DE GRUYTER
OPEN

uniform
distribution
theory

DOI: 10.1515/udt-2016–0016
Unif. Distrib. Theory **11** (2016), no.2, 125–150

ON THE CONSTANT IN THE AVERAGE DIGIT SUM FOR A RECURRENCE-BASED NUMERATION

Christian Ballot

Dedicated to the memory of Pierre Liardet

ABSTRACT. Given an integral, increasing, linear-recurrent sequence A with initial term 1, the greedy algorithm may be used on the terms of A to represent all positive integers. For large classes of recurrences, the average digit sum is known to equal $c_A \log n + O(1)$, where c_A is a positive constant that depends on A. This asymptotic result is re-proved with an elementary approach for a class of special recurrences larger than, or distinct from, that of former papers. The focus is on the constants c_A for which, among other items, explicit formulas are provided and minimal values are found, or conjectured, for all special recurrences up to a certain order.

Communicated by Georges Grekos

1. Introduction

Given a nondecreasing unbounded sequence of integers $A = (a_k)_{k\geq 0}$ with $a_0 = 1$, all positive integers n can be expressed uniquely, using the greedy algorithm, as a sum $\sum_{i=0}^{k} d_i a_i$, where the digits d_i are nonnegative integers and $d_k \geq 1$. This algorithm finds the largest index k such that $a_k \leq n$ and defines d_k as $\lfloor n/a_k \rfloor$. The procedure continues replacing n by $n - d_k a_k$ to find the next nonzero digit. We keep iterating this process until $\sum_{i=0}^{k} d_i a_i = n$, which is bound to happen because $a_0 = 1$. Then the sum-of-digit function s, or s_A, is defined as $s(n) := \sum_{i=0}^{k} d_i$. The *cumulative* sum-of-digit function S, or S_A, is, conforming to tradition, defined as $S(n) := \sum_{k=0}^{n-1} s(k)$, where we conveniently set $s(0) := 0$.

Here we are primarily concerned with sequences A which, in addition to the above hypotheses, are linear recurrent and have a monic integral characteristic

2010 Mathematics Subject Classification: 11A63, 11B37.
Keywords: numeration, digit sum, average, recurrence.

polynomial with a simple *dominant* zero $\alpha > 1$. By dominant we mean that α is real and larger than the modulus of any other zero of the characteristic polynomial. Thus,

$$a_{k+m} = P_1 a_{k+m-1} + P_2 a_{k+m-2} + \cdots + P_m a_k,$$

for all $k \geq 0$, some $m \geq 1$, where the coefficients P_i are integers, usually non-negative, and, as k tends to infinity, $a_k \sim a\alpha^k$ for some real $a > 0$.

Some early papers referenced in [3] were concerned with the geometric case $m = 1$, i.e., $A = (b^k)_{k\geq 0}$, $b \geq 2$ an integer, and showed that $S(n) \sim c_A n \log n$ as n tends to infinity, or, more precisely, that

$$S(n) = c_A n \log n + O(n), \tag{1}$$

with $c_A = (b-1)/(2\log b)$.

Later Trollope [20] improved (1) for $b = 2$ by showing that

$$S(n) = c_A n \log n + nG\left(\frac{\log n}{\log b}\right), \tag{2}$$

where G is a continuous function of period 1 of which a fully explicit description was found. Given that (2) holds with $c_A = (2\log 2)^{-1}$, the knowledge that G is of period 1 readily implies that $S(2n) = 2S(n) + n$. Conversely, the relations $s(2n) = s(n)$ and $s(2n+1) = s(n)+1$ imply $S(2n) = 2S(n)+n$, which yields the 1-periodicity of G. Using other techniques—Fourier analysis and combinatorics––and thus obtaining a different expression for G, Delange [7] proved (2) for a general base b with G continuous of period one and nowhere differentiable. Much later, Delange's results were re-proved using Mellin transforms and the Perron formula [12]. Incidentally, in 1999, Cooper and Kennedy [6] emulated the method of Trollope for a general base b.

Some ten years after Delange's work, Coquet and van den Bosch [5] produced a concise and beautiful paper dealing with the sequence $(a_k)_{k\geq 0}$, where $a_k = F_{k+2}$ and $(F_k)_{k\geq 0}$ is the Fibonacci sequence defined by $F_0 = 0$, $F_1 = 1$ and $F_{k+2} = F_{k+1} + F_k$, for all $k \geq 0$. They proved that

$$S(n) = c_A n \log n + nG\left(\frac{\log n}{\log \alpha}\right) + O(\log n), \tag{3}$$

where G is a continuous, nowhere differentiable function of period 1 and $c_A = \frac{3-\alpha}{5\log\alpha}$, α being the dominant zero of $x^2 - x - 1$.

Pethö and Tichy [16] extended (3) with G bounded, but not necessarily continuous, nor periodic to all recurrences A that satisfy

$$\begin{cases} P_1 \geq P_2 \geq \cdots \geq P_m > 0, \\ a_k > P_1(a_0 + \cdots + a_{k-1}), \ k = 1, \ldots, m-1. \end{cases} \tag{4}$$

The condition $P_1 \geq P_2 \geq \cdots \geq P_m > 0$ guaranties the existence of a dominant zero $\alpha > 1$ of the characteristic polynomial of A. A year later, Grabner and Tichy [13] widened the validity of (3) to a larger class of recurrences, namely those satisfying

$$\begin{cases} P_1 \geq P_2 \geq \cdots \geq P_m > 0, \\ a_k \geq P_1 a_{k-1} + \cdots + P_k a_0 + 1, \ k = 1, \dots, m-1. \end{cases} \tag{5}$$

They corrected a few mistakes of the paper [16], and determined exactly for which recurrences the function G in (3) is continuous of period 1 when $m \geq 2$. This occurs for so-called *canonical* recurrences, i.e., if and only if the inequality on the second line of (5) is an equality. Thus, G is continuous of period 1 for instance when $a_k = F_{k+2}$, as proved in [5], or when $a_k = t_{k+3}$ and $(t_k)_{k\geq 0}$ is the tribonacci sequence[1], i.e., the fundamental sequence of $x^3 - x^2 - x - 1$ (with initial values 0, 0, and 1). The function G is bounded but neither continuous, nor periodic, in the other cases. Most importantly for our purpose, they prove, using a result of Parry on the frequency of α-adic digits, an explicit formula for c_A when (5) holds, which with $f(x) = x^m - P_1 x^{m-1} - \cdots - P_m$ is

$$c_A = (\alpha f'(\alpha) \log \alpha)^{-1} \left(\sum_{k=2}^{m} \left(P_k \alpha^{m-k} \sum_{j=1}^{k-1} P_j \right) + \frac{1}{2} \sum_{k=1}^{m} P_k^2 \alpha^{m-k} - \frac{1}{2} \alpha^m \right). \tag{6}$$

Much work has been done on the distribution of the sum-of-digit function often proving its asymptotic normality. For each $n \geq 1$ one defines the discrete random variable X_n by its probability function

$$Pr\ (X_n = j) := \#\{k < n;\ s_A(k) = j\}/n.$$

Its expectation EX_n is $S_A(n)/n$ and its variance $\frac{1}{n}\sum_{k<n}(s_A(k) - EX_n)^2$. We merely point out two relevant papers which contain a wealth of references [10], and more recently [14] which studies the distribution of summands on intervals $[a_k, a_{k+1})$, for all canonical recurrences (a_k) with nonnegative P_i's and $P_1 P_m > 0$.

This paper is mostly concerned with establishing that $S_A(n) = c_A n \log n + O(n)$ for a class of recurrences larger than (5), with a pedestrian and self-contained approach, which yields a formula equivalent to (6). We then determine, or conjecture, what the least constant c_A is for all recurrences in this class whose order is less than an arbitrary bound. However, our approach also provides conclusions for recurrences which, for instance, do not necessarily have characteristic polynomials with nonnegative coefficients.

---

[1]Its code number in the OEIS, [19], is A000073

A first draft of this paper was written, while the author was unaware of the work done beyond that of Coquet and van den Bosch [5]. This first draft was written with the idea of improving on the work of the two papers [18, 3]. In [18], Pihko considered the sequence $a_k = F_{k+2}$ and, using elementary arguments, reproved the known fact that $S(n) \sim c_F n \log n$, with $c_F = (\gamma\sqrt{5}\log\gamma)^{-1} \simeq 0.574$, γ being the largest zero of $x^2 - x - 1$. The function $R(n) := \sum_{i=0}^{k} d_i S(a_i)$ is introduced and shown to satisfy $R(n) \sim c_F n \log n$ when n tends to $+\infty$. An easy induction shows the difference $S(n) - R(n)$ is $O(n)$ and the result follows. The asymptotic result $R(n) \sim c_F n \log n$ is first established for n running through the Fibonacci numbers $F_2, F_3, \dots, F_k, \dots$ Then, by some rather lengthy and idiosyncratic calculations, it is established for a general n. One purpose of [3] was to move from the asymptotics of $S(F_k)$ to those of $S(n)$, for general n, more swiftly by using a Cesàro-like theorem. Also, in [3], a new elementary proof that $S(n) \sim cn \log n$ for geometric sequences $a_k = b^k$, one closely analogous to the proof given for F_{k+2}, is written. Our intention was to devise a truly common proof, along the method of [3], establishing (1), i.e., $S_A(n) = c_A n \log n + O(n)$, for many recurring sequences, not only the Fibonacci and the geometric sequences. This endeavor now appears in Section 2. All that is required is that A be nondecreasing and have a characteristic polynomial with a simple dominant zero $\alpha > 1$, *and* that (1) holds when n runs through the sequence (a_k). This is Theorem 2. That $R(n)$ is proportional to $n \log n$ up to a $O(n)$-function comes from Theorem 1, which is the Cesàro-like theorem of [3]. That the difference $S(n) - R(n)$ is $O(n)$ is proved in Theorem 2 again by an inductive argument, where the function R is generally defined by

$$R(n) := \sum_{i=0}^{k} d_i S(a_i), \quad \text{whenever} \quad n = \sum_{i=0}^{k} d_i a_i. \tag{7}$$

Looking at the papers [5, 16, 13], one sees that our route is not all that different from theirs. It is usually proved that $S(a_k) = bka_k + O(a_k)$ and then, rather than using the function $R(n)$ as we do, the deviation of $S(n)$ from a sum that involves $\sum_{i=0}^{k} d_i \alpha^i$, is calculated to eventually measure the asymptotics of $S(n)$. Thus we give another proof of Theorem 2, not based on induction, using a technique of these papers.

In Section 3, we put to use the method of Section 2 to *special* recurrences. By this, we mean an integral linear recurrence $A = (a_k)_{k\ge 0}$ annihilated by a *special* polynomial $f(x) = x^m - P_1 x^{m-1} - P_2 x^{m-2} - \cdots - P_m$, $(m \ge 1)$, that is, such that

$$\begin{cases} a_{m-1} \ge a_{m-2} \ge \cdots \ge a_1 > a_0 = 1, \\ P_1 + P_2 + \cdots + P_i \ge P_j + P_{j+1} + \cdots + P_{j+i-1}, \end{cases} \tag{8}$$

for $i \geq 1$, $j \geq 1$ and $i + j \leq m + 1$, where the P_i's are nonnegative integers, $\sum_{i=1}^{m} P_i \geq 2$. Thus, P_1 is at least as large as any P_j, $P_1 + P_2$ is at least as large as the sum of any two consecutive P_j's, etc. Note that special recurrences are nondecreasing.

In particular, all recurrences for which (5) holds are special, but $x^4 - 3x^3 - x^2 - 3x - 1$ or $x^5 - x^4 - 1$ are also special.

The upshot of Section 3 is Theorem 4 with an explicit description of the constants c_A for all special recurrences.

Echoing a concern raised in the introduction of [3] about the least constant c_A that may occur, the minimal constants are determined for all special recurrences of order 1, 2 and 3 in Section 4. In higher order cases, we only state a conjecture which appears in Section 5. An infinite family of recurrences, namely $x^q - x^{q-1} - 1$, $(q \geq 1)$, is conjectured to offer the minimal constant c_A within the class of all special recurrences of degree at most q. The dominant zero of $x^q - x^{q-1} - 1$ is estimated with enough precision so as to show that the corresponding constant c_A is asymptotically equivalent to $(\log q)^{-1}$ as q tends to infinity.

The method of Section 2 implies that if $b_k := S_A(a_k)$ is annihilated by the square of the characteristic polynomial of (a_k), a condition that holds for special recurrences, then $S_A(n) = c_A n \log n + O(n)$. A final sixth section brings some degree of flexibility on this condition without affecting the result (1) on $S_A(n)$, except that the formula for c_A is more general; see Theorem 6 and the equation (26). Some examples of non-special recurrences are treated. Remarkably, some recurrences with characteristic polynomial $x^3 - x - 1$, i.e., with dominant zero the least Pisot number, lend themselves to our extended method, and yield a constant c_A noticeably smaller than the least constant associated with special recurrences of degree ≤ 3; see Theorem 9.

We do not know how far the method can be pushed. Is it possible to relax the general hypotheses (8) on special polynomials further than what we did in Theorem 6? What are the limits of validity of the formulas for c_A of Theorem 4? The constants c_A obtained in Section 6 for non-special recurrences sometimes do, but mostly do not fit formula (6), or the equivalent general formula of Theorem 4. Nondecreasing recurrences associated with the non-special polynomials $x^2 - Px - (P + 1)$, $P \geq 1$, still obey those formulas according to the remark that follows Theorem 7.

Also it would be interesting to investigate the asymptotics of the cumulative sum-of-digit functions when other representations than that provided by the greedy algorithm are used. One may consult the paper [17] and its references for possibilities. The distribution of the sum-of-digit for the far-difference representation of integers which uses distinct signed Fibonacci summands—at least four indices apart if of the same sign and at least three otherwise—was studied on intervals $(\sigma_{k-1}, \sigma_k]$, where $\sigma_k = \sum_{i=0}^{\lfloor k/4 \rfloor} F_{k-4i}$ in [15]. Suppose $m \geq 1$ is an integer. Put $a_0 = 1$ and, for $n = km + r$, $1 \leq r \leq m$, $a_n = 2r(m+1)^k$. Then, if at most one summand from each subset $\{a_0\}$, or $\{a_{im+1}, a_{im+2}, \ldots \ldots, a_{im+m}\}$, $(i \geq 0)$, is allowed, each positive integer has unique representation [9]. The authors [9] studied the distribution of summands, from various points of view, on intervals $[0, 2(m+1)^k)$. In particular, they showed that, for $N = 2(m+1)^k$, $S_A(N) = \frac{m}{m+1} Nk + \frac{N}{2}$. Finally, given an increasing integral recurrence $A = (a_k)_{k \geq 0}$, results on the number of representations $\sum_{i \geq 0} d_i a_i$ of an integer n when the digits d_i are nonnegative and bounded and on the average number of such representations may be found in [11].

Throughout, we use the symbols E and I to denote respectively the shift operator, i.e., $E \cdot a_k = a_{k+1}$ and the identity operator. We use the term *characteristic polynomial* of a sequence A to mean the monic and least-degree annihilating polynomial of A. The fundamental sequence associated with a polynomial f of degree m is the recurrence with characteristic polynomial f and initial values $0, \ldots, 0, 1$ $(m-1$ zeros).

2. Moving from $S_A(a_k)$ to $S_A(n)$

In this section, we do not yet assume that recurrences are special. We establish a few lemmas that will be handy throughout the paper before proving our main theorem, which roughly states that if for a nondecreasing integral recurrence $A = (a_k)_{k \geq 0}$ with $a_k \sim a\alpha^k$, $a > 0$, $\alpha > 1$, we have $S_A(n) = cn \log n + O(n)$ for $n = a_k$, then $S_A(n) = cn \log n + O(n)$ is true for all n.

Lemma 1. *Let $A = (a_k)_{k \geq 0}$ be a recurring sequence of positive real numbers such that the characteristic polynomial of A has a simple real dominant zero $\alpha > 1$. Then the set of quotients a_{k+1}/a_k is bounded above.*

Proof. By the hypothesis, there is a complex number a such that $a_k - a\alpha^k = o(\alpha^k)$. Conjugating yields $a_k - \bar{a}\alpha^k = o(\alpha^k)$ with $\bar{a}$ the complex conjugate of a. Hence, $(a - \bar{a})\alpha^k = o(\alpha^k)$ which implies that $a = \bar{a}$, i.e., a is real (> 0). Thus, for an ε in $(0, 1)$ we find, by the hypothesis, that for all k large enough either $a_{k+1} \leq a_k$, or

$$a\alpha^k(1-\varepsilon) \leq a_k < a_{k+1} < a\alpha^{k+1}(1+\varepsilon), \text{ so that } \frac{a_{k+1}}{a_k} \leq \alpha \frac{1+\varepsilon}{1-\varepsilon}. \qquad \square$$

The next lemma has been variously noticed and used in the literature, e.g., [6, Lemma 1] or [16, proof of Lemma 3].

Lemma 2. *Suppose $a_0 = 1$ and $A = (a_k)_{k\geq 0}$ is a nondecreasing sequence of integers. Suppose d is a nonnegative integer satisfying $da_k < a_{k+1}$ for some $k \geq 0$. Then*

$$S(da_k) = d\,S(a_k) + \frac{d(d-1)}{2}a_k, \tag{9}$$

where S is the cumulative digit sum with respect to A.

Proof. There is nothing to prove if $d \leq 1$ so assume $d \geq 2$. With s the digit-sum function, we may write $S(da_k) = S\big((d-1)a_k\big) + \sum_{j=(d-1)a_k}^{da_k-1} s(j)$. For integers j in the latter sum, we have $s(j) = d-1+s\big(j-(d-1)a_k\big)$ because $j < da_k < a_{k+1}$. Thus, $S(da_k) = S\big((d-1)a_k\big) + (d-1)a_k + S(a_k)$. Summing over all differences $S(ea_k) - S\big((e-1)a_k\big)$, $2 \leq e \leq d$, e integral, leads to

$$S(da_k) = dS(a_k) + \left(\sum_{j=1}^{d-1} j\right)a_k\,. \qquad \square$$

We repeat the proof of the Cesàro-like theorem of paper [3] but with notation more appropriate to this paper.

Theorem 1. *Let $(a_k)_{k\geq 0}$ and $(b_k)_{k\geq 0}$ be sequences of real numbers satisfying*

(i) $a_k \sim a\,\alpha^k, \quad a > 0,\ \alpha > 1,$ *and*

(ii) $b_k = b\,k\,a_k + O(a_k),$ *for some* $b > 0,$ *as k tends to infinity.*

Define

$$A_k = \sum_{i=0}^{k} d_i a_i\,, \qquad B_k = \sum_{i=0}^{k} d_i b_i\,,$$

where the d_i's lie in some interval $[0, d^+]$, $d^+ \geq 1$.

Then, regardless of the choice of the d_i's as long as $d_k \geq 1$, we have

$$B_k = b\,k\,A_k + O(A_k), \text{ as } k \text{ tends to infinity.}$$

Proof. By the triangular inequality, we see that

$$\begin{aligned}\left|\frac{B_k}{kA_k} - b\right| &= (kA_k)^{-1}\sum_{i=0}^{k} d_i\,|b_i - bka_i| \\ &\leq (kA_k)^{-1}\left(\sum_{i=0}^{k} d_i\,|b_i - bia_i| + b\sum_{i=0}^{k} d_i a_i(k-i)\right).\end{aligned}$$

By hypothesis, there is a positive K such that $|b_i - bia_i| \le Ka_i$ for all i's. Therefore the first sum in the previous expression is bounded above by KA_k. For the second sum we see that

$$\sum_{i=0}^{k} d_i a_i (k-i) \ll ad^+ \sum_{i=0}^{k} \alpha^i (k-i) \ll \sum_{j=0}^{k} j\alpha^{k-j} < \alpha^k \sum_{j\ge 0} j\alpha^{-j}.$$

That is,

$$\sum_{i=0}^{k} d_i a_i (k-i) \ll a_k \le A_k. \text{ Hence, } \left|\frac{B_k}{kA_k} - b\right| \ll \frac{1}{k}, \text{ i.e., } B_k = bkA_k + O(A_k). \quad \square$$

Theorem 2. *Let $A = (a_k)_{k\ge 0}$ be a nondecreasing linear recurrent sequence of integers with $a_0 = 1$. Suppose the characteristic polynomial of A has a simple dominant zero $\alpha > 1$. Let S denote the cumulative digit sum with respect to A and assume $S(a_k) = bka_k + O(a_k)$ for some positive b. Then, as n tends to infinity, we have*

$$S(n) = c_A n \log n + O(n), \text{ where } c_A = b/\log\alpha.$$

Proof. By Lemma 1, digits in the greedy-algorithm representation of integers (with respect to A) are bounded. Let $d^+ \ge 1$ be the largest possible digit. Define $b_k := S(a_k)$. By hypothesis, b_k is $bka_k + O(a_k)$. Let $n \ge 1$ be an integer. There is a unique $k \ge 0$ such that $a_k \le n < a_{k+1}$. Thus, the greedy-algorithm representation of n is of the form $\sum_{i=0}^{k} d_i a_i$ with $d_k > 0$. Define the arithmetic function $R(n)$ as $\sum_{i=0}^{k} d_i S(a_i)$. We may observe that $R(n) = R(d_k a_k) + R(n - d_k a_k)$. Also, by application of Theorem 1, we see that $R(n) = bkn + O(n)$.

Now, by strong induction on k, we prove that $0 \le S(n) - R(n) < Mn$, where M is a real number greater than both $a_1/2$ and d^+. If $k = 0$, i.e., if $a_0 = 1 \le n < a_1$, then $S(n) = n(n+1)/2$, while $R(n) = R(n \cdot a_0) = n \cdot S(a_0) = n$. Hence, $0 \le S(n) - R(n) \le n(n-1)/2 < na_1/2 < Mn$. Assume $0 \le S(j) - R(j) < Mj$ holds for all j less than a_k and let n be an integer satisfying $a_k \le n < a_{k+1}$. There is a unique integer d in $[1, d^+]$ satisfying $da_k \le n < (d+1)a_k$. So we may express the difference $S(n) - R(n)$ as $S(da_k) + d(n - da_k) + S(n - da_k) - \big(R(da_k) + R(n - da_k)\big)$. Hence, by the inductive hypothesis,

$$L \le S(n) - R(n) < L + M(n - da_k), \tag{10}$$

where $L = S(da_k) - dS(a_k) + d(n - da_k)$. By Lemma 2, L is equal to the sum $d(d-1)a_k/2 + d(n - da_k)$. As both summands in L are nonnegative, we find that $0 \le S(n) - R(n)$. Now $L + M(n - da_k) = d(d-1)a_k/2 + (d+M)(n - da_k)$, which, as $n - da_k < a_k$ and $M > d^+$, is $< d(d-1)a_k/2 + da_k + Mn - d^2 a_k = Mn - (d^2 - d)a_k/2 \le Mn$. Hence, $S(n) - R(n) = O(n)$ and, as $R(n) = bkn + O(n)$, we conclude that $S(n) = bkn + O(n)$. As $a_i \sim a\alpha^i$ for some $a > 0$, we see that

$$a\alpha^k \sim a_k \le n = \sum_{i=0}^{k} d_i a_i \sim a \sum_{i=0}^{k} d_i \alpha^i < \frac{ad^+\alpha}{\alpha - 1}\alpha^k,$$

which implies that $\log n = k \log \alpha + O(1)$ as $k \to \infty$. Thus, $k = \log n / \log \alpha + O(1)$ so that $S(n) = (b/\log \alpha) n \log n + O(n)$. □

We sketch out another proof of the fact that $S(n) - R(n)$ is $O(n)$. It relies on an exact formula for $S_A(n)$ used in the first lines of the proof of Lemma 2 of [5], generalized in (3.5) of [16] and corrected in (2.1) of [13].

The second proof of Theorem 2. We have, relying on the fact some truncations of the greedy algorithm's representation of an integer remain greedy representations, that

$$\begin{aligned} S(n) = S\left(\sum_{i=0}^{k} d_i a_i\right) &= \sum_{j=0}^{k} \left(S\left(\sum_{i=j}^{k} d_i a_i\right) - S\left(\sum_{i=j+1}^{k} d_i a_i\right)\right) \\ &= \sum_{j=0}^{k} \sum_{m<d_j a_j} s\left(m + \sum_{j<i\le k} d_i a_i\right) \\ &= \sum_{j=0}^{k} \sum_{m<d_j a_j} \left(s(m) + \sum_{j<i\le k} d_i\right) \\ &= \sum_{j=0}^{k} \left(S(d_j a_j) + d_j a_j \sum_{j<i\le k} d_i\right) = R(n) + H(n), \end{aligned}$$

where, by Lemma 2, $H(n) = \sum_{j=0}^{k} d_j a_j \big((d_j - 1)/2 + \sum_{j<i\le k} d_i\big)$.

Now

$$H(n) \le \sum_{j=0}^{k} d^+ a_j \big(d^+ + (k-j)d^+\big) \ll \sum_{j=0}^{k} a_j (k-j) \ll \sum_{j=0}^{k} \alpha^j (k-j).$$

As seen at the end of the proof of Theorem 1, the latter sum is $\ll \alpha^k$. But $\alpha^k \ll a_k \le n$. □

Thus, in order to show that $S_A(n) = c_A n \log n + O(n)$ for some classes of recurrences A, we will need to prove that $b_k := S_A(a_k)$ is equal to $bka_k + O(a_k)$. To be able to do so in some generality we will require, in the following section, that A be special, although the last section of our paper shows the method applies to other cases as well.

3. Application of the method to special recurrences

We begin by proving a couple of lemmas that guarantee the existence of a simple dominant zero α in any special[2] polynomial f and that the characteristic polynomial of any special recurrence annihilated by f must have α as a zero. These lemmas are very close to Lemma 3.1 of [10]. Our proofs differ and we include them so our text be self-contained.

Lemma 3. *Suppose $f(x) = x^m - P_1x^{m-1} - P_2x^{m-2} - \cdots - P_m$ is a polynomial of degree $m \geq 1$, where the P_i's are nonnegative integers, $\sum_{i=1}^m P_i \geq 2$ and some P_j, $1 \leq j \leq m$, j odd, is nonzero. Then f has a simple dominant real zero $\alpha > 1$.*

Proof. By Descartes' rule of signs $f(x)$ has a unique positive real zero α. If $P_i = 0$, for all i, $i \neq j$, then $P_j \geq 2$ and $\alpha^m = P_j\alpha^{m-j}$ implies $\alpha = P_j^{1/j} > 1$. Otherwise, $\alpha^m > P_j\alpha^{m-j}$. Thus, $\alpha^j > P_j \geq 1$. Thus, in all cases, $\alpha > 1$. Note that $\alpha f'(\alpha) - P_m$ equals

$$m\alpha^m - (m-1)P_1\alpha^{m-1} - (m-2)P_2\alpha^{m-2} - \cdots - P_{m-1}\alpha - P_m > mf(\alpha) = 0.$$

Hence, $\alpha f'(\alpha) > P_m \geq 0$. Therefore, $f'(\alpha)$ is positive and α is a simple zero. Also, if $x > \alpha$, then $f(x) > 0$. Suppose z is another zero of f of modulus ρ. Then, by the triangular inequality, $\rho^m \leq \sum_{i=1}^m P_i\rho^{m-i}$, where this inequality is strict if z is nonreal. However, if $\rho > \alpha$, then $f(\rho) > 0$ so that $\rho^m > \sum_{i=1}^m P_i\rho^{m-i}$, contradicting the previous inequality. If $\rho = \alpha$, then $f(\rho) = 0$ and $\rho^m = \sum_{i=1}^m P_i\rho^{m-i}$ so we still get a contradiction unless $z = -\alpha$. But $(-1)^m\alpha^m = \sum P_i(-1)^{m-i}\alpha^{m-i}$ implies that $\alpha^m = \sum P_i(-1)^i\alpha^{m-i}$ which may only happen if P_i were 0 for all odd i's. But $P_j > 0$. $\square$

Lemma 4. *Let $f(x) = x^m - P_1x^{m-1} - P_2x^{m-2} - \cdots - P_m$, where the P_i's are nonnegative integers, $\sum_{i=1}^m P_i \geq 2$ and some P_j, $1 \leq j \leq m$, j odd, is nonzero. Let $A = (a_k)_{k\geq 0}$ be a linear recurring sequence annihilated by f. If the initial values $a_0, \ldots, a_{m-1}$ of A are positive, then there is a positive constant a such that, as k tends to infinity,*

$$a_k \sim a\alpha^k, \qquad \text{where } \alpha \text{ is the dominant zero of } f.$$

Proof. By Lemma 3, f has a simple dominant zero $\alpha > 1$ and $f'(\alpha) > 0$. Define $g(x) := \sum_{i=0}^{m-1} Q_ix^{m-1-i}$ as the cofactor of $x - \alpha$ in $f(x)$, i.e., $f(x) = (x-\alpha)g(x)$. Solving for the Q_i's in terms of the P_i's we get that $Q_0 = 1$ and, for $i \geq 1$, $-\alpha Q_{i-1} + Q_i = -P_i$. Therefore, we find recursively that $Q_i = \alpha^i - P_1\alpha^{i-1} - P_2\alpha^{i-2} - \cdots - P_i$. Thus, as $\alpha^{m-i}Q_i = P_{i+1}\alpha^{m-i-1} + P_{i+2}\alpha^{m-i-2} + \cdots + P_m \geq 0$,

[2] As defined in (8).

$Q_i \geq 0$, for all i, $1 \leq i < m$. Now, as $a_k = a\alpha^k + r_k$, where r_k is annihilated by g, we see that

$$af'(\alpha) = a\prod_{i=2}^{m}(\alpha - \alpha_i) = ag(\alpha) = a\sum_{i=0}^{m-1} Q_i\alpha^i = \sum_{i=0}^{m-1} Q_i(a_i - r_i) = \sum_{i=0}^{m-1} Q_i a_i > 0,$$

implies $a > 0$, where the α_i's are the zeros of g. $\square$

Example 1. Take $f(x) = x^3 - 2x^2 - 2x - 3 = (x^2 + x + 1)(x - 3)$ with dominant zero $\alpha = 3$. Choose any recurring sequence A annihilated by $x^2 + x + 1$ with a_0 and a_1 positive. Then $a_2 < 0$. Indeed, otherwise, as A is annihilated by f, Lemma 4 would imply $a_k \sim a3^k$ for some $a > 0$. Of course, as $x^2 + x + 1$ annihilates (a_k), we see directly that $a_2 = -a_1 - a_0 < 0$.

Again we define b_k as $S(a_k)$, for all $k \geq 0$. Here, $A = (a_k)_{k \geq 0}$ is a special recurrence annihilated by the (special) polynomial

$$f(x) = x^m - P_1x^{m-1} - P_2x^{m-2} + \cdots - P_m.$$

Lemma 5. *We have the general identity valid for all j, $1 \leq j \leq m$, and all $k \geq j - 1$,*

$$\begin{aligned} S\left(\sum_{i=j}^{m} P_i a_{k+m-i}\right) &= P_j b_{k+m-j} + 0.5P_j(P_j - 1)a_{k+m-j} \\ &+ P_j \sum_{j<i\leq m} P_i a_{k+m-i} + S\left(\sum_{j<i\leq m} P_i a_{k+m-i}\right). \end{aligned}$$

P r o o f. The key point is to show the nonnegativity of the expression

$$(P_1 - P_j)a_{k+m-j} + (P_2 - P_{j+1})a_{k+m-j-1} + \cdots + (P_{m-j+1} - P_m)a_k. \tag{11}$$

The sum of the first two terms of (11) satisfies

$$(P_1 - P_j)a_{k+m-j} + (P_2 - P_{j+1})a_{k+m-j-1} \geq (P_1 + P_2 - P_j - P_{j+1})a_{k+m-j-1},$$

a nonnegative number because f is special. Therefore, the sum of the first three terms is $\geq (P_1 + P_2 + P_3 - P_j - P_{j+1} - P_{j+2})a_{k+m-j-2}$ again a nonnegative number. Adding one more term at a time, we end up with the nonnegative lower bound $(P_1 + P_2 + \cdots + P_{m-j+1} - P_j - P_{j+1} - \cdots - P_m)a_k$ of (11) proving the point.

Now suppose n satisfies $P_j a_{k+m-j} \leq n < P_j a_{k+m-j} + \sum_{i=j+1}^{m} P_i a_{k+m-i}$. Adding (11) to $P_j a_{k+m-j} + \sum_{i=j+1}^{m} P_i a_{k+m-i}$ yields

$$\begin{aligned} \big(P_j + (P_1 - P_j)\big)a_{k+m-j} &+ \big(P_{j+1} + (P_2 - P_{j+1})\big)a_{k+m-j-1} \\ &+ \cdots + \big(P_m + (P_{m-j+1} - P_m)\big)a_k, \end{aligned}$$

which is $P_1a_{k+m-j}+P_2a_{k+m-j-1}+\cdots+P_{m-j+1}a_k$, a quantity bounded above by $a_{k+m-j+1}=P_1a_{k+m-j}+P_2a_{k+m-j-1}+\cdots+P_ma_{k-j+1}$. Therefore, we find that $s(n)=P_j+s(n-P_ja_{k+m-j})$. Thus, summing over all such n's, we see that

$$S\left(\sum_{i=j}^{m}P_ia_{k+m-i}\right)=S(P_ja_{k+m-j})+P_j\sum_{i=j+1}^{m}P_ia_{k+m-i}+S\left(\sum_{i=j+1}^{m}P_ia_{k+m-i}\right),$$

which using Lemma 2 on the term $S(P_ja_{k+m-j})$ yields the lemma. □

This allows us to state a first theorem which gives the exact recursion followed by the sequence (b_k).

Theorem 3. *Suppose $(a_k)_{k\geq 0}$ is a special recurrence annihilated by $f(x)=x^m-P_1x^{m-1}-P_2x^{m-2}-\cdots-P_m$. Then, for all $k\geq m-1$, the terms of the two sequences (a_k) and (b_k) satisfy the equation*

$$f(E)\cdot b_k=\frac{1}{2}\sum_{j=1}^{m}\left(P_j(P_j-1)a_{k+m-j}+2P_j\sum_{j<i\leq m}P_ia_{k+m-i}\right). \tag{12}$$

Hence, the sequence $(b_k)_{k\geq m-1}$ is an integral linear recurrent sequence annihilated by f^2.

Writing $f(x)$ as $x-\alpha$, x^2-Px-Q and x^3-Px^2-Qx-R for m respectively equal to $1,2$ and 3, equation (12) takes, in those respective cases, the simpler explicit forms

$$2f(E)\cdot b_k=\begin{cases}\alpha(\alpha-1)a_k,\\ P(P-1)a_{k+1}+2PQa_k+Q(Q-1)a_k,\\ P(P-1)a_{k+2}+Q(2P+Q-1)a_{k+1}+R(2Q+2P+R-1)a_k.\end{cases} \tag{13}$$

Proof. The identity (12) is obtained by iterating Lemma 5 for $j=1$, then $j=2$ till $j=m$ and putting all (b_k) terms on the LHS. □

Theorem 4. *Let $f(x)=x^m-P_1x^{m-1}-P_2x^{m-2}-\cdots-P_m$ be a special polynomial as defined in (8). Suppose $A=(a_k)_{k\geq 0}$, $a_0=1<a_1\leq\cdots\leq a_{m-1}$ is an integral linear recurrence annihilated by f. Then*

$$S_A(n)=c_An\log n+O(n)\ \text{ with } c_A=\frac{\sum_{\ell=1}^{m}P_\ell\big(P_\ell-1+2\sum_{1\leq j<\ell}P_j\big)\alpha^{m-\ell}}{2\alpha f'(\alpha)\log\alpha}, \tag{14}$$

where α is the dominant zero and f' the derivative of f.

P r o o f. By Lemma 3, f has a simple dominant zero $\alpha > 1$. By Lemma 4, there is a positive constant a such that $a_k \sim a\alpha^k$ as k tends to $+\infty$. Because the coefficients of the various a_{k+m-i} on the RHS of (12) are all nonnegative and not all zero, the RHS of (12) is of the form $\lambda\alpha^k + o(\alpha^k)$ for some positive λ. Hence, $(x-\alpha)^2$ must be a factor of the characteristic polynomial of (b_k). Since the dominant zero of f is the dominant double zero of f^2, which annihilates (b_k), there is a positive constant c such that $b_k = ck\alpha^k + O(a_k)$ as k tends to $+\infty$. Hence, by Theorem 2, $S_A(n) = c_A n \log n + O(n)$ with $c_A = c/(a\log\alpha)$. To find out the value of c/a, we compare both sides of (12). If Δ represents the derivation operator, then $k\alpha^k = \alpha(\Delta x^k)|_{x=\alpha}$. Thus, as E and Δ commute, $f(E)\cdot b_k \sim f(E)\cdot c\alpha(\Delta x^k)|_{x=\alpha} = c\alpha\,\big(\Delta\cdot f(E)\big)x^k|_{x=\alpha} = c\alpha\,\Delta\big(x^k f(x)\big)|_{x=\alpha} = c\alpha\big(k\alpha^{k-1}f(\alpha)+\alpha^k f'(\alpha)\big) = 0 + c\alpha f'(\alpha)\cdot\alpha^k$. On the other hand, the RHS of (12) is asymptotically equivalent to

$$\frac{a}{2}\sum_{j=1}^{m}\left(P_j(P_j-1)\alpha^{m-j} + 2P_j\sum_{i=j+1}^{m}P_i\alpha^{m-i}\right)\alpha^k. \tag{15}$$

Therefore, comparing (15) to $c\alpha f'(\alpha)\cdot\alpha^k$ and solving for c/a yields

$$c_A = \frac{\sum_{j=1}^{m}\big(P_j(P_j-1)\alpha^{m-j} + 2P_j\sum_{i=j+1}^{m}P_i\alpha^{m-i}\big)}{2\alpha f'(\alpha)\log\alpha}.$$

But the coefficient of $\alpha^{m-\ell}$ in $\sum_{j=1}^{m}\big(P_j(P_j-1)\alpha^{m-j} + 2P_j\sum_{i=j+1}^{m}P_i\alpha^{m-i}\big)$ being $P_\ell(P_\ell-1) + \sum_{1\le j\le \ell-1}2P_jP_\ell$ the expression for c_A given in the theorem follows. $\square$

Corollary 6. *The value of c_A in Theorem 4 is independent of the choice of the a_i, $1 \le i < m$, as long as $a_{m-1} \ge a_{m-2} \ge \cdots \ge a_1 > a_0 = 1$.*

4. Search of the minimal constant c_A for special recurrences of order ≤ 3

In this section, we determine the least constant c_A, as A varies through all special recurrences of order less than, or equal to m, for $1 \le m \le 3$.

Thus, $A = (a_k)_{k\ge0}$ is assumed to be a special recurrence of order $\le m$ annihilated by the special polynomial $f(x) = x^m - P_1x^{m-1} - P_2x^{m-2} - \cdots - P_m$ of degree m whose dominant zero is denoted by α.

Case 1. The first-order recurrences.

If $A = (a_k)_{k\ge0}$ is a first-order special recurrence, then, by equation (14), $c_A = (\alpha-1)/(2\log\alpha)$. The function $x \mapsto (x-1)/\log x$ is increasing on $]1,+\infty[$. Thus, the least c_A corresponds to $\alpha = 2$. It is $(2\log 2)^{-1} \simeq 0.721$.

Case 2. Special recurrences at most of the second order.

In this particular case, we re-state Theorem 4 as a corollary.

Corollary 7. *If $A = (a_k)_{k\ge 0}$ satisfies $a_{k+2} = Pa_{k+1} + Qa_k$ for all $k \ge 0$, where $P \ge Q \ge 0$, $P + Q \ge 2$, $a_0 = 1$ and $a_1 > 1$ are integers, then*

$$S_A(n) = c_A n \log n + O(n), \text{ with } c_A = \frac{P(P-1)\alpha + Q(Q-1) + 2PQ}{2\alpha\sqrt{D}\log\alpha}, \tag{16}$$

and $D = P^2 + 4Q$.

Note that if $Q = 0$, then as $P \ge 2$ we may choose $a_1 = P$ and A is the geometric sequence $(P^k)_{k\ge 0}$. In that case, we recover the fact that $S_A(n) = c_A n \log n + O(n)$ with $c_A = (P-1)/(2\log P)$, since $Q = 0$ implies $\alpha = P = \sqrt{D}$.

In [3] we had wondered whether the most economical recurrence-based numeration system, taking the size of the constant c_A as our gross criterion, took place for the Zeckendorf representation, i.e., the representation derived from $a_k = F_{k+2}$. This appears to be true at least within the class of special second-order recurrences. Note that for $a_k = F_{k+2}$, we may recover that $c_A = c_F = (\gamma\sqrt{5}\log\gamma)^{-1} \simeq 0.574$ using Corollary 7.

Corollary 8. *The minimal c_A, as A varies through all the first-order and second-order special recurrences is achieved when $a_k = F_{k+2}$, where F_k is the kth Fibonacci number.*

Proof. By (16), we need to show the inequality

$$2\alpha\sqrt{D}\log\alpha \le (\gamma\sqrt{5}\log\gamma)\bigl(P(P-1)\alpha + Q(Q-1) + 2PQ\bigr). \tag{17}$$

Suppose $P \ge 8$. As $Q \le P \le P^2$, we claim that the LHS of (17) is bounded above by $(\gamma\sqrt{5}\log\gamma)P(P-1)\alpha$. Indeed, $\sqrt{D} \le P\sqrt{5}$ and $\alpha \le P\gamma$, so $2\alpha\sqrt{D}\log\alpha \le 2\alpha P\sqrt{5}\log(P\gamma)$. Hence, our claim holds if

$$\frac{\log(P\gamma)}{\log\gamma} \le \frac{\gamma}{2}(P-1),$$

which is true for all $P \ge 8$. The twenty-seven remaining values of c_A with $1 \le Q \le P \le 7$, excluding $P = Q = 1$, are easily checked to satisfy (17) with some mathematical software. (If $Q = 0$, then $c_A = (P-1)/(2\log P)$, where $a_n = P^n$, $P \ge 2$. We saw that $c_A \ge (2\log 2)^{-1} \simeq 0.721$, which is larger than c_F.) ☐

Table 1. $c_A(P,Q)$ for $1 \le Q \le P \le 5$.

$P \setminus Q$	1	2	3	4	5
1	0.574	*	*	*	*
2	0.733	0.813	*	*	*
3	0.907	0.948	1.009	*	*
4	1.076	1.097	1.135	1.185	*
5	1.236	1.247	1.272	1.306	1.348

We provide the values of c_A rounded up to the nearest third decimal place for P and Q positive and at most five.

Remark 1. The Lucas numbers L_n, defined by $L_0 = 2$, $L_1 = 1$ and $L_{n+2} = L_{n+1}+L_n$, would also yield $c_L = c_F$ by Corollary 6 provided we took $a_n = L_{n+1}$, $n \ge 0$.

Case 3. Special recurrences at most of the third order.

Here we assume that $A = (a_k)_{k\ge 0}$ is at most a third-order linear recurrence with $a_2 \ge a_1 > a_0 = 1$ integral and, for all $k \ge 0$, $a_{k+3} = Pa_{k+2}+Qa_{k+1}+Ra_k$, where $P \ge 1$, $Q \ge 0$, $R \ge 0$ are integers that satisfy $P + Q + R \ge 2$ and $P \ge \max\{Q, R\}$. We restate Theorem 4 for these recurrences in a corollary.

Corollary 9. *Let $f(x) = x^3 - Px^2 - Qx - R$ be a special polynomial. Suppose $A = (a_k)_{k\ge 0}$ is an integral recurrence annihilated by the polynomial f. Assume $a_0 = 1$ and $a_2 \ge a_1 > 1$. Then $S_A(n) = c_A n \log n + O(n)$, where*

$$c_A = \frac{P(P-1)\alpha^2 + 2PQ\alpha + Q(Q-1)\alpha + 2R(P+Q) + R(R-1)}{2\alpha f'(\alpha)\log\alpha}, \tag{18}$$

with f' and α, respectively, the derivative and the dominant zero of f.

Now we find the recurrence f which provides the least constant c_f.

Corollary 10. *The minimal c_A, as A varies through all recurrences that satisfy $a_2 \ge a_1 > a_0 = 1$ and are annihilated by some special characteristic polynomials $x^3 - Px^2 - Qx - R$ is achieved when $a_k = N_{k+4}$, where $N_0 = 0$, $N_1 = 0$, $N_2 = 1$ and $N_{k+3} = N_{k+2} + N_k$, i.e., when A is the Narayana sequence.*[3]

Proof. If $P = R = 1$ and $Q = 0$, then $c_A = c_N = (\eta f'(\eta)\log\eta)^{-1}$, where $\eta \simeq 1.46557$ is the dominant zero of $x^3 - x^2 - 1$. We find that $c_N \simeq 0.508$. The value of c_N is less than c_F proving the truth of the theorem with regard to the first and the second-order recurrences of Corollary 7. So we may assume $R \ge 1$.

[3] See [1] for historic background on this sequence; its code number in the OEIS, [19], is A000930.

By (18), it suffices to show the inequality

$$2\alpha f'(\alpha)\log\alpha,\ \text{i.e.,}\ 2\alpha(3\alpha^2-2P\alpha-Q)\log\alpha \le c_N^{-1}\ P(P-1)\alpha^2,$$

or more simply to show that

$$(6\alpha-4P)\log\alpha \le c_N^{-1}\ P(P-1). \tag{19}$$

Because $\alpha > 1$, we obtain $\alpha^3 = P\alpha^2+Q\alpha+R \le P(\alpha^2+\alpha+1) \le 3P\alpha^2$ so that $\alpha \le 3P$. Thus, $(6\alpha-4P)\log\alpha \le 14P\log(3P)$. Hence, to prove (19) it suffices to have

$$14c_N(\log 3+\log P) \le P-1,$$

which is true for all $P \ge 34$. The remaining values of c_A for $P \le 33$, $0 \le Q \le P$ and $1 \le R \le P$ are finitely many and a program tells us they all exceed c_N. (Of course, there are many ways to reduce further the numerical search. For instance, one can observe that $\alpha \le 3P$ implies that $\alpha^3 \le P\alpha^2+3PQ+R \le P\alpha^2+3P^2+P^2 < (P+4)\alpha^2$, since $\alpha > P$. Thus, $\alpha < P+4$. Hence, $(6\alpha-4P)\log\alpha < (24+2P)\log(P+4)$. So it suffices to have $P-1 \ge c_N(24/P+2)\log(P+4)$ for (19) to hold. But this latter inequality holds for all $P \ge 8$. This leaves only $\sum_{i=1}^{7} i(i+1) = 168$ values of c_A to compute and compare to c_N.) $\square$

We provide a couple of tables, one with $Q=0$, another with $Q=1$, with all c_f's rounded up to the nearest third decimal for all special polynomials $f(x) = x^3-Px^2-Qx-R$ when P does not exceed five.

Table 2. $c_A(P, Q=0, R)$ for $1 \le R \le P \le 5$.

$P \setminus R$	1	2	3	4	5
1	0.508	*	*	*	*
2	0.682	0.718	*	*	*
3	0.883	0.884	0.903	*	*
4	1.065	1.060	1.063	1.074	*
5	1.232	1.226	1.225	1.229	1.235

Table 3. $c_A(P, Q=1, R)$ for $1 \le R \le P \le 5$.

$P \setminus R$	1	2	3	4	5
1	0.626	*	*	*	*
2	0.740	0.779	*	*	*
3	0.902	0.913	0.935	*	*
4	1.069	1.070	1.078	1.092	*
5	1.230	1.229	1.231	1.237	1.245

Remark 2. Beyond c_N, it seems that c_F and c_t are respectively the second and the third least constants c_A among all constants considered in Corollary 10. We have $c_t \simeq 0.626$, where (t_n) is the shift of the tribonacci sequence with initial values 1, 2 and 4. We also have that the dominant zeros α_N, α_F and α_t of the characteristic polynomials of the Narayana, Fibonacci and tribonacci sequences satisfy $\alpha_N < \alpha_F < \alpha_t$. However, it is not true always that $\alpha_u < \alpha_v$ implies $c_u < c_v$. For instance,

$$\alpha_{x^3-4x^2-1} < \alpha_{x^3-4x^2-2}, \quad \text{but} \quad c_{x^3-4x^2-1} > c_{x^3-4x^2-2}$$

as Table 2 shows.

5. Conjectured minimal constants for general order special recurrences

This section focuses on a family of recurrences of which a study was made in [4]. This family was also shown [8] to provide a general unique integer representation with positive and negative summands which generalizes the far-difference representation of Fibonacci numbers [2].

Let $q \geq 1$ be an integer and $G = (g_k)_{k\geq 0}$ be the fundamental sequence of $f(x) = x^q - x^{q-1} - 1$. That is, $g_{k+q} = g_{k+q-1} + g_k$, $k \geq 0$, and the initial values of G are $0, \ldots, 0, 1, 1, \ldots, 1, 2$, with $q-1$ initial zeros followed by q ones. Then we consider $A = (a_k)_{k\geq 0}$ defined by $a_k = g_{2q-2+k}$. The sequence A is a shift of G that starts with the last term of the sequence G equal to 1. Note that for $q = 1, 2$ and 3, a_k is respectively 2^k, F_{k+2} and N_{k+4}.

We conjecture that c_q is the least constant c_A when A varies through all recurrences of order at most q that satisfy Theorem 4. We saw this is true if $q = 1$, 2 and 3. The method used to prove the cases $q = 1$, 2 and 3 could be used for any specific q, but would leave some numerical verification for small values of the coefficients. It cannot provide a proof for all q.

In this section, we are curious of the behavior of c_q as q tends to infinity. We begin with an estimate of the dominant zero of $f(x)$ and use the notation $\log_2 x$ to denote $\log\log x$.

Lemma 11. *The dominant zero α of $x^q - x^{q-1} - 1$ satisfies*

$$1 + \frac{\log q}{q} - \frac{\log_2 q}{q} + \frac{\log_2^2 q}{q \log q} > \alpha \geq 1 + \frac{\log q}{q} - \frac{\log_2 q}{q},$$

where the leftmost inequality holds for q large enough and the rightmost inequality is valid for all $q \geq 3$.

P r o o f. By Lemma 3, $f(x)$ has a simple dominant zero $\alpha > 1$. Assume $q \geq 3$ and define

$$\nu := \frac{\log_2 q}{\log \frac{q}{\log q}} \quad \text{and} \quad \omega := \frac{\log_2 q}{\log q}.$$

Put $\alpha = 1 + \varepsilon$ and, reasoning by contradiction, assume $\varepsilon < \frac{\log q}{(1+\nu)q}$. Then

$$1 = (1+\varepsilon)^{q-1}\varepsilon < (1+\varepsilon)^{q-1}\frac{\log q}{(1+\nu)q}, \quad \text{so that} \quad \frac{(1+\nu)q}{\log q} < (1+\varepsilon)^{q-1}.$$

Taking logarithms yields

$$\log\big((1+\nu)q\big) - \log_2 q < (q-1)\varepsilon < \frac{\log q}{1+\nu}.$$

Hence,

$$\log(1+\nu) + \frac{\nu}{1+\nu}\log q < \log_2 q.$$

As $\log(1+\nu) > 0$ for $q \geq 3$, this implies

$$\frac{\nu}{1+\nu}\log q < \log_2 q,$$

which, since $\nu/(1+\nu) = \log_2 q/\log q$, leads to $1 < 1$. Therefore, for all $q \geq 3$,

$$\alpha \geq 1 + \frac{\log q}{(1+\nu)q} = 1 + \frac{\log q}{q} - \frac{\log_2 q}{q}.$$

Put $x_q = 1 + \frac{\log q}{(1+\omega)q}$. Note that $f(x_q) = y_q - 1$, where $y_q = x_q^{q-1}(x_q - 1)$. We will see that for q large enough $y_q > 1$, which as α is the only positive zero of f proves that $x_q > \alpha$. Because $x_q < 1 + \frac{\log q}{q}(1 - \omega + \omega^2)$, we will obtain the first inequality of the lemma. Now

$$\log\frac{x_q y_q}{x_q - 1} = q\log x_q = \frac{\log q}{1+\omega} + O\Big(\frac{\log^2 q}{q}\Big).$$

Thus,

$$\frac{x_q y_q}{x_q - 1} = e^{\log q\left(1-\omega+\omega^2(1+o(1))\right)+O\left(\frac{\log^2 q}{q}\right)} = \frac{q}{\log q}e^{\frac{\log_2^2 q}{\log q}\left(1+o(1)\right)}.$$

Therefore,

$$y_q = \frac{1}{1+\omega}\cdot\frac{1}{1+\frac{\log q}{(1+\omega)q}}\cdot e^{\frac{\log_2^2 q}{\log q}\left(1+o(1)\right)},$$

which, as $\frac{\log q}{q}$ and ω are both $o\big(\frac{\log_2^2 q}{\log q}\big)$, yields that

$$y_q = 1 + \frac{\log_2^2 q}{\log q}\big(1+o(1)\big) > 1, \text{ when } q \to \infty.$$

$\square$

Theorem 5. *Let $q \geq 1$. If A is the shift of the sequence G defined above, we find that*

$$S_A(n) = c_q\, n \log n + O(n), \ \text{ where } c_q = (\alpha f'(\alpha) \log \alpha)^{-1},$$

α is the dominant zero of $f(x) = x^q - x^{q-1} - 1$ and f' is the derivative of f. Moreover, the constants c_q become arbitrarily small as q tends to infinity. In fact,

$$c_q \sim (\log q)^{-1}, \ \text{ as } q \to +\infty. \tag{20}$$

P r o o f. Note that $x^q - x^{q-1} - 1$ is a special polynomial as defined right after (8). Thus, the hypotheses of Theorem 4 hold. The statement on the asymptotics of S_A follows and the value of c_q is obtained by setting $P_1 = P_q = 1$ and all other P_i's equal to 0 with $m = q$ in formula (14). Expressing $\alpha f'(\alpha)$ in terms of $f(\alpha) = \alpha^q - \alpha^{q-1} - 1$, we obtain $\alpha f'(\alpha) = q\alpha^q - (q-1)\alpha^{q-1} = q + \alpha^{q-1}$. Therefore, for all $q \geq 1$, we find that

$$c_q = \big((\alpha^{q-1} + q) \log \alpha\big)^{-1}.$$

Thus, we need to show that $(\alpha^{q-1} + q) \log \alpha \sim \log q$ as q tends to infinity. By Lemma 11, for q large enough we find that

$$\alpha^{q-1} < \alpha^q = e^{q \log \alpha} < e^{\log q - \log_2 q + \varepsilon_q \log_2 q} = \frac{q}{\log q} \log^{\varepsilon_q} q,$$

where $0 \leq \varepsilon_q < \frac{\log_2 q}{\log q}$. As $\log(\log^{\varepsilon_q} q) = \varepsilon_q \log_2 q \to 0$ with $q \to \infty$, we see that $\alpha^{q-1} < q\big(1 + o(1)\big)/\log q$. Therefore, we conclude that

$$(\alpha^{q-1} + q) \log \alpha \sim q \log \alpha \sim q \cdot \frac{\log q}{q} = \log q, \ \text{ as } q \to +\infty.$$

$\square$

Numerical data. We compare c_q and $\log^{-1} q$ in a small table, rounded up to the nearest third decimal place, for some q's. The values of $c_q \log q$ seem to increase steadily as q varies from 2 to 119 surpassing 1 at about $q = 40$ and, perhaps somewhat surprisingly in view of the fact that $c_q \sim (\log q)^{-1}$, reaching $\simeq 1.058$ when $q = 119$.

Table 4.

q	3	4	5	6	7	30	60	99	119
c_q	0.508	0.468	0.440	0.419	0.403	0.288	0.251	0.229	0.221
$\log^{-1} q$	0.910	0.721	0.621	0.558	0.514	0.294	0.244	0.218	0.209

6. Further results on non-special recurrences close to special ones

We open this section with two lemmas and a theorem which, among other applications, show a certain degree of stability in the value of the constant c_A under small variations of A off from a special recurrence.

Lemma 12. *Suppose $\alpha_1,\dots,\alpha_{m-1}$ are complex numbers with $|\alpha_i|<\theta<\alpha$, $1\le i\le m-1$, where $\alpha>\theta>1$ are real numbers. Assume $(s_k)_{k\ge0}$ and $(r_k)_{k\ge0}$ are complex sequences satisfying*

$$(E-\alpha_1 I)\circ(E-\alpha_2 I)\circ\cdots\circ(E-\alpha_{m-1}I)\circ(E-\alpha I)\cdot s_k=r_k,\quad \textit{where } r_k=O(\theta^k).$$

Then, for all $k\ge1$,

$$s_k=s_0\alpha^k+\alpha^{k-1}t_0+\alpha^{k-2}t_1+\cdots+t_{k-1},\quad \textit{where } t_k=O(\theta^k). \tag{21}$$

P r o o f. First we observe that if $(E-\beta I)\cdot s_k=r_k$, where $|\beta|<\theta$, then $s_k=O(\theta^k)$. Indeed, using inductively the relation $s_{k+1}=\beta s_k+r_k$, one sees that a solution to $(E-\beta I)\cdot s_k=r_k$ satisfies

$$s_k=\beta^k s_0+\beta^{k-1}r_0+\beta^{k-2}r_1+\cdots+r_{k-1}, \tag{22}$$

for all $k\ge1$. If $|r_k|\le B\theta^k$, for $k\ge0$, then

$$|s_k|\le|s_0||\beta|^k+B(|\beta|^{k-1}+\theta|\beta|^{k-2}+\cdots+\theta^{k-1}).$$

But

$$\sum_{i=0}^{k-1}\theta^{k-1-i}|\beta|^i\le\theta^{k-1}\sum_{i\ge0}(|\beta|/\theta)^i=O(\theta^k).$$

For $j=1,\dots,m-1$, put $s_k^{(j)}:=$ the solution to $\prod_{i=1}^{j}(E-\alpha_i I)\cdot s_k^{(j)}=r_k$. Then

$$\begin{aligned}
(E-\alpha_1 I)\cdot s_k^{(1)} &= r_k,\\
(E-\alpha_2 I)\cdot s_k^{(2)} &= s_k^{(1)},\\
&\ \vdots\\
(E-\alpha_{m-1} I)\cdot s_k^{(m-1)} &= s_k^{(m-2)},\\
(E-\alpha I)\cdot s_k &= s_k^{(m-1)}.
\end{aligned}$$

But

$$s_k^{(1)}=O(\theta^k)\implies s_k^{(2)}=O(\theta^k)\implies\cdots\implies s_k^{(m-1)}=O(\theta^k).$$

Now, by (22), $(E-\alpha I)\cdot s_k=s_k^{(m-1)}$ implies (21) with $t_k=s_k^{(m-1)}$. $\square$

Lemma 13. *Suppose $A = (a_k)_{k\geq 0}$ is a real-valued linear recurrent sequence with $a_k \sim a\alpha^k$, as k tends to infinity, where $\alpha > 1$ and $a \neq 0$ are real numbers. Let $(s_k)_{k\geq 0}$ and $(r_k)_{k\geq 0}$ be two sequences of real numbers satisfying*

$$g(E) \cdot s_k = \lambda a_k + r_k, \tag{23}$$

where g is a polynomial over the reals with simple dominant zero α, $\lambda \neq 0$ and $r_k = O(\theta^k)$ for some θ less than α but larger than the moduli of the other zeros of g. Then

$$s_k = bk\alpha^k + c\alpha^k + O(\theta^k) + (\alpha^{k-1}t_0 + \alpha^{k-2}t_1 + \cdots + t_{k-1}), \tag{24}$$

for some nonzero b, some constant c and some sequence (t_k) with $t_k = O(\theta^k)$.

Proof. Since $a_k \sim a\alpha^k$ we infer that the characteristic polynomial f of A is of the form $(x-\alpha)\ell(x)$, where the zeros of ℓ have moduli less than α. Any solution (s_k) to (23) is the sum of a solution (x_k) to $g(E) \cdot x_k = \lambda a_k$ and a solution (y_k) to $g(E) \cdot y_k = r_k$. A solution to $g(E) \cdot x_k = \lambda a_k$ must be a linear recurrence of the form $bk\alpha^k + c_1\alpha^k + O(\theta^k)$ for a nonzero b, because (x_k) is annihilated by fg, but not by ℓg. The solution to $g(E) \cdot y_k = r_k$ is of the type $y_k = y_0\alpha^k + \alpha^{k-1}t_0 + \alpha^{k-2}t_1 + \cdots + t_{k-1}$ by Lemma 12. The result follows. □

The next theorem may be seen as a further generalization of our main theorems, Theorems 3 and 4, on special recurrences.

Theorem 6. *Let $A = (a_k)_{k\geq 0}$ be a nondecreasing integral linear recurrent sequence with $1 = a_0 < a_1 \leq \cdots \leq a_{m-1}$ and $a_k \sim a\alpha^k$, as k tends to infinity, for some real numbers $\alpha > 1$ and $a > 0$. Suppose*

$$g(E) \cdot b_k = \lambda a_k + r_k, \tag{25}$$

where g is a polynomial over the integers with simple dominant zero α, $b_k = S_A(a_k)$, $|r_k|$ is bounded above by $B\theta^k$, for some real number $\theta < \alpha$ but larger than the moduli of the other zeros of g and of the characteristic polynomial of A, and $\lambda > 0$. Then

$$S_A(n) = c_A n \log n + O(n), \text{ where } c_A = \lambda(\alpha g'(\alpha)\log\alpha)^{-1}. \tag{26}$$

Proof. By Lemma 13, $b_k = bk\alpha^k + c\alpha^k + O(\theta^k) + (\alpha^{k-1}t_0 + \alpha^{k-2}t_1 + \cdots + t_{k-1})$ with $t_k = O(\theta^k)$ and b positive in this context. Clearly, $g(E) \cdot O(\theta^k) = O(\theta^k)$ and $g(E) \cdot \alpha^k = 0$. Now if $z_k = \alpha^{k-1}t_0 + \alpha^{k-2}t_1 + \cdots + t_{k-1}$, then

$$\begin{aligned}(E - \alpha I) \cdot z_k &= z_{k+1} - \alpha z_k \\ &= (\alpha^k t_0 + \alpha^{k-1}t_1 + \cdots + t_k) - (\alpha^k t_0 + \cdots + \alpha t_{k-1}) = t_k.\end{aligned}$$

Thus, $g(E) \cdot z_k = O(\theta^k)$. Hence, because $r_k = o(\alpha^k)$ and $a_k \sim a\alpha^k$, we must have, by (25), $g(E) \cdot bk\alpha^k = \lambda a \alpha^k$. Indeed, as seen in the proof of Theorem 4,

$$g(E) \cdot (k\alpha^k) = \alpha g'(\alpha)\alpha^k.$$

Thus, $b\alpha g'(\alpha) = \lambda a$. We saw in the proof of Lemma 12 that the expression in (21) is $O(\theta^k)$; the same argument gives that $z_k = O(\alpha^k)$. Hence, $b_k = (b/a)ka_k + O(a_k)$ and we find, by Theorem 2, that our claim holds with $c_A = b/(a \log \alpha)$. $\square$

We turn to families of the non-special second-order recurrences, which include $a_k = 2^{k+1} - 1$, yet fall under our method. Again we define b_k as $S(a_k)$.

Corollary 14. *Let $\alpha \geq 2$ be an integer. Suppose the sequence $A = (a_k)_{k\geq 0}$, with a_1 integral and $a_1 > a_0 = 1$, is annihilated by $x^2 - (\alpha+1)x + \alpha$. Then*

$$S_A(n) = \frac{\alpha - 1}{2\log\alpha} \cdot n \log n + O(n).$$

Proof. A simple induction would show that $(a_k)_{k\geq 0}$ is increasing. Then we note that $a_{k+1} = \alpha a_k + (a_1 - \alpha)$ for all $k \geq 0$. If $a_1 = \alpha$, then we fall back on the well-known geometric case. So assume first $a_1 > \alpha$. Suppose $a_k \leq n < a_{k+1}$. If $\alpha a_k \leq n < a_{k+1}$, then $s(n) = \alpha + s(n - \alpha a_k)$. Thus, we see that

$$b_{k+1} = S(\alpha a_k) + K,$$

where K is the constant $\alpha(a_1 - \alpha) + S(a_1 - \alpha)$. Using Lemma 2, this leads to

$$b_{k+1} - \alpha b_k = 0.5\alpha(\alpha - 1)a_k + K, \tag{27}$$

for all $k \geq 1$. The conclusion comes from Theorem 6 with a constant c_A equal to $0.5\alpha(\alpha-1)(\alpha\log\alpha)^{-1}$.

If $1 < a_1 < \alpha$, then instead we find that $b_{k+1} = S(\alpha a_k) - \sum_{i<\alpha-a_1} s(i + a_{k+1})$. Because Lemma 2 assumes $\alpha a_k \leq a_{k+1}$, it would be wrong to conclude that (27) held with K negative and equal to $-(\alpha - a_1) - S(\alpha - a_1)$. However, the error we make in writing $S(\alpha a_k) = 0.5\alpha(\alpha-1)a_k + \alpha b_k$ only comes from the last $\alpha - a_1$ integers before αa_k. For an integer n, $a_k \leq n < a_{k+1}$, we see that $1 \leq s(n) < (k+1)d^+$, where d^+ is the maximal digit that may occur; see Lemma 1. Hence, the difference between b_{k+1} and $0.5\alpha(\alpha-1)a_k + \alpha b_k$ is at most $K + (\alpha - a_1)(k+2)d^+$. Hence, $b_{k+1} - \alpha b_k = 0.5\alpha(\alpha-1)a_k + r_k$ with $r_k = O(k)$. We conclude again with Theorem 6. $\square$

Corollary 15. *Let $\alpha \geq 2$ be an integer. Suppose the sequence $A = (a_k)_{k\geq 0}$, with $a_1 > a_0 = 1$ integral, is annihilated by $x^2 - (\alpha-1)x - \alpha$. Then*

$$S_A(n) = \frac{\alpha - 1}{2\log\alpha} \cdot n \log n + O(n).$$

Proof. The sequence A is seen to be increasing. Moreover, $a_{k+1} = \alpha a_k + (-1)^k(a_1 - \alpha)$. Suppose k is even and $a_1 > \alpha$, or k is odd and $a_1 < \alpha$. Thus, $a_{k+1} = \alpha a_k + |a_1 - \alpha|$. Hence, by Lemma 2, $b_{k+1} = S(\alpha a_k) + K = 0.5\alpha(\alpha-1)a_k + \alpha b_k + K$, where K is the constant $\alpha|a_1 - \alpha| + S(|a_1 - \alpha|)$. In the other cases, i.e., k odd and $a_1 > \alpha$, or k even and $a_1 < \alpha$, we find that $a_{k+1} = \alpha a_k - |a_1 - \alpha|$. Thus, $b_{k+1} = S(\alpha a_k) - K_1$ with $K_1 = |a_1 - \alpha| + S(|a_1 - \alpha|)$. By the argument used in the proof of Corollary 14, we see that $|S(\alpha a_k) - 0.5\alpha(\alpha-1)a_k - \alpha b_k| \leq K_1 + (k+2)d^+|a_1 - \alpha|$. Therefore, for all k and $a_1 - \alpha$, we see that $b_{k+1} = 0.5\alpha(\alpha-1)a_k + \alpha b_k + r_k$, where $r_k = O(k)$. Theorem 6 offers the conclusion. $\square$

We gather in a theorem the previous two corollaries.

Theorem 7. *Let $\alpha \geq 2$ be an integer. Suppose $A = (a_k)_{k\geq 0}$, with $1 = a_0 < a_1$ integral, is annihilated by $x^2 - (\epsilon + \alpha)x + \epsilon\alpha$, where $\epsilon = \pm 1$. Then*

$$S_A(n) = \frac{\alpha - 1}{2\log\alpha} \cdot n\log n + O(n).$$

Remark 3. None of the second-order recurrences of Theorem 7 are special. Interestingly, formula (16) still yields the right constant c_A for all the recurrences of Corollary 15. However, this is not true for recurrences of Corollary 14, e.g., for $a_k = 2^{k+1} - 1$, when putting $P = 3$, $Q = -2$, $\alpha = 2$ and $D = 1$ in (16) leads to a wrong value of c_A. In any case all second-order recurrences in Theorem 7 are very close to the geometric sequence (α^k) with which they share the same constant c_A.

We give a few more typical applications of Theorem 6 with a second-order and a third-order recurrences with dominant zero 3 and other zeros of moduli larger than 1, and a fourth-order recurrence with a non-integral dominant zero, namely the Golden ratio γ.

Theorem 8. *For each of the three recurrences $A = (a_k)_{k\geq 0}$, where*

i) $a_k = 3^{k+1} - 2^{k+1}$; ii) $a_k = 3^k + k2^k$; *and* iii) $a_k = F_{k+4} - k - 2$,

we find that

$$S_A(n) = c_A n\log n + O(n),$$

where

$$c_A = \begin{cases} (\log 3)^{-1}, & \textit{in cases i) and ii);} \\ (\gamma\sqrt{5}\log\gamma)^{-1} = c_F, & \textit{in the third case.} \end{cases}$$

Proof. The three sequences are increasing and have their a_0 term equal to 1. In case i), $a_{k+1} = 3(3^{k+1} - 2^{k+1}) + 3\cdot 2^{k+1} - 2^{k+2} = 3a_k + 2^{k+1}$. Hence, $b_{k+1} = S(3a_k) + 3\cdot 2^{k+1} + S(2^{k+1})$. Since $2^{k+1} < a_{k+1}$, we see that $s(n) \leq (k+1)d^+$, where d^+ is the maximal digit that can occur in the numeration based on A, for all $n < 2^{k+1}$. Therefore, using Lemma 2, we find that $b_{k+1} - 3b_k = 3a_k + r_k$, where $r_k = 3 \cdot 2^{k+1} + S(2^{k+1}) = O(\theta^k)$ with, say, $\theta = 5/2$. We conclude with the help of Theorem 6. For the second sequence, we find that $a_{k+1} = 3a_k - (k-2)2^k$. Thus, for $k \geq 2$, we have

$$b_{k+1} = S(3a_k) - \sum_{a_{k+1} \leq n < 3a_k} s(n) = S(3a_k) - \Big((k-2)2^k + S\big((k-2)2^k\big)\Big).$$

As at the end of the proof of Corollary 14, evaluating $S(3a_k)$ with Lemma 2 gives $3a_k + 3b_k + O(k^2 2^k)$, where the error term $O(k^2 2^k)$ stems from the $(k-2)2^k$ largest integers between a_{k+1} and $3a_k$ whose sum of digits is at most $(k+2)d^+$ each. Taking into account the two terms $-(k-2)2^k$ and $-S\big((k-2)2^k\big)$, we obtain that $b_{k+1} - 3b_k = 3a_k + r_k$, where, as in case i), $r_k = O((5/2)^k)$.

For our third sequence, since it is increasing we find that $b_{k+2} = b_{k+1} + (a_{k+2} - a_{k+1}) + S(a_{k+2} - a_{k+1})$. Since $a_k \leq a_{k+2} - a_{k+1} = F_{k+4} - 1 \leq a_{k+1}$ and there are k integers in the interval $[a_k, F_{k+4} - 1[$, we see that $S(a_{k+2} - a_{k+1}) = b_k + O(k(k+1)d^+)$. Hence, $b_{k+2} - b_{k+1} - b_k = a_k + k + 1 + O(k^2) = a_k + O(\theta^k)$ with $\theta = 3/2 < \gamma$. Again the conclusion comes from Theorem 6. □

We investigate recurrences annihilated by $f(x) = x^3 - x - 1$, a polynomial which has a substantially smaller dominant zero than special cubics. Though f is not a special polynomial our method works! As usual b_k denotes $S(a_k)$. The constant c_A turns out to be much smaller than all constants associated with the recurrences of Corollary 10.

Theorem 9. *Suppose $A = (a_k)_{k\geq 0}$ is an integral recurrence annihilated by $f(x) = x^3 - x - 1$ with $a_0 = 1 < a_1 \leq a_2 \leq a_1 + 1 = a_3$. Then*

$$S_A(n) = c_A n \log n + O(n), \text{ where } c_A = \big(\alpha(\alpha^2 - \alpha + 1)f'(\alpha)\log\alpha\big)^{-1} \simeq 0.440,$$

α is the dominant zero of f and f' is its derivative.

Proof. Suppose $k \geq 4$. Note that $a_{k+1} - a_k = (a_{k-1} + a_{k-2}) - (a_{k-2} + a_{k-3}) = a_{k-1} - a_{k-3} = a_{k-4} < a_k$. Thus, if $a_k \leq n < a_{k+1}$, then $s(n) = 1 + s(n - a_k)$. Hence, $b_{k+1} - b_k = a_{k+1} - a_k + S(a_{k+1} - a_k)$. That is $b_{k+1} - b_k - b_{k-4} = a_{k-4}$. Or, for all $k \geq 0$,

$$b_{k+5} - b_{k+4} - b_k = a_k. \tag{28}$$

That is, $g(E) \cdot b_k = a_k$ with $g(x) = x^5 - x^4 - 1 = (x^2 - x + 1)(x^3 - x - 1)$. The zeros of $x^2 - x + 1$ are roots of unity and $g'(\alpha) = (\alpha^2 - \alpha + 1)f'(\alpha)$. Thus, Theorem 6 is applicable and yields the claim. □

Remark 4. Note that the sequence with $a_0 = 1$, $a_1 = a_2 = 2$ is a shift of the fundamental sequence associated with $x^3 - x - 1$ (i.e., a shift of the Padovan sequence[4]) which satisfies the hypotheses of Theorem 9. Formula (18) is not valid for the Padovan or other sequences satisfying Theorem 9 as putting $P = 0$ and $Q = R = 1$ into (18) yields $\bigl(\alpha f'(\alpha) \log \alpha\bigr)^{-1}$, a larger constant ($\simeq 0.629$).

Acknowledgements. A new draft of the paper was written after the referee mentioned some results presented were already known. We thank that anonymous referee for pointing out relevant references and for his encouraging report.

REFERENCES

[1] ALLOUCHE, J.-P.—JOHNSON, T.: *Narayana's cows and delayed morphisms*, Cahiers du GREYC, JIM 96 **4** (1996), 2–7;
`http://recherche.ircam.fr/equipes/repmus/jim96/actes/Allouche.ps`

[2] ALPERT, H. : *Differences of multiple Fibonacci numbers*, Integers **9** (2009), A57, 745–749.

[3] BALLOT, C.: *On Zeckendorf and base b digit sums*, Fibonacci Quart. **51** (2013), 319–325.

[4] ______ *A family of recurrences*, submitted preprint.

[5] COQUET, J.—VAN DEN BOSCH, P.: *A summation formula involving Fibonacci digits*, J. Number Theory **22** (1986), 139–146.

[6] COOPER, C.—KENNEDY, R. E.: *A generalization of a result by Trollope on digital sums*, J. Inst. Math. Comput. Sci. Math. Ser. **12** (1999), 17–22.

[7] DELANGE, H.: *Sur la fonction sommatoire de la fonction 'Somme des Chiffres'*, Enseignement Math. **21** (1975), 31–47.

[8] DEMONTIGNY, P.—DO, T.—KULKARNI, A.—MILLER, S. J.—VARMA, U.: *A generalization of Fibonacci far-difference representations and Gaussian behavior*, Fibonacci Quart. **52** (2014), 247–273.

[9] DORWARD, R. —FORD, P. L.—FOURAKIS, E.—HARRIS, P. E.—MILLER, S. J.—PALSSON, E.— PAUGH, H.: *A generalization of Zeckendorf's theorem via circumscribed m-gons*, submitted preprint.

[10] DRMOTA, M.—GAJDOSIK, J.: *The distribution of the sum-of-digits function*, J. Théor. Nombres Bordeaux **10** (1998), 17–32.

[11] FENG, D.-J.—LIARDET, P.—THOMAS, A.: *Partition functions in numeration systems with bounded multiplicity*, Unif. Distrib. Theory **9** (2014), 43–77.

[12] FLAJOLET, P.—GRABNER, P.—KIRSCHENHOFER, P.— PRODINGER, H.—TICHY, R. F.: *Mellin transforms and asymptotics: digital sums*, Theoret. Comput. Sci. **123** (1994), 291–314.

[13] GRABNER, P. J.—TICHY, R. F.: *Contributions to digit expansions with respect to linear recurrences*, J. Number Theory **36** (1990), 160–169.

[4] Code number in the OEIS is A000931.

[14] KOLOĞLU, M.—KOPP, G. S.—MILLER, S. J.—WANG, Y.: *On the number of summands in Zeckendorf decompositions*, Fibonacci Quart. **49** (2011), 116–130.

[15] MILLER, S. J.—WANG, Y.: *From Fibonacci numbers to central limit theorems*, J. Combin. Theory Ser. A **119** (2012), 1398–1413.

[16] PETHÖ, A.—TICHY, R. F.: *On digit expansions with respect to linear recurrences*, J. Number Theory **33** (1989), 243–256.

[17] HAMLIN, N.—WEBB, W.: *Representing positive integers as a sum of linear recurrence sequences*, Fibonacci Quart. **50** (2012), 99–105.

[18] PIHKO, J.: *On the average number of summands in the Zeckendorf representation*, Congr. Numer. **200** (2010), 317–324.

[19] SLOANE, N. J. A.: *The Online Encyclopedia of Integer Sequences*, `http://oeis.org`

[20] TROLLOPE, J. R.: *An explicit expression for binary digital sums*, Math. Mag. **41** (1968), 21–25.

Received August 12, 2015
Accepted November 24, 2015

Christian Ballot
Université de Caen
Département de Mathématiques et Mécanique
Campus 2
Blvd Maréchal Juin
14032 Caen
FRANCE
E-mail: christian.ballot@unicaen.fr