$, and conversely to any cover of $E(\alpha) \cap [0,1)$ and to any interval (a, a') of this cover, correspond two cylinder sets $[w], [w']$ such that $(a, a') \subset I_w \cup I_{w'}$, where the words w, w' have length k such that $\frac{1}{b^{k+1}} \leq a' - a < \frac{1}{b^k}$. One deduce easily that (43) holds.

The measure $\eta_{b,p}$ is obviously symmetrical with respect to the middle of its support $\left[0, \frac{d-1}{b-1}\right]$. So $E(\alpha)$ is a symmetric subset of $\left[0, \frac{d-1}{b-1}\right]$ of $\eta_{b,p}$, and it remains to prove that $E(\alpha) \cap (q, q+1) = q + E(\alpha) \cap (0,1)$ for any positive integer $q < \frac{1}{2}\frac{d-1}{b-1}$ (hence $q \neq a$). Given a positive integer k_0, we consider the b-adic intervals $I_{\varepsilon_1 \ldots \varepsilon_k}$ such that $k \geq k_0 + a$ and $\varepsilon_1 \ldots \varepsilon_{k_0} \neq 0^{k_0}$.

The entries of $M_{\varepsilon_1}\ldots M_{\varepsilon_{k_0+a}}$ are positive, except possibly the ones of its last row: this is due to the fact that the entries of dM_j are at least equal to the ones of the $(0,1)$-matrix

$$(\delta_{ij})_{\substack{0\leq i\leq a\\0\leq j\leq a}} = \begin{pmatrix} 1 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & \ldots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \end{pmatrix}$$

such that $\delta_{ij} = 1 \Leftrightarrow i \neq a$ and $|i-j| \leq 1$, except the second entry of the first row of dM_0 which is 0. Denoting by $C(k_0)$ the largest entry of the matrix (with positive integral entries) $d^{k_0+a}E_q M_{\varepsilon_1}\ldots M_{\varepsilon_{k_0+a}}$ for any $q \in \{0,\ldots,a-1\}$ and $\varepsilon_1\ldots\varepsilon_{k_0+a} \in \{0,\ldots,b-1\}^{k_0+a}$ such that $\varepsilon_1\ldots\varepsilon_{k_0} \neq 0^{k_0}$, the formula (31) implies

$$\frac{1}{C(k_0)} \leq \frac{\eta(q+I_{\varepsilon_1\ldots\varepsilon_k})}{\eta(I_{\varepsilon_1\ldots\varepsilon_k})} \leq C(k_0).$$

Since it is true for any $k_0 \in \mathbb{N}$ and $q \in \{0,\ldots,a-1\}$, one deduce

$$E(\alpha) \cap (q,q+1) = q + E(\alpha) \cap (0,1) \quad \text{for } 0 < q < a,$$

and (38) follows from (42), (43) and from the symmetry of $E(\alpha)$. □

6. The general framework in Pisot base

The normalization map [15] in the integral or Pisot base $\beta > 1$, associates to each sequence $(\omega_i)_{i\in\mathbb{N}} \in \{0,1,\ldots,d-1\}^{\mathbb{N}}$ the sequence $(\varepsilon_i)_{i\in\mathbb{Z}} \in \{0,1,\ldots\ldots,\lceil\beta\rceil-1\}^{\mathbb{Z}}$ satisfying both conditions:

$$\sum_{i\in\mathbb{N}} \frac{\omega_i}{\beta^i} = \sum_{i\in\mathbb{Z}} \frac{\varepsilon_i}{\beta^i},$$

$$\forall i \in \mathbb{Z}, \quad \sum_{j>i} \frac{\varepsilon_j}{\beta^j} < \frac{1}{\beta^i} \text{ (Parry } \beta\text{-admissibility condition [21]).} \tag{44}$$

6.1. The tranducer $\mathcal{T}$ associated to β and d

The states of the transducer are the carries of the normalization of the sequences $(\omega_i)_{i\in\mathbb{N}}$ with digits in $\{0,1,\ldots,d-1\}$. More precisely suppose that (44) holds and put $\omega_i = 0$ for any $i \leq 0$; then for each $i \in \mathbb{Z}$ the sum $\sum_{j>i} \frac{\omega_j}{\beta^j}$ is equal to $\sum_{j>i} \frac{\varepsilon_j}{\beta^j}$ plus a real number that we denote by $\frac{q_i}{\beta^i}$. So we call "the ith carry", the real q_i defined by both relations

$$\frac{q_i}{\beta^i} = \sum_{j>i} \frac{\omega_j - \varepsilon_j}{\beta^j} = \sum_{j\leq i} \frac{\varepsilon_j - \omega_j}{\beta^j}. \tag{45}$$

The first relation implies
$$q_i \in (-1, \alpha] \quad \text{with} \quad \alpha := \frac{d-1}{\beta-1}.$$

The second relation implies
$$q_i = \beta q_{i-1} - \omega_i + \varepsilon_i \tag{46}$$

and implies that q_i belongs to the set

$$S_{\beta,d} := \left\{ \sum_{j=0}^{i-1} \alpha_j \beta^j \ : \ i \in \mathbb{N}, \ \alpha_j \in (-d, \lceil \beta \rceil) \cap \mathbb{Z}, \right\} \cap (-1, \alpha].$$

Garsia's separation lemma [16] ensuring that $S_{\beta,d}$ is finite, we can chose $S_{\beta,d}$ as set of states of $\mathcal{T}$ and assume that the arrows have the form

$$q \xrightarrow{\quad \omega \ / \ \varepsilon \quad} \beta q - \omega + \varepsilon.$$

Notice that the arrows are in the opposite direction of the ones considered in integral base because, as can be seen for instance in Example 22, the Euclidean division that we use for the normalization in integral base do not have analogue in non-integral base; more precisely, the relation

$$q' + \omega = \beta q + \varepsilon \quad \text{with} \quad \varepsilon \in \{0, 1, \ldots, \lceil \beta \rceil - 1\}$$

may hold for several values of q and ε when q' and ω are fixed.

In practice we construct at the same time a suitable set of states $S'_{\beta,d} \subset S_{\beta,d}$ and the transducer: the set of states $S'_{\beta,d}$ is by definition the smallest set S containing 0 and containing $\beta q - \omega + \varepsilon$, whenever the three reals

$$q \in S, \quad \omega \in \{0, 1, \ldots, d-1\}, \quad \varepsilon \in \{0, 1, \ldots, \lceil \beta \rceil - 1\}$$

satisfy the condition $\qquad \beta q - \omega + \varepsilon \in (-1, \alpha].$

EXAMPLE 22. The transducer $\mathcal{T}$ when $\beta^2 = 3\beta - 1$ and $d = 3$:
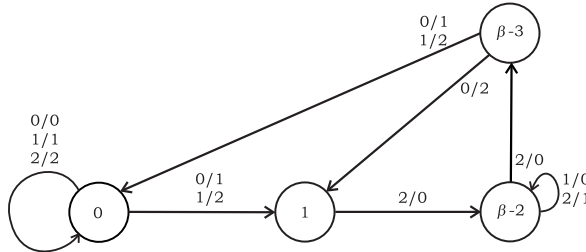
E.g., $3 - \beta = -\beta^2 + 2\beta + 2 = \frac{1}{\beta}$ belongs to $S_{\beta,d}$, but does not belong to $S'_{\beta,d}$.

6.2. How to normalize a sequence $(\omega_i)_{i\in\mathbb{N}}$?

We first notice that in a non-integral base there do not exist a literal transducer of normalization. For instance in base $\beta = \frac{1+\sqrt{5}}{2}$, if we normalize from the left to the right a sequence of the form $(01)^n x$ without knowing if the digit x is 0 or 1, we obtain $(01)^n 0$ if $x = 0$ and $1(00)^n$ if $x = 1$. So all the terms of the normalized sequence depend on the value of x, while the successive carries cannot depend on x because x is at the right. As well if we normalize from the right to the left a sequence of the form $0x(11)^n$ without knowing if the digit x is 0 or 1, we obtain $0(10)^n 0$ if $x = 0$ and $(10)^n 01$ if $x = 1$.

PROPOSITION 23. *Let $(\omega_i)_{i\in\mathbb{N}} \in \{0,1,\ldots,d-1\}^{\mathbb{N}}$. The normalized sequence defined in (44) is the unique β-admissible sequence $(\varepsilon_i)_{i\in\mathbb{Z}}$ with*

$$\#\{i \leq 0 \,:\, \varepsilon_i \neq 0\} < \infty,$$

which is the output label of a bi-infinite path of input label $\bar{0}\omega_1\omega_2\omega_3\ldots$

P r o o f. If (44) holds, the states q_i defined by (45) satisfy (46), so $(\varepsilon_i)_{i\in\mathbb{Z}}$ is the output label of a bi-infinite path of input label $\bar{0}\omega_1\omega_2\omega_3\ldots$

Conversely, if $(\varepsilon_i)_{i\in\mathbb{Z}}$ is β-admissible and is the output label of a bi-infinite path of input label $\bar{0}\omega_1\omega_2\omega_3\ldots$, one has for any $i < i'$ in $\mathbb{Z}$

$$\frac{q_i}{\beta^i} = \sum_{i<j\leq i'} \frac{\omega_j - \varepsilon_j}{\beta^j} + \frac{q_{i'}}{\beta^{i'}} \tag{47}$$

which implies

$$\sum_{i\in\mathbb{N}} \frac{\omega_i}{\beta^i} = \sum_{i\in\mathbb{Z}} \frac{\varepsilon_i}{\beta^i}$$

because $\lim_{i'\to+\infty} \frac{q_{i'}}{\beta^{i'}} = 0$ (obvious) and $\lim_{i\to-\infty} \frac{q_i}{\beta^i} = 0$ (for $|q_{i-1}| < |q_i|$ when $\omega_i = \varepsilon_i = 0$). $\qquad\square$

6.3. The number of redundant representations

For any finite sequence $\varepsilon_1\ldots\varepsilon_k \in \{0,1,\ldots,d-1\}^k$ we denote by $\mathcal{N}(\varepsilon_1\ldots\varepsilon_k)$ the number of $\omega_1\ldots\omega_k \in \{0,1,\ldots,d-1\}^k$ such that

$$\omega_1\beta^{k-1} + \cdots + \omega_k\beta^0 = \varepsilon_1\beta^{k-1} + \cdots + \varepsilon_k\beta^0. \tag{48}$$

Notice that, this time, the carries $q_i = \sum_{j=i+1}^{k} \frac{\omega_j - \varepsilon_j}{\beta^{j-i}}$ do not belong to $(-1,\alpha]$ but to $(-\alpha, \alpha)$. The transducer $\mathcal{T}'$ we consider has the same arrows as $\mathcal{T}$, but its set of states $\{i_0 = 0, i_1, \ldots, i_a\}$ is the smallest set S containing 0 and containing $\beta q - \omega + \varepsilon$ for any $q \in S$, $\omega \in \{0,1,\ldots,d-1\}$, $\varepsilon \in \{0,1,\ldots,\lceil\beta\rceil-1\}$ such that $\beta q - \omega + \varepsilon \in (-\alpha, \alpha)$.

Proposition 24. $\mathcal{N}(\varepsilon_1 \ldots \varepsilon_k)$ *is the number of paths in* $\mathcal{T}'$, *from* 0 *to* 0, *with output label* $\varepsilon_1 \ldots \varepsilon_k$. *Equivalently,*
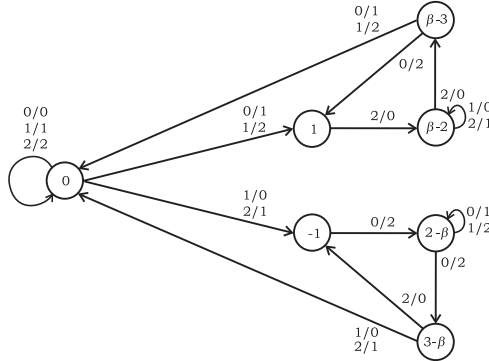
$$\mathcal{N}(\varepsilon_1 \ldots \varepsilon_k) = \begin{pmatrix} 1 & 0 & \ldots & 0 \end{pmatrix} N_{\varepsilon_1} \ldots N_{\varepsilon_k} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where the $(0, 1)$-*matrices* $N_\ell = (n_{ij}^\ell)_{\substack{0 \leq i \leq a \\ 0 \leq j \leq a}}$, $\ell \in \{0, 1, \ldots, d-1\}$ *are defined by*

$$n_{ij}^\ell = 1 \Leftrightarrow \beta \mathsf{i}_i - \mathsf{i}_j + \ell \in \{0, 1, \ldots, d-1\}.$$

P r o o f. There exists a path from 0 to 0 with output label $\varepsilon_1 \ldots \varepsilon_k$, if and only if (46) holds for some $q_0 = 0, q_1, \ldots, q_k = 0$ in $\{\mathsf{i}_0, \ldots, \mathsf{i}_a\}$ and $\omega_1, \ldots, \omega_k$ in $\{0, 1, \ldots, d-1\}$. This is equivalent to $\sum_{0 < j \leq k} \frac{\omega_j - \varepsilon_j}{\beta^j} = 0$, because (46) is equivalent to $\frac{q_{i-1}}{\beta^{i-1}} = \frac{\omega_i - \varepsilon_i}{\beta^i} + \frac{q_i}{\beta^i}$. $\qquad\square$

Example 25. If $\beta^2 = 3\beta - 1$ and $d = 3$, the transducer $\mathcal{T}'$ is

6.4. The linear representation of the Bernoulli convolutions in Pisot base

The Bernoulli convolution [6, 22] in real base $\beta > 1$, associated to a positive probability vector $p = (p_0, \ldots, p_{d-1})$, $d \geq \beta$, is the probability measure $\eta = \eta_{\beta, p}$ defined by setting, for any interval $I \subset \mathbb{R}$

$$\eta_{\beta, p}(I) := P_p \left(\left\{ (\omega_k)_{k \in \mathbb{N}} \ : \ 0 \leq \omega_k \leq d-1, \ \sum_k \frac{\omega_k}{\beta^k} \in I \right\} \right), \qquad (49)$$

where P_p the product probability defined on $\{0, \ldots, d-1\}^{\mathbb{N}}$ from the probability vector p.

In order to compute the value of $\eta_{\beta,p}$ on some suitable intervals, we recall the Parry condition of β-admissibility [15, Theorem 2.3.11].

DEFINITION 26. A sequence $(\varepsilon_k)_{k \in \mathbb{N}}$ is β-admissible, i.e., is the Rényi expansion of a real $x \in [0, 1)$, iff $\forall k$, $\varepsilon_k \varepsilon_{k+1} \cdots \prec \alpha_1 \alpha_2 \ldots$ for the lexicographical order, where $(\alpha_k)_{k \in \mathbb{N}}$ is the quasi-expansion of the unity that can be defined by

$$1 = \sum_{i \in \mathbb{N}} \frac{\alpha_i}{\beta^i} \quad \text{and} \quad \forall i \geq 0, \ 0 < \sum_{j > i} \frac{\alpha_j}{\beta^j} \leq \frac{1}{\beta^i}.$$

From now we assume that the Pisot number β has a finite Rényi expansion [24], or equivalently that $(\alpha_k)_{k \in \mathbb{N}}$ has a period T.

LEMMA 27. (i) *A sequence is in the closure of* Adm_β *(set of the β-admissible sequences), if and only if it is a infinite concatenation of words belonging to the set $\mathcal{W}$ defined by*

$$w \in \mathcal{W} \Leftrightarrow \begin{array}{l} w = \alpha_1 \ldots \alpha_T \ or \\ \exists i \in \{1, \ldots, T\}, \alpha_i' \in \{0, \ldots, \alpha_i - 1\}, \ w = \alpha_1 \ldots \alpha_{i-1} \alpha_i'. \end{array}$$

(ii) *If $\varepsilon_1 \ldots \varepsilon_k$ is a concatenation of words of $\mathcal{W}$, the β-adic interval $I_{\varepsilon_1 \ldots \varepsilon_k}$, i.e., the set of the $x \in [0, 1)$ whose Rényi expansion begins by $\varepsilon_1 \ldots \varepsilon_k$, is simply*

$$I_{\varepsilon_1 \ldots \varepsilon_k} = \left[\sum_{i=1}^{k} \frac{\varepsilon_i}{\beta^i}, \ \sum_{i=1}^{k} \frac{\varepsilon_i}{\beta^i} + \frac{1}{\beta^k} \right).$$

P r o o f. (i) If $\varepsilon_1 \varepsilon_2 \cdots \in \overline{\text{Adm}}_\beta$ one has $\varepsilon_1 \ldots \varepsilon_T \preceq \alpha_1 \ldots \alpha_T$, so there exists $k \leq T$ such that
$$\varepsilon_1 \ldots \varepsilon_k \in \mathcal{W} \text{ and } \varepsilon_{k+1} \varepsilon_{k+2} \cdots \in \overline{\text{Adm}}_\beta.$$
By iteration, $\varepsilon_1 \varepsilon_2 \cdots = w_1 w_2 \ldots$ with $\forall i$, $w_i \in \mathcal{W}$.

Conversely suppose that $\varepsilon_1 \varepsilon_2 \cdots = w_1 w_2 \ldots$ with $\forall i$, $w_i \in \mathcal{W}$. Then for any i there exists j and a nonempty suffix w_j' of w_j such that

$$\varepsilon_{i+1} \varepsilon_{i+2} \cdots = w_j' w_{j+1} \ldots$$

Clearly, from the definition of $\mathcal{W}$ one has $w_j w_{j+1} \cdots \preceq \alpha_1 \alpha_2 \ldots$ But there exists k such that $w_j = \alpha_1 \ldots \alpha_k w_j'$, so one deduce

$$w_j' w_{j+1} \cdots \preceq \alpha_{k+1} \alpha_{k+2} \cdots \preceq \alpha_1 \alpha_2 \ldots,$$

proving that $\varepsilon_1 \varepsilon_2 \cdots \in \overline{\text{Adm}}_\beta$.

(ii) If $\varepsilon_1 \ldots \varepsilon_k$ is a concatenation of words of $\mathcal{W}$, from (i) the eventually periodic sequence $\varepsilon_1 \ldots \varepsilon_k \overline{\alpha_1 \ldots \alpha_T}$ belongs to $\overline{\text{Adm}}_\beta$. Denoting by $\varepsilon_1 \varepsilon_2 \ldots$ this sequence, the closed interval $\overline{I_{\varepsilon_1 \ldots \varepsilon_k}}$ contains $\sum_{i=1}^{\infty} \frac{\varepsilon_i}{\beta^i} = \sum_{i=1}^{k} \frac{\varepsilon_i}{\beta^i} + \frac{1}{\beta^k}$. $\qquad \square$

In order to define the matrices associated to $\eta_{\beta,p}$, we consider a set of reals that may be slightly different from the set $S'_{\beta,d}$ defined in Subsection 6.1. Let $\{j_0 = 0, \ldots, j_{a'}\}$ be the smallest set S containing 0 and containing $\beta q - \omega + \varepsilon$, whenever the three reals $q \in S$, $\omega \in \{0, 1, \ldots, d-1\}$, $\varepsilon \in \{0, 1, \ldots, \lceil \beta \rceil - 1\}$ satisfy the condition $\beta q - \omega + \varepsilon \in (-1, \alpha)$. The matrices $M_\ell = (m_{ij}^\ell)_{\substack{0 \le i \le a' \\ 0 \le j \le a'}}$, $\ell \in \{0, 1, \ldots, \lceil \beta \rceil - 1\}$, are defined by

$$m_{ij}^\ell = \begin{cases} 0 & \text{if } \beta j_i - j_j + \ell \notin \{0, 1, \ldots, d-1\}, \\ p_{\beta j_i - j_j + \ell} & \text{else}. \end{cases}$$

The following theorem gives a linear representation of the Bernoulli convolutions associated to Pisot numbers with finite Rényi expansions. Nevertheless, Feng [8, Theorem 3.3] gives by an other method a matrix-product characterization of Bernoulli convolutions associated with general Pisot numbers.

THEOREM 28. *If $\varepsilon_1 \ldots \varepsilon_k$ is a concatenation of words of $\mathcal{W}$, one has for any $i \in \{0, \ldots, a'\}$*

$$\eta(j_i + I_{\varepsilon_1 \ldots \varepsilon_k}) = E_i M_{\varepsilon_1} \ldots M_{\varepsilon_k} C, \tag{50}$$

where $E_0, E_1, \ldots, E_{a'}$ are the canonical basis $(a'+1)$-dimensional row vectors and C is a suitable positive eigenvector of the irreducible matrix $\sum_{w \in \mathcal{W}} M_w$. Consequently, one can define a sofic measure on the symbolic space $\mathcal{W}^{\mathbb{N}}$ by setting

$$\eta_i[w_1 \ldots w_k] := \frac{\eta(j_i + I_{w_1 \ldots w_k})}{\eta(j_i + [0, 1))}. \tag{51}$$

P r o o f. Let $I_j = \left[\sum_{i=j+1}^{k} \frac{\varepsilon_i}{\beta^{i-j}}, \sum_{i=j+1}^{k} \frac{\varepsilon_i}{\beta^{i-j}} + \frac{1}{\beta^{k-j}} \right)$ for any $0 \le j \le k$; so one has $I_0 = I_{\varepsilon_1 \ldots \varepsilon_k}$ (by Lemma 27 (ii)), $I_k = [0, 1)$ and for any $j \ne 0$,

$$\beta I_{j-1} = \varepsilon_j + I_j.$$

Assuming by convention that $p_i = 0$ for any $i \notin \{0, 1, \ldots, d-1\}$,

$$\begin{aligned} \eta(j_i + I_{j-1}) &= \sum_{\iota=0}^{d-1} P\left(\left\{ \omega_1 = \iota \text{ and } \sum_{k=1}^{\infty} \frac{\omega_{k+1}}{\beta^k} \in \beta j_i + \beta I_{j-1} - \iota \right\} \right) \\ &= \sum_{\iota=0}^{d-1} p_\iota P\left(\left\{ \sum_{k=1}^{\infty} \frac{\omega_{k+1}}{\beta^k} \in \beta j_i - \iota + \varepsilon_j + I_j \right\} \right) \\ &= \sum_{i'=0}^{a'} p_{\beta j_i - j_{i'} + \varepsilon_j}\, \eta\left(j_{i'} + I_j \right) \end{aligned}$$

so one has

$$
\begin{pmatrix} \eta(j_0 + I_{j-1}) \\ \vdots \\ \eta(j_{a'} + I_{j-1}) \end{pmatrix} = M_{\varepsilon_j} \begin{pmatrix} \eta(j_0 + I_j) \\ \vdots \\ \eta(j_{a'} + I_j) \end{pmatrix}
$$

and (50) follows by induction, with

$$
C := \begin{pmatrix} \eta([j_0, j_0 + 1)) \\ \vdots \\ \eta([j_{a'}, j_{a'} + 1)) \end{pmatrix}. \tag{52}
$$

From Lemma 27 (i) the intervals I_w, $w \in \mathcal{W}$, form a partition of $[0, 1)$. Denoting by M_w the product matrix $M_{\varepsilon_1} \ldots M_{\varepsilon_k}$ for any $w = \varepsilon_1 \ldots \varepsilon_k$ one deduce that the column vector C, defined as in (52), satifies

$$
\sum_{w \in \mathcal{W}} M_w C = C.
$$

The entries of C are positive because the intervals $[j_i, j_i + 1)$ intersect the interior of $[0, \alpha)$ (the support of η). Now $\sum_{w \in \mathcal{W}} M_w$ is irreducible because, from the definition of $\{j_0, \ldots, j_{a'}\}$, there exists a path from any j_i to any $j_{i'}$ in the transducer $\mathcal{T}_\eta$ of set of states $\{j_0, \ldots, j_{a'}\}$ and arrows $q \xrightarrow{\omega \ / \ \varepsilon} \beta q - \omega + \varepsilon$. □

EXAMPLE 29. This is the transducer $\mathcal{T}_\eta$ in the case $\beta = \frac{1+\sqrt{5}}{2}$ and $d = 2$, the set of states being $\{j_0 = 0, j_1 = 1, j_2 = \beta - 1\}$:
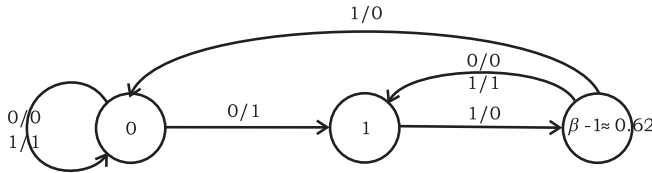
For $\ell \in \{0, 1\}$ and $i, j \in \{0, 1, 2\}$, the (i, j)-entry of M_ℓ is p_ω if $\omega = \beta j_i - j_j + \ell$ is an integer, i.e., when there exists an arrow $j_i \xrightarrow{\omega \ / \ \ell} j_j$:
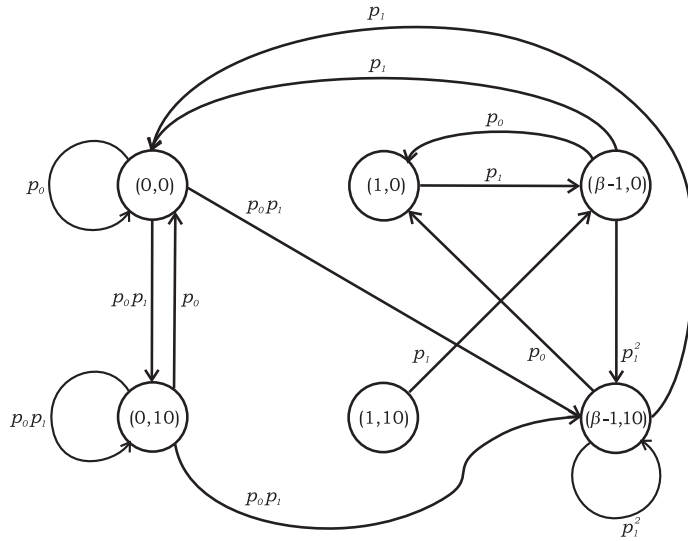
$$
M_0 = \begin{pmatrix} p_0 & 0 & 0 \\ 0 & 0 & p_1 \\ p_1 & p_0 & 0 \end{pmatrix} \quad \text{and} \quad M_1 = \begin{pmatrix} p_1 & p_0 & 0 \\ 0 & 0 & 0 \\ 0 & p_1 & 0 \end{pmatrix}.
$$

We have $\mathcal{W} = \{0, 10\}$. So we can construct from $\mathcal{T}_\eta$ a new transducer with the output labels 0 and 10:
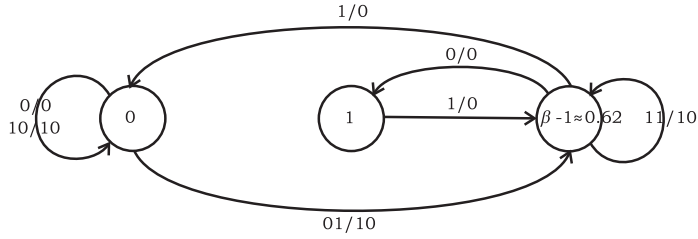
We deduce the graph of the Markov chain associated to each of the sofic measures η_0, η_1, η_2:

and its transition matrix:

$$\begin{pmatrix} M_0 & M_{10} \\ M_0 & M_{10} \end{pmatrix} = \begin{pmatrix} p_0 & 0 & 0 & p_0 p_1 & 0 & p_0 p_1 \\ 0 & 0 & p_1 & 0 & 0 & 0 \\ p_1 & p_0 & 0 & 0 & 0 & p_1^{\,2} \\ p_0 & 0 & 0 & p_0 p_1 & 0 & p_0 p_1 \\ 0 & 0 & p_1 & 0 & 0 & 0 \\ p_1 & p_0 & 0 & 0 & 0 & p_1^{\,2} \end{pmatrix}.$$

7. More about the representations in base $b = 2$ with three digits

Let $\mathcal{N}(n)$ be the number of representations, with digits in $\{0, 1, 2\}$, of the integer n. We denote as follows the canonical expansion of n in base 2:

$$n =_2 1^{a_s} 0^{a_{s-1}} \cdots 0^{a_1} 1^{a_0},$$

where $a_0 \geq 0$ ($a_0 = 0$, if and only if n is even), and $a_1, \ldots, a_s \geq 1$.

According to [5, Proposition 6.11], $\mathcal{N}(n)$ is the denominator q_s of the continued fraction

$$[0; a_1, \ldots, a_s] := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_s}}}.$$

From [11, Theorem 19] there exists $\alpha_0 > 0$ and $S \subset \mathbb{N}$ of natural density 1 such that

$$\lim_{n \in S,\ n \to \infty} \frac{\log \mathcal{N}(n)}{\log n} = \alpha_0. \tag{53}$$

One obtains $\alpha_0 \approx 0.56$ by computing $\mathcal{N}$ by a recurrence relation:

f := proc(n): if n=0 then 1 elif irem(n,2)=0 then f(n/2)+f(n/2-1) else f(n/2-1/2) end if end proc:

s := seq(log(f(n)), n = 1 .. 16383):

α_0:= evalf(sum(s[n], n = 8192 .. 16383)/(8192*log(8192)))

The value of α_0 can also be obtained, without recurrence relation, from the classical formula for the denominators of the continued fractions:

$$\alpha_0 = \lim_{k \to \infty} \frac{1}{2^k \log(2^k)} \sum_{s, a_1, \ldots, a_s} \log \left(\sum_{h, i_0, \ldots, i_{h+1}} \prod_{j=0}^{h} a_{i_j, i_{j+1}} \right),$$

summing for $s, a_1, \ldots, a_s \in \mathbb{N}$ such that $\sum_{i=1}^{s} a_i \leq k$ and for $h \in \{0, \ldots, s\}$, $0 = i_0 < i_1 < \cdots < i_h < i_{h+1} = s + 1$, assuming that

$$a_{s+1} = 1 \text{ and } a_{i',i} = \begin{cases} a_i & \text{if } i - i' \text{ is an odd positive integer,} \\ 0 & \text{else.} \end{cases}$$

PROPOSITION 30. *The constant α_0 defined in (53) is also the Lebesgue-a.e. value of*

$$\lim_{s \to \infty} \frac{\log q_s(x(t))}{s \log 4},$$

where $x(t) = [0; a_1, a_2, \ldots]$ for any $t =_2 0.1^{a_1} 0^{a_2} 1^{a_3} \cdots \in \left[\frac{1}{2}, 1\right)$, and $q_s(x(t))$ is the denominator of both continued fractions $[0; a_1, \ldots, a_s]$ and $[0; a_s, \ldots, a_1]$.

P r o o f. From [11, Remark 21], $\alpha_0 \log 2$ is the constant γ defined in [13, Theorem 1.1]. Now by [13, Corollary 1.2], $\frac{\log 3}{\log 2} - \alpha_0$ is the almost sure value of the local dimension $d_{\mathrm{loc}}(t)$ of the Bernoulli convolution $\eta = \eta_{2,3}$ (in base 2 with three digits).

In the sequel we denote by $0.1^{a_1} 0^{a_2} 1^{a_3} \ldots$ or by $0.\varepsilon_1 \varepsilon_2 \ldots$ the expansion of the real $t \in \left[\frac{1}{2}, 1\right)$ in base 2, with $a_i \in \mathbb{N}$ and $\varepsilon_i \in \{0, 1\}$, and we suppose that ε_i is not eventually 0 nor eventually 1. For any $k \geq 1$ we put

$$n(t, k) := \varepsilon_1 2^{k-1} + \cdots + \varepsilon_k 2^0.$$

If $k = a_1 + \cdots + a_s$ for some even integer s, the formula of [5, Proposition 6.11] gives

$$\mathcal{N}\big(n(t, k)\big) = q_s\big(x(t)\big). \tag{54}$$

By (34),

$$\mathcal{N}\big(n(t, k)\big) + \mathcal{N}\big(n(t, k) - 1\big) = 2 \cdot 3^k \eta\big(I_{\varepsilon_1 \ldots \varepsilon_k}\big). \tag{55}$$

But for any integer n, $\mathcal{N}(n-1)$ is in a sense close to $\mathcal{N}(n)$ because, by [11, Lemma 9 (ii)], one has

$$\frac{1}{C \log n} \leq \frac{\mathcal{N}(n)}{\mathcal{N}(n-1)} \leq C \log n.$$

So (54) and (55) imply

$$\lim_{\substack{s \text{ even} \\ s \to \infty}} \frac{\log q_s(x(t))}{a_1 + \cdots + a_s} = \lim_{k \to \infty} \frac{\log(2 \cdot 3^k \eta(I_{\varepsilon_1 \ldots \varepsilon_k}))}{k} = \log 3 - d_{\mathrm{loc}}(t) \log 2. \tag{56}$$

Since, from the law of large numbers, $a_1 + \cdots + a_s$ is equivalent to $2s$ for Lebesgue-a.e. t, (56) implies that, for Lebesgue-a.e. $t \in \left[\frac{1}{2}, 1\right)$,

$$\lim_{s \to \infty} \frac{\log q_s(x(t))}{2s \log 2} = \frac{\log 3}{\log 2} - d_{\mathrm{loc}}(t) = \alpha_0.$$

$\square$

REMARK 31. Using the matricial expression of the denominators of the continued fractions, Proposition 30 means that $\alpha_0 \log 2$ is the Lyapunov exponent of the set of matrices $\left\{ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}$.

REMARK 32. The Levy constant

$$\lim_{s \to \infty} \frac{\log q_s(x)}{s} = \frac{\pi^2}{12 \log 2} \approx 1.18657 \quad \text{for Lebesgue-a.e. } x \in (0, 1) \tag{57}$$

is larger than the value of

$$\lim_{s \to \infty} \frac{\log q_s(x(t))}{s} = \alpha_0 \log 4 \approx 0.78 \quad \text{for Lebesgue-a.e. } t \in \left[\frac{1}{2}, 1\right). \tag{58}$$

On the other side, it is well-known that the partial quotients $a_i(x)$ of Lebesgue-a.e. real x satisfy

$$\lim_{s \to \infty} \frac{a_1(x) + \ldots a_s(x)}{s} = \infty \tag{59}$$

For any t such that $x(t)$ satisfies (57) and (59), the l.h.s. in (56) tends to 0, consequently the local dimension at t is maximal: $d_{\mathrm{loc}}(t) = \frac{log 3}{\log 2} \approx 1.585$. Since $\frac{log 3}{\log 2} - \alpha_0 \approx 1.025$ and $\frac{log 3}{\log 2} - \frac{log \frac{1+\sqrt{5}}{2}}{\log 2} \approx 0.891$ (the minimal local dimension because $q_s \leq \left(\frac{1+\sqrt{5}}{2}\right)^{a_1 + \cdots + a_s}$ by induction), the graph of the singularity spectrum of $\eta_{2,3}$ has the following form:
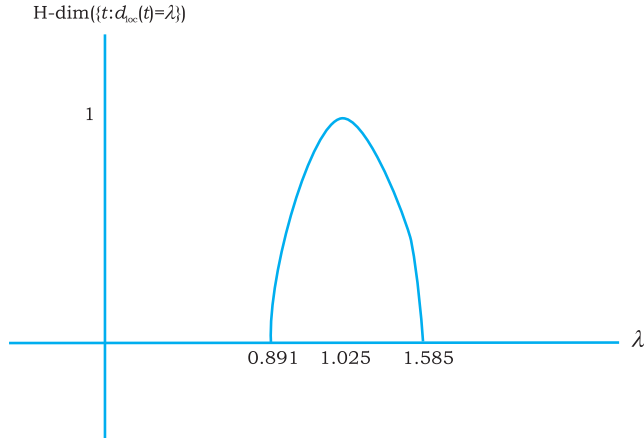
REFERENCES

[1] BLANCHARD, F.: *Systèmes codés*, Theor. comput. sci. **44** (1986), 17–49;
http://ac.els-cdn.com/0304397586901088/1-s2.0-0304397586901088-main.pdf?_tid=
41cf34ba-8e58-11e5-ad0a-00000aab0f6c&acdnat=1447894721_a3eec8fd742db41891f4efe
165b0ef8b

[2] BOYLE, M.—PETERSEN, K.: *Hidden Markov processes in the context of symbolic dynamics*, London Mathematical Society Lecture Note Series **385** (2011), 5–71;
http://www.unc.edu/math/Faculty/petersen/Papers/06Jan2010.pdf

[3] MARCUS, B.—PETERSEN, K.—WEISSMAN, T. (EDS.): *Entropy of Hidden Markov Processes and Connections to Dynamical Systems.* Papers from the Banff International Research Station workshop, Banff, Canada, October 2007;
`http://books.google.fr/books?id=1Zz_zHpfbIAC&pg=PA223&lpg=PA223&dq=%22Entropy+`
`of+hidden+Markov+processes+and+connections+to+dynamical+systems%22&source=bl&`
`ots=2wm0Ng1-tM&sig=PKwt4sipcqY1Vh7mX8n7zMOegkg&hl=en&sa=X&ei=PGhYVOqJD4LfPZyE`
`gIAN&ved=0CCoQ6AEwBA#v=onepage&q=%22Entropy%20of%20hidden%20Markov%20process`
`es%20and%20connections%20to%20dynamical%20systems%22&f=false`

[4] DHARMADHIKARI, S. W.: *Functions of finite Markov chains,* Ann. Math. Stat. **34** (1963), 1022–1032;
`http://projecteuclid.org/download/pdf_1/euclid.aoms/1177704025`

[5] DUMONT, J.-M.—SIDOROV, N.—THOMAS, A.: *Number of representations related to linear recurrent basis,* Acta Arithmetica **88** (1999), 371–396;
`http://matwbn.icm.edu.pl/ksiazki/aa/aa88/aa8847.pdf`

[6] ERDÖS, P.: *On a family of symmetric Bernoulli convolutions,* Amer. J. of Math., **61** (1939), 974–976; `http://www.renyi.hu/~p_erdos/1939-05.pdf`

[7] ERICKSON, R. V.: *Functions of Markov chains,* Ann. Math. Statist. **41** (1970), 843–850;
`https://projecteuclid.org/download/pdf_1/euclid.aoms/1177696962`

[8] FENG, D. J.: *Smoothness of the L^q-spectrum of self-similar measures with overlaps,* J. Lond. Math. Soc. **68** (2003) 102–118;
`http://www.math.cuhk.edu.hk/d̃jfeng/fengpapers/jlms2003/jlmsf.pdf`

[9] FENG, D.-J.: *Lyapunov exponents for products of matrices and multifractal analysis. Part I: Positive matrices,* Israel J. of Math., **138** (2003), 353–376;
`http://www.math.cuhk.edu.hk/%7Edjfeng/djfeng1.html`

[10] FENG, D.-J.: *Lyapunov exponents for products of matrices and multifractal analysis. Part II: General matrices,* Israel J. of Math. **170** (2009), 355–394;
`http://www.math.cuhk.edu.hk/ djfeng/fengpapers/Israel2007/feng-final-new.pdf`

[11] FENG, D.-J.—LIARDET, P.—THOMAS, A.: *Partition functions in numeration systems with bounded multiplicity,* Uniform Distribution Theory **9** (2014), 43–77;
`https://math.boku.ac.at/udt/vol09/no1/05FengLiardetThomas.pdf`

[12] FENG, D.-J.—OLIVIER, E.: *Multifractal analysis of weak Gibbs measures and phase transition-application to some Bernoulli convolutions,* Ergodic Theory Dyn. Syst. **23** (2003), 1751–1784;
`http://www.paper.edu.cn/index.php/default/selfs/downpaper/fengdejun361929-self`
`-200811-14`

[13] FENG, D.-J.—SIDOROV, N.: *Growth rate for beta-expansions,* Monatsh. Math. **162** (2011), 41–60;
`http://www.math.cuhk.edu.hk/~djfeng/fengpapers/Feng-Sidorov/fs-final.pdf`

[14] FISCHER, R.: *Sofic systems and graphs,* Monatsh. Math. **80** (1975), 179–186.

[15] FROUGNY, C.—SAKAROVITCH, J.: *Number representation and finite automata,* Berthé, V. (ed.) et al., Combinatorics, automata, and number theory. Cambridge: Cambridge University Press. Encyclopedia of Mathematics and its Applications **135** (2010), 34–107; `http://www.liafa.univ-paris-diderot.fr/~cf/publications/cant-ch1.pdf`

[16] GARSIA, A. M.: *Entropy and Singularity of infinite convolutions*, Pac. J. Math. **13** (1963), 1159–1169;
`http://projecteuclid.org/DPubS/Repository/1.0/Disseminate?view=body&id=pdf_1 &handle=euclid.pjm/1103034552`

[17] HANSEL, G.—PERRIN, D.: *Rational probability measures*, Theor. Comput. Sci. **65** (1989), 171–188;
`http://www.sciencedirect.com/science/article/pii/030439758990042X`

[18] KITCHENS, B.—TUNCEL, S.: *Finitary measures for subshifts of finite type and sofic systems*, Mem. Am. Math. Soc. **338** (1985), 68 p;
`https://books.google.fr/books?id=azzUCQAAQBAJ&lpg=PR4&dq=Finitary+measures +for+subshifts+of+finite+type+and+sofic+systems&source=bl&ots=GHyH_461gY&si g=3OKN_439-4uB79BGWDCttkWMgQQ&hl=fr&sa=X&output=reader&pg=GBS.PR1`

[19] KRIEGER, W.: *On subshifts and topological Markov chains*, (Althöfer, Ingo ed. et al.). In: Numbers, Information and Complexity. Dedicated to Rudolf Ahlswede on the occasion of his 60th birthday. Kluwer Academic Publishers, Dordrecht, 2000, pp. 453–472.

[20] LEYSIEFFER, F. W.: *Functions of finite Markov chains*, Ann. Math. Stat. **38** (1967), 206–212;
`https://projecteuclid.org/download/pdf_1/euclid.aoms/1177699072`

[21] PARRY, W.: *On the β-expansions of real numbers*, Acta Math. Acad. Sci. Hungar. **11** (1960), 401–416.

[22] PERES, Y.—SCHLAG, W.—SOLOMYAK, B.: *Sixty years of Bernoulli convolutions*, (Bandt, Christoph (ed.) et al.,) In: Fractal geometry and stochastics II. (Proceedings of the 2nd conference, Greifswald / Koserow, Germany, August 28--September 2, 1998.) Prog. Probab. Vol. 46, 2000, Basel: Birkhuser, pp. 39–65;
`http://www.stat.berkeley.edu/~peres/mine/sixty.ps`

[23] PROTASOV, V. YU.: *Asymptotic behavior of the partition function*, Sbornick Math. **191** (2000), 381–414.

[24] RÉNYI, A.: *Representations for real numbers and their ergodic properties*, Acta Math. Acad. Sci. Hungar. **8** (1957), 477–493.

[25] WEISS, B.: *Subshifts of finite type and sofic systems*, Monats. Math. **77** (1973), 462–474.

[26] WILLIAMS, S. G.: *Introduction to symbolic dynamics*. In: Symbolic dynamics and its applications, Proc. Sympos. Appl. Math. Vol. 60. Amer. Math. Soc., Providence, RI, 2004. pp. 1–11.

Alain Thomas
448, allée des Cantons,
83640 Plan d'Aups Sainte Baume
FRANCE
E-mail: alain-yves.thomas@laposte.net