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YET ANOTHER FOOTNOTE TO *THE LEAST NON ZERO DIGIT OF* $n!$ *IN BASE* 12

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À la mémoire de Pierre LIARDET

ABSTRACT. We continue the study, initiated with Imre Ruzsa, of the last non zero digit $\ell_{12}(n!)$ of $n!$ in base 12, showing that for any $a \in \{3, 4, 6, 8, 9\}$, the set of those integers n for which $\ell_{12}(n!) = a$ is not 3-automatic.

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1. Introduction

We pursue the study, initiated in [5], prolongated in [4][1], of the sequence $\big(\ell_{12}(n!)\big)_n$, where $\ell_b(n)$ denotes the last (or final) non zero digit of the integer n in base b. In other words, if $v_b(n)$ denotes the largest integer v such that b^v divides n, then $\ell_b(n)$ is the integer in $\{1, 2, \dots, b-1\}$ congruent to $n/b^{v_b(n)}$ modulo b.

We do not repeat the introduction of the previous papers, but just recall in short, that $\ell_{10}(n!)$ is even as soon as $n \geq 2$ because $v_2(n!)$ is larger than $v_5(n!)$, and moreover the sequence $\big(\ell_{12}(n!)\big)_n$ is 5-automatic. In the case of the sequence $\big(\ell_{12}(n!)\big)_n$, the number $v_4(n!)$ is usually larger than $v_3(n!)$, but may be also smaller. A key point in the study is that the sequence $\big(\ell_{12}(n!)\big)_n$ coincides on a set of asymptotic density 1 with a 3-automatic sequence taking only the values 4 and 8, each with asymptotic density 1/2; in [5] some doubts were expressed on the fact that the sequence $\big(\ell_{12}(n!)\big)_n$ could be automatic itself. Our aim is to prove the following result

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[1]On lit en première page de cet article *Communicated by Pierre Liardet*, formule qui masque l'enthousiasme et la précision que Pierre mettait dans son activité éditoriale, comme dans toutes ses entreprises.

Theorem 1. *For $a \in \{4, 8\}$ the sequence $\{n\,;\, \ell_{12}(n!) = a\}$ is not 3-automatic. For $a \in \{3, 6, 9\}$ the sequence $\{n\,;\, \ell_{12}(n!) = a\}$ is not automatic.*

We first prove the second point, showing that each of the three considered sequences has no gap, which permits to apply a criterion of Minsky and Pappert for non automatic sequences, revisited by Cobham (Proposition 3 below). We then show that each of the sequences considered in the first point is a 3-automatic sequence, modified by a non automatic sequence.

2. The tools for the proof of Theorem 1

The following proposition recalls points established in the previous papers [5] and [4].

Proposition 1. *Let $u_0u_1u_2\dots u_n\dots$ be the fixed point, starting with a 4 of the substitution $4 \to 448884884$, $8 \to 884448448$. The following holds true*

(i) *If $v_4(n!) > v_3(n!)$, a relation which occurs on a set of asymptotic density 1, one has $\ell_{12}(n!) = u_n$.*

Let n be an integer divisible by 144 for which $v_3(n!) \geq v_4(n!) + 2$, then

(ii) *for $k \in \{0, 1, 2, 3\}$, we have $\ell_{12}\big((n+k)!\big) \in \{3, 6, 9\}$,*

(iii) *for $a \in \{3, 6, 9\}$, there exists $k \in \{0, 2, 3, 7\}$ such that $\ell_{12}\big((n+k)!\big) = a$.*

The second ingredient, already used in our previous papers is Legendre's relation connecting $v_{p^a}(n!)$ and the sum of the digits $s_p(n)$ of n in the prime base p, namely

$$v_{p^a}(n!) = \left\lfloor \frac{n - s_p(n)}{a(p-1)} \right\rfloor. \tag{1}$$

The third ingredient is a straightforward corollary of the main result of [9].

Proposition 2. *There exists a constant C such that, for $n \geq 25$ one has*

$$s_2(n) + s_3(n) > \frac{\log\log n}{\log\log\log n + C} - 1. \tag{2}$$

Our last ingredient is a sufficient criterion of Minsky and Pappert ([2], Theorem 10) implying that a sequence is not automatic.

Proposition 3. *An infinite sequence of integers $(a_n)_n$ with zero asymptotic density (i.e., a_n/n tends to infinity) and no gap (i.e., a_{n+1}/a_n tends to 1) is automatic in no base.*

3. Proof of Theorem 1

We first prove that there are many integers n for which $s_3(n)$ is small.

Lemma 1. *Let $\big(h(n)\big)_n$ be an increasing sequence of integers tending to infinity. The sequence $\{n;\, s_3(n) \le h(n)\}$ is not lacunary, in the sense that the sequence of the quotients of its consecutive terms tends to* 1.

Proof. Without loss of generality we may assume that $h(n)$ is an even number, say $h(n) = 2g(n)$, and that $h(n) \le \log n/2\log 3$. We consider the set $\mathcal{S}_h$ of integers n for which $h(n) \ge 4$ and $s_3(n) \le h(n)$. Let n be in $\mathcal{S}_h$; we are going to prove that the interval

$$\left(n\,,\, n\left(1+3^{1-g(n)}\right)\right]$$

contains an element of $\mathcal{S}_h$, which is enough to prove the lemma.

- If $s_3(n) \le h(n) - 1$, we have $s_3(n+1) \le s_3(n) + 1 \le h(n) \le h(n+1)$ and so $(n+1)$ belongs to $\mathcal{S}_h$.
- If $s_3(n) = h(n) = 2g(n)$, we let $n = \eta_K 3^K + \eta_{K-1}3^{K-1} + \cdots + \eta_0$ be the proper representation of n in base 3, with $\eta_K \ne 0$ and $K = \lfloor \log n/\log 3 \rfloor$. Writing g for $g(n)$, we have
$$\eta_K + \eta_{K-1} + \cdots + \eta_{K-g+2} \le 2(g-1) = h(n) - 2,$$
and so we have
$$\eta_0 + \eta_1 + \cdots + \eta_{K-g+1} \ge 2.$$

- If $\eta_{K-g+1} = 2$, we consider the integer $n + 3^{K-g+1}$; it is in the prescribed interval and satisfies, since there is a carry over,
$$s_3\left(n+3^{K-g+1}\right) \le s_3(n) = h(n) \le h\left(n+3^{K-g+1}\right).$$
- If $\eta_{K-g+1} \le 1$, there exists $\ell < K-g+1$ with $\eta_\ell \ge 1$; we then consider the integer $n + 3^{K-g+1} + 3^\ell$, which is in the prescibed interval and satisfies
$$s_3\left(n+3^{K-g+1} - 3^\ell\right) \le s_3(n) = h(n) \le h\left(n+3^{K-g+1} - 3^\ell\right).$$

This proves Lemma 1. □

In the sequel, we shall apply Lemma 1 with

$$h(n) = 2\left\lfloor \frac{\log\log n}{18\log\log\log n}\right\rfloor$$

and so the sequence

$$\mathcal{S} = \big\{n \ge 25\,;\, s_3(n) \le h(n)\big\} \quad \text{is not lacunary.} \tag{3}$$

Since the expansion of 144 in base 3 is 12100, for n in $\mathcal{S}$ and large enough, we have

$$s_3(144n) \leq 4s_3(n) \leq 4h(n) \leq \frac{4\log\log(144n)}{9\log\log\log(144n)}.$$

This, together with (2) implies that when n is in $\mathcal{S}$ and is large enough, we have

$$s_2(144n) \geq s_3(144n) + 4.$$

We now combine this last relation with Legendre's formula (1) and get that for sufficiently large $n \in \mathcal{S}$, one has

$$v_3\big((144n)!\big) \geq v_4\big((144n)!\big) + 2. \tag{4}$$

For $1 \leq a \leq 11$, we let $\mathcal{L}_a = \big\{n\,;\,\ell_{12}(n!) = a\big\}$.

We start by considering the case when $a \in \{3, 6, 9\}$. By Proposition 1 (i), the set $\mathcal{L}_a$ has zero asymptotic density; by (4), Proposition 1 (iii) and (3), it is not lacunary; so, by Proposition 3, it is not automatic and the second part of Theorem 1 is settled.

We turn now our attention to $\mathcal{L}_a$ for $a \in \{4, 8\}$; we let $\mathcal{U}_a = \{n\,;\,u_n = a\}$. By Proposition 1 (i), $\mathcal{L}_a \subset \mathcal{U}_a$ and those two sets have both density 1/2 and coincide on a set of density 1/2. Another feature of the 3-automatic sequence $\mathcal{U}_a$, easily seen on its definition, is that out of 4 consecutive integers, one at least belongs to it. By (4), Proposition 1 (ii) and (3), $\mathcal{U}_a \backslash \mathcal{L}_a$ is a non lacunary infinite sequence of asymptotic density 0, and thus, is not automatic. This implies the first part of Theorem 1 and ends its proof.

We finally remark, that the non automaticity in all bases for the sequences $\mathcal{L}_4$ and $\mathcal{L}_8$ is still to be proved.

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