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Existence of Solutions for Some Nonlinear Elliptic Anisotropic Unilateral Problems with Lower Order Terms

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Abstract. In this paper, we prove the existence of entropy solutions for anisotropic elliptic unilateral problem associated to the equations of the form

$$-\sum_{i=1}^{N}\partial_i a_i(x,u,\nabla u)-\sum_{i=1}^{N}\partial_i\phi_i(u)=f,$$

where the right hand side f belongs to $L^1(\Omega)$. The operator $-\sum_{i=1}^{N}\partial_i a_i(x,u,\nabla u)$ is a Leray-Lions anisotropic operator and $\phi_i\in C^0(\mathbb{R},\mathbb{R})$.

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1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N\geq 2)$ with smooth boundary and let $1<p_1,...,p_N<+\infty$ be a N real numbers and $\overrightarrow{p}=(p_1,...,p_N)$. We consider the obstacle problem associated with the following elliptic equations

$$\begin{cases} Au-\mathrm{div}\phi(u)=f \text{ in } \Omega \\ u=0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

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where A is a Leray-Lions operator from anisotropic space $W_0^{1,\overrightarrow{p}}(\Omega)$ into its dual $W^{-1,\overrightarrow{p'}}(\Omega)$ defined by $Au = -\mathrm{div} a(x,u,\nabla u)$ and $\phi = (\phi_1,...,\phi_N)$ belongs to $C^0(\mathbb{R},\mathbb{R})^N$. As regards the second member, we assume that the datum f belongs to $L^1(\Omega)$.

In recent years an increasing interest has turned towards anisotropic elliptic and parabolic equations. A special interest in the study of such equations is motivated by their applications to the mathematical modeling of physical and mechanical processes in anisotropic continuous medium. We refer to the recent works [4, 5, 21] where it is possible to find some references.

In [1, 7, 13] the authors proved the existence of the solutions for some unilateral nonlinear elliptic problem in the classical Sobolev space $W_0^{1,p}(\Omega)$ and in the Orlicz spaces with $f \in L^1(\Omega) + W_0^{1,\overrightarrow{p'}}(\Omega)$. L. Boccardo in [12] proved the existence of solutions of some nonlinear Dirichlet problem in L^1 involving lower order terms in divergence form.

Boccardo et al. in [11] studied the existence of weak solutions for nonlinear elliptic problem (1.1) with $Au = -\sum_{i=1}^N \frac{\partial}{\partial x_i}(|\frac{\partial u}{\partial x_i}|^{p_i-2}\frac{\partial u}{\partial x_i})$, $\phi_i(u) = 0$ for $i = 1,...,N$ and the right-hand side is a bounded Radon measure on Ω. In the case where $Au = -\sum_{i=1}^N \frac{\partial}{\partial x_i} a_i(x, \frac{\partial u}{\partial x_i})$, $\quad \phi_i(u) = 0$ for $i = 1,...,N$ and the right hand side $f = (f_1,...,f_m)^\top$ is vector-valued Radon measure on Ω of finite mass, existence solutions of (1.1) is proved by Bendahmane et al. in [5]. We cite some papers that have dealt with the equation (1.1) or similar problems, see [4, 5, 14, 15, 16, 21]. Note that in the isotropic case, there are large works in the direction of problem (1.2) can be found in [3, 6, 7, 8, 23].

The objective of our article is to study the anisotropic unilateral nonlinear elliptic problem associated with the nonlinear problem (1.1). More precisely, we prove the existence of entropy solutions for the following unilateral anisotropic problem.

$$\begin{cases} u \geq \psi \ a.e. \text{ in } \Omega, \\ T_k(u) \in W_0^{1,\overrightarrow{p}}(\Omega) \quad \forall k > 0, \\ \sum_{i=1}^N \int_\Omega a_i(x,u,\nabla u)\partial_i T_k(u-v)dx + \sum_{i=1}^N \int_\Omega \phi_i(u)\partial_i T_k(u-v)dx \leq \int_\Omega f T_k(u-v)dx, \\ \forall v \in K_\psi \cap L^\infty(\Omega), \end{cases} \tag{1.2}$$

where $K_\psi = \{u \in W_0^{1,\overrightarrow{p}}(\Omega),\ u \geq \psi \ a.e. \text{ in } \Omega\}$ with ψ is a measurable function on Ω such that $\psi^+ \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega)$ and T_k is the usual truncation function. Note that the existence result is proved by assuming only ϕ is continuous function. If we take $\psi = -\infty$, we obtain the existence results of problem (1.2) in the case of equation. The integrals in (1.2) are well defined: Indeed under the condition (3.2), the function $a_i(x,u,\nabla u)$ is belongs to $L^{p_i'}(\Omega)$ and since $\partial_i T_k(u-v)$ is belongs to $L^{p_i}(\Omega)$, the first integral in the left hand in (1.2) is well defined. For the second integral in the left hand in (1.2), since $\phi_i(u)\partial_i T_k(u-v) = 0$ on $\{|u| > \|v\|_\infty + k\}$ and $\phi_i \in C^0(\mathbb{R},\mathbb{R})$, $\phi_i(u)$ is bounded in $\{|u| \leq \|v\|_\infty + k\}$, then the second integral is well defined. Moreover since $f \in L^1(\Omega)$ and $T_k(u-v) \in L^\infty(\Omega)$, the integral in the right hand is well defined.

Since the function $\phi_i(u)$ does not belong to $L^1_{loc}(\Omega)$ in general, the problem (1.1) does not admit weak solutions. To overcome this difficulty, we use the entropy solutions in this work which introduced for the first time by Bnilan et al. in [8]

This paper is organized as follows: Section 2 is devoted to introduce some preliminary results including a brief discussion on the anisotropic Sobolev spaces. Section 3 is devoted to give some important Lemmas. Section 4 contains the main result. Section 5 will be devoted to show the principal proposition concerning the existence of solutions for approximate problems.

2. Preliminaries

Let Ω be a bounded open subset of $\mathbb{R}^N$ ($N \geq 2$) with smooth boundary and let $1 < p_1,...,p_N < \infty$ be N real numbers, $p^+ = \max\{p_1,...,p_N\},\ p^- = \min\{p_1,...,p_N\}$ and $\overrightarrow{p} = (p_1,...,p_N)$. We denote $\partial_i = \dfrac{\partial}{\partial x_i}$.

The anisotropic Sobolev space (see [22])

$$W^{1,\overrightarrow{p}}(\Omega) = \left\{u \in W^{1,1}(\Omega), \partial_i u \in L^{p_i}(\Omega), i = 1, 2, ..., N\right\}$$

is a Banach space with respect to norm

$$\|u\|_{W^{1,\overrightarrow{p}}(\Omega)} = \|u\|_{L^1(\Omega)} + \sum_{i=1}^{N} \|\partial_i u\|_{L^{p_i}(\Omega)}. \tag{2.1}$$

The space $W_0^{1,\overrightarrow{p}}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to this norm. Let us recall the Sobolev type inequalities, for $u \in W_0^{1,\overrightarrow{p}}(\Omega)$, there exists a constant and C (see [22]) such that

$$\|u\|_{L^q(\Omega)} \leq C_s \prod_{i=1}^{N} \left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(\Omega)}^{\frac{1}{N}}, \tag{2.2}$$

where $q = \overline{p}^* = \frac{N\overline{p}}{N-\overline{p}}$ if $\overline{p} < N$ or $q \in [1, +\infty[$ if $\overline{p} \geq N$, which implies by (2.2)

$$\|u\|_{L^q(\Omega)} \leq \frac{C_s}{N} \sum_{i=1}^{N} \left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}(\Omega)} \tag{2.3}$$

When $\overline{p} < N$, by (2.3), we have the continuous embedding of $W_0^{1,\overrightarrow{p}}(\Omega)$ into $L^q(\Omega)$ for every $q \in [1, \overline{p}^*]$. The space $W_0^{1,\overrightarrow{p}}(\Omega)$ is separable and reflexive Banach space which satisfies the continuous imbedding $W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow W_0^{1,p_-}(\Omega)$ and its dual $(W_0^{1,\overrightarrow{p}}(\Omega))'$ is denoted by $W^{-1,\overrightarrow{p'}}(\Omega)$.

Remark 2.1. *As a consequence of the Sobolev imbedding and the continuous imbedding $W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow W_0^{1,p^-}(\Omega)$, the imbedding $W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^{p^-}(\Omega)$ is compact.*

Moreover, we consider the space

$$\mathcal{T}_0^{1,\overrightarrow{p}}(\Omega) = \{u \text{ measurable in } \Omega, T_k(u) \in W_0^{1,\overrightarrow{p}}(\Omega), \forall k > 0\},$$

where

$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k \\ k\dfrac{s}{|s|} & \text{if } |s| > k. \end{cases}$$

3. Assumptions and Lemmas :

In this section, we give the assumptions of our problem and some technical lemmas. Let Ω be a bounded open subset of $\mathbb{R}^N$ ($N \geq 2$) with Lipschitz continuous boundary $\partial\Omega$.
The functions a_i: $\Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are Carathéodory functions satisfying the following conditions, for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N, \xi' \in \mathbb{R}^N$ and a. e. in Ω,

$$\sum_{i=1}^{N} a_i(x, s, \xi)\xi_i \geq \alpha \sum_{i=1}^{N} |\xi_i|^{p_i}, \tag{3.1}$$

$$|a_i(x, s, \xi)| \leq \beta[j_i(x) + |s|^{\frac{1}{p_i'}} + |\xi_i|^{p_i-1}], \tag{3.2}$$

$$(a_i(x, s, \xi) - a_i(x, s, \xi'))(\xi_i - \xi_i') > 0 \quad \text{for } \xi_i \neq \xi_i', \tag{3.3}$$

where α, β are some positive constants and j_i is a positive function in $L^{p_i'}(\Omega)$.
Moreover, we suppose that

$$\phi_i \in C^0(\mathbb{R}, \mathbb{R}) \text{ for } i = 1, ..., N. \tag{3.4}$$

and

$$f \in L^1(\Omega). \tag{3.5}$$

We consider the convex set

$$K_\psi = \{u \in W_0^{1,\overrightarrow{p}}(\Omega),\ u \geq \psi \ a.e. \text{ in } \Omega\}$$

where ψ is a measurable function with values in $\overline{\mathbb{R}}$ such that

$$\psi^+ \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega). \tag{3.6}$$

Lemma 3.1. *([19]) Let $g \in L^r(\Omega)$ and let $g_n \in L^r(\Omega)$, $\|g_n\|_{L^r(\Omega)} < c$, $1 < r < +\infty$. If $g_n(x) \to g(x)$ a. e. in Ω, then $g_n \rightharpoonup g$ weakly in $L^r(\Omega)$.*

The following lemma generalizes lemma 5 in [13] to the anisotropic case. We utilize the method used in [2] and [13].

Lemma 3.2. *: Assume that (3.1)-(3.3) hold and let $(u_n)_n$ be a sequence in $W_0^{1,\overrightarrow{p}}(\Omega)$ such that $u_n \rightharpoonup u$ in $W_0^{1,\overrightarrow{p}}(\Omega)$ and*

$$\lim_{n\to+\infty} \int_\Omega \Big(a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u)\Big)\nabla(u_n - u)dx = 0. \tag{3.7}$$

Then $u_n \to u$ strongly in $W_0^{1,\overrightarrow{p}}(\Omega)$ for a subsequence.

Proof Let $D_n = \Big[a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u)\Big]\nabla(u_n - u)$, by (3.3), D_n is a positive function and by (3.7) , we have $D_n \to 0$ in $L^1(\Omega)$ as $n \to +\infty$. Since $u_n \rightharpoonup u$ in $W_0^{1,\overrightarrow{p}}(\Omega)$, using Remark 2.1, we have $u_n \to u$ strongly in $L^{p^-}(\Omega)$. Then $u_n \to u$ a. e. in Ω and $D_n \to 0$ a. e. in Ω for a subsequence. Thus there exists a subset B of Ω, of zero measure, such that for $x \in \Omega \setminus B$, $u(x) < +\infty$, $|\nabla u(x)| < +\infty$, $|j_i(x)| < +\infty$, $u_n(x) \to u(x)$ and $D_n(x) \to 0$.
We have

$$\begin{aligned}
D_n(x) &= \sum_{i=1}^N \Big[a_i(x, u_n, \nabla u_n) - a_i(x, u_n, \nabla u)\Big][\partial_i u_n - \partial_i u] \\
&= \sum_{i=1}^N \Big[a_i(x, u_n, \nabla u_n)\partial_i u_n + a_i(x, u_n, \nabla u)\partial_i u - a_i(x, u_n, \nabla u_n)\partial_i u - a_i(x, u_n, \nabla u)\partial_i u_n\Big] \\
&\geq \alpha \sum_{i=1}^N |\partial_i u_n|^{p_i} + \alpha \sum_{i=1}^N |\partial_i u|^{p_i} - \beta \sum_{i=1}^N \Big[j_i(x) + |u_n|^{\frac{1}{p_i'}} + |\partial_i u_n|^{p_i-1}\Big]|\partial_i u| \\
&\qquad\qquad -\beta \sum_{i=1}^N \Big[j_i(x) + |u_n|^{\frac{1}{p_i'}} + |\partial_i u|^{p_i-1}\Big]|\partial_i u_n|. \\
&\geq \alpha \sum_{i=1}^N |\partial_i u_n|^{p_i} - c(x)\Big[1 + \sum_{i=1}^N |\partial_i u_n|^{p_i-1} + \sum_{i=1}^N |\partial_i u_n|\Big]. \\
&\geq \sum_{\{i=1,\dots,N: \partial_i u_n \neq 0\}} |\partial_i u_n|^{p_i}\Big[\alpha - \frac{c(x)}{N|\partial_i u_n|^{p_i}} - \frac{c(x)}{N|\partial_i u_n|} - \frac{c(x)}{N|\partial_i u_n|^{p_i-1}}\Big],
\end{aligned}$$

where $c(x)$ is a function which doesn't depend on n.
Since $D_n(x) \to 0$ a. e. in Ω, the last inequality implies that $\Big(\partial_i u_n\Big)_n$ is bounded uniformly with respect to n.
Letting ξ_i^* be an accumulation point of $\Big(\partial_i u_n\Big)_n$ for $i = 1, ..., N$, we have $|\xi_i^*| < +\infty$ and by the continuity of $a_i(x, ., .)$, we obtain

$$\Big(a_i(x, u, \xi^*) - a_i(x, u, \nabla u)\Big)\Big(\xi_i^* - \partial_i u\Big) = 0.$$

Using (3.3), we obtain $\xi_i^* = \partial_i u$ for $i = 1, ..., N$. The uniqueness of the accumulation point implies that $\nabla u_n \to \nabla u$ a. e. in Ω. Since the sequence $\Big(a_i(x, u_n, \nabla u_n)\Big)_n$ is bounded in $L^{p_i'}(\Omega)$ and $a_i(x, u_n, \nabla u_n) \to a_i(x, u, \nabla u)$ a. e. in Ω, by Lemma 3.1, we have $a_i(x, u_n, \nabla u_n)$ converges to $a_i(x, u, \nabla u)$ weakly in $L^{p_i'}(\Omega)$ and a. e. in Ω.
As in [13], we have

$$a_i(x, u_n(x), \nabla u_n(x))\partial_i u_n \rightharpoonup a_i(x, u, \nabla u)\partial_i u \text{ weakly in } L^1(\Omega).$$

For fixed $i = 1, ..., N$, we set $y_n^i = \frac{1}{\alpha} a_i(x, u_n, \nabla u_n)\partial_i u_n$ and $y^i = \frac{1}{\alpha} a_i(x, u, \nabla u)\partial_i u$, using Fatou's lemma, we get

$$\int_\Omega 2y^i dx \le \liminf_{n\to+\infty} \int_\Omega \Big(y_n^i + y^i - \frac{1}{2^{p_i-1}}|\partial_i u_n - \partial_i u|^{p_i}\Big)dx.$$

Then, we have $0 \le -\limsup_{n\to+\infty} \int_\Omega |\partial_i u_n - \partial_i u|^{p_i} dx$. We deduce $\int_\Omega |\partial_i u_n - \partial_i u|^{p_i} dx \to 0$ as $n \to +\infty$.

Consequently, we conclude that $u_n \to u$ in $W_0^{1,\overrightarrow{p}}(\Omega)$, the proof is complete.

Lemma 3.3. *If $u \in W_0^{1,\overrightarrow{p}}(\Omega)$, then $\sum_{i=1}^N \int_\Omega \partial_i u dx = 0$.*

Proof: Since $u \in W_0^{1,\overrightarrow{p}}(\Omega)$, there exists $u_k \in C_0^\infty(\Omega)$ such that $u_k \to u$ strongly in $W_0^{1,\overrightarrow{p}}(\Omega)$.
Moreover, since $u_k \in C_0^\infty(\Omega)$, by Green's Formula, we have

$$\sum_{i=1}^N \int_\Omega \partial_i u_k dx = \int_{\partial\Omega} u_k.\overrightarrow{n} ds = 0. \tag{3.8}$$

Since $\partial_i u_k \to \partial_i u$ strongly in $L^{p_i}(\Omega)$, we have $\partial_i u_k \to \partial_i u$ strongly in $L^1(\Omega)$.
We pass to limit in (3.8), we conclude that $\sum_{i=1}^N \int_\Omega \partial_i u dx = 0$.

4. Main result

Definition 4.1. *A function $u \in \mathcal{T}_0^{1,\overrightarrow{p}}(\Omega)$ such that $u \ge \psi$ a. e. in Ω is an entropy solution of the problem (1.1) if*

$$\sum_{i=1}^N \int_\Omega \Big[a_i(x, u, \nabla u)\partial_i T_k(u-\varphi)dx + \phi_i(u)\partial_i T_k(u-\varphi)dx\Big] \le \int_\Omega f T_k(u-\varphi)dx$$

for all $\varphi \in K_\psi(\Omega) \cap L^\infty(\Omega)$.

Theorem 4.1. *Assume that (3.1)-(3.6) hold. Then there exists at least an entropy solution of problem (1.1).*

Proof:
Step1. Approximate problems. We consider the following approximate problems

$$\begin{cases} u_n \in K_\psi. \\ \sum_{i=1}^N \int_\Omega a_i(x, u_n, \nabla u_n)\partial_i(u_n - v)dx + \sum_{i=1}^N \int_\Omega \phi_i^n(u_n)\partial_i(u_n - v)dx \le \int_\Omega f_n(u_n - v)dx, \\ \forall v \in K_\psi \text{ and } \forall k > 0, \end{cases} \tag{4.1}$$

where $f_n = T_n(f)$ and $\phi_i^n(s) = \phi_i(T_n(s))$.

Lemma 4.1. *We consider the operator $\Phi_n : K_\psi \to W^{-1,\overrightarrow{p'}}(\Omega)$ defined by*

$$< \Phi_n u, v >= \sum_{i=1}^N \int_\Omega \phi_i(T_n(u))\partial_i v dx \quad \textit{for all } u \in K_\psi \textit{ and } v \in W_0^{1,\overrightarrow{p}}(\Omega).$$

The operator $B_n = A + \Phi_n$ is pseudo-monotone and coercive in the following sense; there exists $v_0 \in K_\psi$ such that $\frac{<B_n v, v-v_0>}{\|v\|_{W_0^{1,\vec{p}}(\Omega)}} \to +\infty$ if $\|v\|_{W_0^{1,\vec{p}}(\Omega)} \to +\infty$ and $v \in K_\psi$.

For the proof of Lemma 4.1, (see Appendix).

Proposition 4.1. *Under the conditions (3.1)-(3.6), there exists at least one solution of the problem (4.1).*

Proof. Thanks to Lemma 4.1 and Theorem 8.2 chapiter 2 in [19], there exists at least one solution to the problem (4.1).

Step2. A priori estimate.

Proposition 4.2. *Assume that (3.1)- (3.6) hold and if u_n is a solution of the approximate problem (4.1). Then there exists a constant C such that*

$$\sum_{i=1}^{N} \int_\Omega |\partial_i T_k(u_n)|^{p_i} dx \le C(k+1) \quad \forall k > 0.$$

Proof. Let $v = u_n - \eta T_k(u_n^+ - \psi^+)$ where $\eta \ge 0$. Since $v \in W_0^{1,\vec{p}}(\Omega)$ and for all η small enough, we have $v \in K_\psi$. We take v as test function in problem (4.1), we have

$$\sum_{i=1}^{N} \int_\Omega a_i(x, u_n, \nabla u_n) \partial_i T_k(u_n^+ - \psi^+) dx + \sum_{i=1}^{N} \int_\Omega \phi_i^n(u_n) \partial_i T_k(u_n^+ - \psi^+) dx \le \int_\Omega f_n T_k(u_n^+ - \psi^+) dx.$$

Which implies that

$$\sum_{i=1}^{N} \int_\Omega a_i(x, u_n, \nabla u_n) \partial_i T_k(u_n^+ - \psi^+) dx \le \int_\Omega f_n T_k(u_n^+ - \psi^+) dx + \sum_{i=1}^{N} \int_\Omega |\phi_i^n(u_n)||\partial_i T_k(u_n^+ - \psi^+)| dx.$$

Since $\partial_i T_k(u_n^+ - \psi^+) = 0$ on the set $\{u_n^+ - \psi^+ > k\}$, we have

$$\sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ \le k\}} a_i(x, u_n, \nabla u_n) \partial_i (u_n^+ - \psi^+) dx \le \int_\Omega f_n T_k(u_n^+ - \psi^+) dx + \sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ \le k\}} |\phi_i^n(u_n)||\partial_i (u_n^+ - \psi^+)| dx,$$

thus, we can write

$$\begin{aligned}\sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ \le k\}} a_i(x, u_n^+, \nabla u_n^+) \partial_i u_n^+ dx \le& \int_\Omega f_n T_k(u_n^+ - \psi^+) dx + \sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ \le k\}} |\phi_i^n(u_n)||\partial_i u_n^+| dx \\ &+ \sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ \le k\}} |\phi_i^n(u_n)||\partial_i \psi^+| dx + \sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ \le k\}} a_i(x, u_n^+, \nabla u_n^+) \partial_i \psi^+ dx\end{aligned}$$

Thanks to Young's inequalities, we obtain

$$\begin{aligned}\sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ \le k\}} a_i(x, u_n^+, \nabla u_n^+) \partial_i u_n^+ dx \le& \int_\Omega f_n T_k(u_n^+ - \psi^+) dx \\ &+ C_1(\alpha) \sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ \le k\}} |\phi_i^n(T_{k+\|\psi\|_\infty}(u_n))|^{p_i'} dx + \frac{\alpha}{6} \sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ \le k\}} |\partial_i u_n^+|^{p_i} dx \\ &+ \sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ \le k\}} |\phi_i^n(T_{k+\|\psi\|_\infty}(u_n))||\partial_i \psi^+| dx \\ &+ \sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ \le k\}} \beta[j_i(x) + |u_n^+|^{\frac{1}{p_i'}} + |\partial_i u_n^+|^{p_i - 1}]|\partial_i \psi^+| dx\end{aligned}$$

Thanks to (3.2), we have

$$\sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ \le k\}} a_i(x, u_n, \nabla u_n) \partial_i u_n^+ dx \le \int_\Omega f_n T_k(u_n^+ - \psi^+) dx$$

$$+C_2(\alpha)\sum_{i=1}^{N}\int_{\{u_n^+-\psi^+\leq k\}}|\phi_i^n(T_{k+\|\psi\|_\infty}(u_n))|^{p_i'}dx+\frac{\alpha}{6}\sum_{i=1}^{N}\int_{\{u_n^+-\psi^+\leq k\}}|\partial_i u_n^+|^{p_i}dx$$
$$+\sum_{i=1}^{N}\int_{\{u_n^+-\psi^+\leq k\}}|\phi_i^n(T_{k+\|\psi\|_\infty}(u_n))||\partial_i\psi^+|dx+\beta\sum_{i=1}^{N}\int_{\{u_n^+-\psi^+\leq k\}}j_i(x)|\partial_i\psi^+|dx$$
$$+\beta\sum_{i=1}^{N}\int_{\{u_n^+-\psi^+\leq k\}}|u_n^+|^{\frac{1}{p_i'}}|\partial_i\psi^+|dx+\beta\sum_{i=1}^{N}\int_{\{u_n^+-\psi^+\leq k\}}|\partial_i u_n^+|^{p_i-1}|\partial_i\psi^+|dx$$

Using young's inequality, we get

$$\sum_{i=1}^{N}\int_{\{u_n^+-\psi^+\leq k\}}a_i(x,u_n,\nabla u_n)\partial_i u_n^+dx\leq\int_\Omega f_nT_k(u_n^+-\psi^+)dx$$
$$+C_3(\alpha)\sum_{i=1}^{N}\int_{\{u_n^+-\psi^+\leq k\}}|\phi_i^n(T_{k+\|\psi\|_\infty}(u_n))|^{p_i'}dx+\frac{\alpha}{6}\sum_{i=1}^{N}\int_{\{u_n^+-\psi^+\leq k\}}|\partial_i u_n^+|^{p_i}dx$$
$$+\sum_{i=1}^{N}\int_{\{u_n^+-\psi^+\leq k\}}|\phi_i^n(T_{k+\|\psi\|_\infty}(u_n))||\partial_i\psi^+|dx+\beta\sum_{i=1}^{N}\int_{\{u_n^+-\psi^+\leq k\}}|j_i(x)||\partial_i\psi^+|dx$$
$$+\beta C_4\sum_{i=1}^{N}\int_{\{u_n^+-\psi^+\leq k\}}|u_n^+|dx+\beta\frac{\alpha}{6\beta}\sum_{i=1}^{N}\int_{\{u_n^+-\psi^+\leq k\}}|\partial_i\psi^+|dx$$
$$+\beta\frac{\alpha}{6\beta}\sum_{i=1}^{N}\int_{\{u_n^+-\psi^+\leq k\}}|\partial_i u_n^+|^{p_i}dx+C_5(\alpha)\sum_{i=1}^{N}\int_{\{u_n^+-\psi^+\leq k\}}|\partial_i\psi^+|^{p_i}dx$$

Using (3.1), (3.4), (3.5) and (3.6), we get

$$\sum_{i=1}^{N}\int_{\{u_n^+-\psi^+\leq k\}}|\partial_i u_n^+|^{p_i}dx\leq Ck+C' \tag{4.2}$$

Since $\{x\in\Omega,\ u^+\leq k\}\subset\{x\in\Omega,\ u^+-\psi^+\leq k+\|\psi^+\|_\infty\}$, then

$$\sum_{i=1}^{N}\int_\Omega|\partial_i T_k(u_n^+)|^{p_i}dx=\sum_{i=1}^{N}\int_{\{u^+\leq k\}}|\partial_i u_n^+|^{p_i}dx\leq\sum_{i=1}^{N}\int_{\{u^+-\psi^+\leq k+\|\psi^+\|_\infty\}}|\partial_i u_n^+|^{p_i}dx.$$

Thus, by (4.2), we obtain

$$\sum_{i=1}^{N}\int_\Omega|\partial_i T_k(u_n^+)|^{p_i}dx\leq(k+\|\psi^+\|_\infty)C+C'\quad\forall k>0. \tag{4.3}$$

Similarly, taking $v=u_n+T_k(u_n^-)$ as test function in approximate problem (4.1), we obtain

$$\sum_{i=1}^{N}\int_\Omega|\partial_i T_k(u_n^-)|^{p_i}dx\leq C''(k+1). \tag{4.4}$$

Combining (4.3) and (4.4), we get

$$\sum_{i=1}^{N}\int_\Omega|\partial T_k(u_n)|^{p_i}dx\leq(k+\|\psi^+\|_\infty+1)C'\quad\forall k>0.$$

Step3. Strong convergence of truncations.

Proposition 4.3. *If u_n is a solution of approximate problem (4.1). Then there exists a measurable function u and a subsequence of u_n such that*

$$T_k(u_n)\to T_k(u)\ \textit{strongly in}\ W_0^{1,\overrightarrow{p}}(\Omega).$$

Proof. Using Proposition 4.1, we obtain

$$\|T_k(u_n)\|_{W_0^{1,\overrightarrow{p}}(\Omega)}\leq C(k+\|\psi^+\|_\infty+1)^{\frac{1}{p_-}}. \tag{4.5}$$

Now, we will prove that $(u_n)_n$ is a Cauchy sequence in measure in Ω. For all $\lambda > 0$, we have
$\{|u_n - u_m| > \lambda\} \subset \{|u_n| > k\} \cup \{|u_m| > k\} \cup \{|T_k(u_n) - T_k(u_m)| > \lambda\}$
which implies that

$$meas\{|u_n - u_m| > \lambda\} \leq meas\{|u_n| > k\} + meas\{|u_m| > k\} \\ +meas\{|T_k(u_n) - T_k(u_m)| > \lambda\}. \tag{4.6}$$

By Hölder's inequality, Remark 2.1 and (4.5), we have

$$\begin{aligned} k.meas\{|u_n| > k\} = \int_{\{|u_n|>k\}} |T_k(u_n)|dx &\leq \int_\Omega |T_k(u_n)|dx \\ &\leq (meas(\Omega))^{\frac{1}{p^{-\prime}}} \|T_k(u_n)\|_{L^{p^-}(\Omega)} \\ &\leq C(meas(\Omega))^{\frac{1}{p^{-\prime}}} \|T_k(u_n)\|_{W_0^{1,\vec{p}}(\Omega)} \\ &\leq C(k + \|\psi^+\|_\infty + 1)^{\frac{1}{p^-}}. \end{aligned}$$

Then $meas\{|u_n| > k\} \leq C\Big(\dfrac{1}{k^{-1+p^-}} + \dfrac{1 + \|\psi^+\|_\infty}{k^{p^-}}\Big)^{\frac{1}{p^-}} \to 0$ as $k \to +\infty$. Which implies that, for all $\varepsilon > 0$, there exists k_0 such that $\forall k > k_0$, we have

$$meas\{|u_n| > k\} \leq \frac{\varepsilon}{3} \text{ and } meas\{|u_m| > k\} \leq \frac{\varepsilon}{3}. \tag{4.7}$$

Moreover, since the sequence $(T_k(u_n))_n$ is bounded in $W_0^{1,\vec{p}}(\Omega)$, there exists a subsequence $(T_k(u_n))_n$ such that $T_k(u_n)$ converges to v_k a.e. in Ω, weakly in $W_0^{1,\vec{p}}(\Omega)$ and strongly in $L^{p^-}(\Omega)$ as n tends to $+\infty$. Then the sequence $(T_k(u_n))_n$ is a Cauchy sequence in measure in Ω, thus $\forall \lambda > 0$, there exists n_0 such that

$$meas\{|T_k(u_n) - T_k(u_m)| > \lambda\} \leq \frac{\varepsilon}{3}, \quad \forall n, m \geq n_0. \tag{4.8}$$

Combining (4.6), (4.7) and (4.8), then for all $\lambda > 0$ and for all $\varepsilon > 0$, we have

$$meas\{|u_n - u_m| > \lambda\} \leq \varepsilon \quad \forall n, m \geq n_0.$$

Then $(u_n)_n$ is a Cauchy sequence in measure in Ω, then there exists a subsequence denoted by $(u_n)_n$ such that u_n converges to a measurable function u a.e. in Ω and

$$T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } W_0^{1,\vec{p}}(\Omega) \text{ and a.e. in } \Omega \quad \forall k > 0. \tag{4.9}$$

It remains to prove that

$$\lim_{n\to\infty} \sum_{i=1}^{N} \int_\Omega \Big(a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))\Big)\Big(\partial_i T_k(u_n) - \partial_i T_k(u)\Big)dx = 0. \tag{4.10}$$

Let us take $v = u_n + T_1(u_n - T_m(u_n))^-$ as test function in approximate problem (4.1), we obtain

$$-\sum_{i=1}^{N} \int_\Omega a_i(x, u_n, \nabla u_n)\partial_i T_1(u_n - T_m(u_n))^- dx - \sum_{i=1}^{N} \int_\Omega \phi_i^n(u_n)\partial_i T_1(u_n - T_m(u_n))^- dx \\ \leq -\int_\Omega f_n T_1(u_n - T_m(u_n))^- dx.$$

Then

$$\sum_{i=1}^{N} \int_{\{-(m+1)\leq u_n \leq -m\}} a_i(x, u_n, \nabla u_n)\partial_i u_n dx + \sum_{i=1}^{N} \int_{\{-(m+1)\leq u_n \leq -m\}} \phi_i(u_n)\partial_i u_n dx \\ \leq -\int_\Omega f_n T_1(u_n - T_m(u_n))^- dx. \tag{4.11}$$

We set $\Phi_i^n(s)=\int_0^s \phi_i^n(t)\chi_{\{-(m+1)\leq t\leq -m\}}dt$. Then by Green's formula, we have
$$\sum_{i=1}^N\int_{\{-(m+1)\leq u_n\leq -m\}}\phi_i(u_n)\partial_i u_n dx=\sum_{i=1}^N\int_\Omega \partial_i\Phi_i^n(u_n)dx=0.$$
Then, we get
$$\sum_{i=1}^N\int_{\{-(m+1)\leq u_n\leq -m\}}a_i(x,u_n,\nabla u_n)\partial_i u_n dx\leq -\int_\Omega f_nT_1(u_n-T_m(u_n))^-dx$$
By Lebesgue's theorem, we have
$$\lim_{m\to+\infty}\limsup_{n\to+\infty}\int_\Omega f_nT_1(u_n-T_m(u_n))^-dx=0$$
Then, we have
$$\lim_{m\to+\infty}\limsup_{n\to+\infty}\sum_{i=1}^N\int_{\{-(m+1)\leq u_n\leq -m\}}a_i(x,u_n,\nabla u_n)\partial_i u_n dx=0. \tag{4.12}$$
Similarly, taking $v=u_n-\eta T_1(u_n-T_m(u_n))^+$ as test function in approximate problem (4.1), we get
$$\lim_{m\to\infty}\limsup_{n\to\infty}\sum_{i=1}^N\int_{\{m\leq u_n\leq m+1\}}a_i(x,u_n,\nabla u_n)\partial_i u_n dx=0. \tag{4.13}$$
We consider the following function of one real variable:
$$h_m(s)=\begin{cases}1 & \text{if } |s|\leq m\\ 0 & \text{if } |s|\geq m+1\\ m+1-|s| & \text{if } m\leq |s|\leq m+1,\end{cases}$$
where $m>k$.
Let $\varphi=u_n-\eta(T_k(u_n)-T_k(u))^+h_m(u_n)$ as test function in approximate problem (4.1), we obtain
$$\begin{aligned}&\sum_{i=1}^N\int_\Omega a_i(x,u_n,\nabla u_n)\partial_i(T_k(u_n)-T_k(u))^+h_m(u_n)dx\\&+\sum_{i=1}^N\int_\Omega a_i(x,u_n,\nabla u_n)(T_k(u_n)-T_k(u))^+\partial_i u_n h_m'(u_n)dx\\&+\sum_{i=1}^N\int_\Omega \phi_i^n(u_n)\partial_i(T_k(u_n)-T_k(u))^+h_m(u_n)dx+\sum_{i=1}^N\int_\Omega \phi_i^n(u_n)\partial_i u_n(T_k(u_n)-T_k(u))^+h_m'(u_n)dx\\&\leq\int_\Omega f_n(T_k(u_n)-T_k(u))^+h_m(u_n)dx\end{aligned} \tag{4.14}$$
By (4.12) and (4.13), we have the second integral in (4.14) converges to zero as n and m tend to $+\infty$.
Since $h_m(u_n)=0$ if $|u_n|>m+1$, we have
$$\sum_{i=1}^N\int_\Omega \phi_i^n(u_n)\partial_i(T_k(u_n)-T_k(u))^+h_m(u_n)dx=\sum_{i=1}^N\int_\Omega \phi_i(T_{m+1}(u_n))h_m(u_n)\partial_i(T_k(u_n)-T_k(u))^+dx.$$
Using Lebesgue's theorem, we have $\phi_i^n(T_{m+1}(u_n))h_m(u_n)\to\phi_i(T_{m+1}(u))h_m(u)$ in $L^{p_i'}(\Omega)$ and $\partial_iT_k(u_n)\rightharpoonup\partial_iT_k(u)$ weakly in $L^{p_i}(\Omega)$ as n tends to $+\infty$, then the third integral in (4.14) converges to zero as n and m tend to $+\infty$.
Using (3.1), (4.12), (4.13) and Lebesgue's theorem, we obtain
$$\lim_{m\to+\infty}\lim_{n\to+\infty}\sum_{i=1}^N\int_{\{-(m+1)\leq u_n\leq -m\}}|\partial_i u_n|^{p_i}(T_k(u_n)-T_k(u))^+dx=0 \tag{4.15}$$

and

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\sum_{i=1}^{N}\int_{\{m\le u_n\le m+1\}}|\partial_i u_n|^{p_i}(T_k(u_n)-T_k(u))^{+}dx=0. \tag{4.16}$$

We deduce that

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\sum_{i=1}^{N}\int_{\Omega}a_i(x,u_n,\nabla u_n)\partial_i(T_k(u_n)-T_k(u))^{+}h_m(u_n)dx\le 0,$$

which implies that

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\sum_{i=1}^{N}\int_{\{T_k(u_n)-T_k(u)\ge 0,|u_n|\le k\}}a_i(x,u_n,\nabla u_n)\partial_i(T_k(u_n)-T_k(u))h_m(u_n)dx$$
$$-\lim_{m\to+\infty}\lim_{n\to+\infty}\sum_{i=1}^{N}\int_{\{T_k(u_n)-T_k(u)\ge 0,|u_n|>k\}}a_i(x,u_n,\nabla u_n)\partial_i T_k(u)h_m(u_n)dx\le 0.$$

Since $h_m(u_n)=0$ in $\{|u_n|>m+1\}$, we have

$$\sum_{i=1}^{N}\int_{\{T_k(u_n)-T_k(u)\ge 0,|u_n|>k\}}a_i(x,u_n,\nabla u_n)\partial_i T_k(u)h_m(u_n)dx$$
$$=\sum_{i=1}^{N}\int_{\{T_k(u_n)-T_k(u)\ge 0,|u_n|>k\}}a_i(x,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\partial_i T_k(u)h_m(u_n)dx.$$

Since $\Big(a_i(x,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\Big)_{n\ge 0}$ is bounded in $L^{p_i'}(\Omega)$, we have $a_i(x,T_{m+1}(u_n),\nabla T_{m+1}(u_n))$ converges to X_m^i weakly in $L^{p_i'}(\Omega)$. Then

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\sum_{i=1}^{N}\int_{\{T_k(u_n)-T_k(u)\ge 0,|u_n|>k\}}a_i(x,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\partial_i T_k(u)h_m(u_n)dx$$
$$=\lim_{m\to+\infty}\sum_{i=1}^{N}\int_{\{|u|>k\}}X_m^i\partial_i T_k(u)h_m(u)dx=0,$$

which implies

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\sum_{i=1}^{N}\int_{\{T_k(u_n)-T_k(u)\ge 0\}}a_i(x,T_k(u_n),\nabla T_k(u_n))\partial_i(T_k(u_n)-T_k(u))h_m(u_n)dx\le 0. \tag{4.17}$$

Moreover, we have $a_i(x,T_k(u_n),\nabla T_k(u))h_m(u_n)\to a_i(x,T_k(u),\nabla T_k(u))h_m(u)$ in $L^{p_i'}(\Omega)$ and $\partial_i(T_k(u_n)-T_k(u))\rightharpoonup 0$ weakly in $L^{p_i}(\Omega)$, then

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\sum_{i=1}^{N}\int_{\{T_k(u_n)-T_k(u)\ge 0\}}a_i(x,T_k(u_n),\nabla T_k(u))\partial_i(T_k(u_n)-T_k(u))h_m(u_n)dx=0. \tag{4.18}$$

Combining (3.3), (4.17) and (4.18), we deduce

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\sum_{i=1}^{N}\int_{\{T_k(u_n)-T_k(u)\ge 0\}}\Big(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\Big)$$
$$\partial_i(T_k(u_n)-T_k(u))h_m(u_n)dx=0. \tag{4.19}$$

Similarly, we take $\varphi=u_n+(T_k(u_n)-T_k(u))^{-}h_m(u_n)$ as test function in approximate problem (4.1), we obtain,

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\sum_{i=1}^{N}\int_{\{T_k(u_n)-T_k(u)\le 0\}}\Big(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\Big)$$
$$\partial_i(T_k(u_n)-T_k(u))h_m(u_n)dx=0. \tag{4.20}$$

Combining (4.19) and (4.20) we get

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\sum_{i=1}^{N}\int_\Omega\Big(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\Big)\partial_i(T_k(u_n)-T_k(u))h_m(u_n)dx=0. \tag{4.21}$$

Now, we prove $\lim_{m\to+\infty}\lim_{n\to+\infty}\sum_{i=1}^{N}\int_\Omega\Big(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\Big)$

$$\partial_i(T_k(u_n)-T_k(u))(1-h_m(u_n))dx=0. \tag{4.22}$$

Let $\varphi=u_n+T_k(u_n)^-(1-h_m(u_n))$ as test function in approximate problem (4.1), we obtain

$$\begin{aligned}-\sum_{i=1}^{N}\int_\Omega a_i(x,u_n,\nabla u_n)\partial_iT_k(u_n)^-(1-h_m(u_n))dx+\sum_{i=1}^{N}\int_\Omega a_i(x,u_n,\nabla u_n)\partial_iu_nT_k(u_n)^-h'_m(u_n)dx\\-\sum_{i=1}^{N}\int_\Omega \phi_i(u_n)\partial_iT_k(u_n)^-(1-h_m(u_n))dx+\sum_{i=1}^{N}\int_\Omega \phi_i^n(u_n)\partial_iu_nT_k(u_n)^-h'_m(u_n)dx\\\le-\int_\Omega f_nT_k(u_n)^-(1-h_m(u_n))dx\end{aligned} \tag{4.23}$$

By (4.12) and (4.13), we have $\lim_{m\to+\infty}\lim_{n\to+\infty}\sum_{i=1}^{N}\int_\Omega a_i(x,u_n,\nabla u_n)\partial_iu_nT_k(u_n)^-h'_m(u_n)dx=0$. Then the second integral in (4.23) converges to zero as n and m tends to $+\infty$. Since $\partial_iT_k(u_n)^-\rightharpoonup\partial_iT_k(u)^-$ in $L^{p_i}(\Omega)$ and $\phi_i(T_k(u_n))(1-h_m(u_n))\to\phi_i(T_k(u))(1-h_m(u))$ strongly in $L^{p'_i}(\Omega)$, we have

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\sum_{i=1}^{N}\int_\Omega\phi_i(u_n)\partial_iT_k(u_n)^-(1-h_m(u_n))dx=\lim_{m\to+\infty}\sum_{i=1}^{N}\int_\Omega\phi_i(T_k(u))\partial_iT_k(u)^-(1-h_m(u))dx.$$

By Lebesgue's theorem, we get

$$\lim_{m\to+\infty}\sum_{i=1}^{N}\int_\Omega\phi_i(T_k(u))\partial_iT_k(u)^-(1-h_m(u))dx=0.$$

Then the third integral in (4.23) converges to zero as n and m tends to $+\infty$.
We set $\Phi_i^n(t)=\int_0^t\phi_i(s)T_k(s)^-h'_m(s)ds$, by Green's Formula, we have

$$\sum_{i=1}^{N}\int_\Omega\phi_i^n(u_n)\partial_iu_nT_k(u_n)^-h'_m(u_n)dx=\sum_{i=1}^{N}\int_\Omega\partial_i\Phi_i^n(u_n)dx=0.$$

Then the fourth integral in (4.23) converges to zero as n and m tend to $+\infty$. By the Lebesgue's theorem, we have the integral on the right hand in (4.23) converges to zero as n and m tend to $+\infty$. We deduce

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\sum_{i=1}^{N}\int_{\{u_n\le0\}}a_i(x,u_n,\nabla u_n)\partial_iT_k(u_n)(1-h_m(u_n))dx=0. \tag{4.24}$$

Besides this, for η small enough, we take $\varphi=u_n-\eta T_k(u_n^+-\psi^+)(1-h_m(u_n))$ as test function in approximate problem (4.1), we obtain

$$\begin{aligned}\sum_{i=1}^{N}\int_\Omega a_i(x,u_n,\nabla u_n)\partial_iT_k(u_n^+-\psi^+)(1-h_m(u_n))dx-\sum_{i=1}^{N}\int_\Omega a_i(x,u_n,\nabla u_n)\partial_iu_nT_k(u_n^+-\psi^+)h'_m(u_n)dx\\+\sum_{i=1}^{N}\int_\Omega\phi_i^n(u_n)\partial_iT_k(u_n^+-\psi^+)(1-h_m(u_n))dx-\sum_{i=1}^{N}\int_\Omega\phi_i^n(u_n)\partial_iu_nT_k(u_n^+-\psi^+)h'_m(u_n)dx\end{aligned}$$

$$\leq \int_\Omega f_n T_k(u_n^+ - \psi^+)(1-h_m(u_n))dx \tag{4.25}$$

By Hölder's inequality, (3.1), (4.12) and (4.13), we have

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\sum_{i=1}^N \int_\Omega \phi_i^n(u_n)\partial_i u_n T_k(u_n^+ - \psi^+)h_m'(u_n)dx = 0.$$

Using young's inequality, we obtain

$$\sum_{i=1}^N \int_\Omega a_i(x,u_n,\nabla u_n)\partial_i T_k(u_n^+ - \psi^+)(1-h_m(u_n))dx \leq$$

$$\sum_{i=1}^N \int_{\{-(m+1)\leq u_n\leq -m\}} a_i(x,u_n,\nabla u_n)\partial_i u_n T_k(u_n^+ - \psi^+)dx + \int_\Omega f_n T_k(u_n^+ - \psi^+)(1-h_m(u_n))dx$$
$$+\sum_{i=1}^N \int_{\{u_n^+ - \psi^+\leq k\}} \phi_i^n(u_n)\partial_i u_n^+(1-h_m(u_n))dx + \sum_{i=1}^N \int_{\{u_n^+ - \psi^+\leq k\}} \phi_i^n(u_n)\partial_i \psi^+(1-h_m(u_n))dx \tag{4.26}$$

By (4.12), we have the first integral on the right hand converges to zero as n and m tend to $+\infty$. Using the Lebesque's theorem, we obtain the second integral in the right hand converges to zero as n and m tend to $+\infty$. Since

$$\sum_{i=1}^N \int_{\{u_n^+ - \psi^+\leq k\}} \phi_i^n(u_n)\partial_i u_n^+(1-h_m(u_n))dx$$
$$= \sum_{i=1}^N \int_\Omega \phi_i^n(T_{\{k+\|\psi^+\|_{L^\infty(\Omega)}\}}(u_n))\partial_i T_{\{k+\|\psi^+\|_{L^\infty(\Omega)}\}}(u_n^+)(1-h_m(u_n))dx.$$

Since $\partial_i T_{\{k+\|\psi^+\|_{L^\infty(\Omega)}\}}(u_n^+) \rightharpoonup \partial_i T_{\{k+\|\psi^+\|_{L^\infty(\Omega)}\}}(u^+)$ weakly in $L^{p_i}(\Omega)$ and $\phi_i^n(T_{\{k+\|\psi^+\|_{L^\infty(\Omega)}\}}(u_n))(1-h_m(u_n)) \to \phi_i(T_{\{k+\|\psi^+\|_{L^\infty(\Omega)}\}}(u))(1-h_m(u_n))$ strongly in $L^{p_i'}(\Omega)$, we have

$$\sum_{i=1}^N \int_\Omega \phi_i^n(T_{\{k+\|\psi^+\|_{L^\infty(\Omega)}\}}(u_n))\partial_i T_{\{k+\|\psi^+\|_{L^\infty(\Omega)}\}}(u_n^+)(1-h_m(u_n))dx$$
$$= \sum_{i=1}^N \int_\Omega \phi_i(T_{\{k+\|\psi^+\|_{L^\infty(\Omega)}\}}(u))\partial_i T_{\{k+\|\psi^+\|_{L^\infty(\Omega)}\}}(u)(1-h_m(u))dx + \varepsilon(n).$$

By Lebesgue's theorem, we have

$$\lim_{m\to\infty}\sum_{i=1}^N \int_\Omega \phi_i(T_{\{k+\|\psi^+\|_{L^\infty(\Omega)}\}}(u))\partial_i T_{\{k+\|\psi^+\|_{L^\infty(\Omega)}\}}(u)(1-h_m(u))dx = 0.$$

Then, we have the third integral converges to zero as n and m tend to $+\infty$. Similarly as (4.24), we obtain

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\sum_{i=1}^N \int_{\{u_n>0\}} a_i(x,u_n,\nabla u_n)\partial_i T_k(u_n)(1-h_m(u_n)) = 0. \tag{4.27}$$

Combining (4.24) and (4.27), we get

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\sum_{i=1}^N \int_\Omega a_i(x,u_n,\nabla u_n)\partial_i T_k(u_n)(1-h_m(u_n))dx = 0. \tag{4.28}$$

Furthermore, we have

$$\sum_{i=1}^N \int_\Omega \Big(a_i(x,T_k(u_n),\nabla T_k(u_n)) - a_i(x,T_k(u_n),\nabla T_k(u))\Big)\Big(\partial_i T_k(u_n) - \partial_i T_k(u)\Big)dx$$

$$= \sum_{i=1}^{N} \int_\Omega \Big(a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u)) \Big) \Big(\partial_i T_k(u_n) - \partial_i T_k(u) \Big) h_m(u_n) dx$$

$$+ \sum_{i=1}^{N} \int_\Omega \Big(a_i(x, T_k(u_n), \nabla T_k(u_n)) \Big) \partial_i T_k(u_n)(1 - h_m(u_n)) dx$$

$$- \sum_{i=1}^{N} \int_\Omega \Big(a_i(x, T_k(u_n), \nabla T_k(u_n)) \Big) \partial_i T_k(u)(1 - h_m(u_n)) dx$$

$$- \sum_{i=1}^{N} \int_\Omega \Big(a_i(x, T_k(u_n), \nabla T_k(u)) \Big) \Big(\partial_i T_k(u_n) - \partial_i T_k(u) \Big)(1 - h_m(u_n)) dx.$$

By (4.21) and (4.28), the first and the second integrals on the right hand side converge to zero as n and m tend to $+\infty$.
Since $\Big(a_i(x, T_k(u_n), \nabla T_k(u_n)) \Big)_n$ is bounded in $L^{p_i'}(\Omega)$ and $\partial_i T_k(u)(1 - h_m(u_n))$ converge to zero in $L^{p_i}(\Omega)$ as n and m tend to $+\infty$, hence the third integral on the right hand side converge to zero as n and m tend to $+\infty$.
So, since $a_i(x, T_k(u_n), \nabla T_k(u_n))(1 - h_m(u_n))$ converges to $a_i(x, T_k(u), \nabla T_k(u))(1 - h_m(u))$ strongly in $L^{p_i'}(\Omega)$ and $\partial_i T_k(u_n) \rightharpoonup \partial_i T_k(u)$ weakly in $L^{p_i}(\Omega)$, we obtain the fourth integral on the right hand side converge to zero as n and m tend to $+\infty$. Then, we get (4.10).
Using (4.9), (4.10) and Lemma 3.2, we obtain

$$T_k(u_n) \to T_k(u) \text{ strongly in } W_0^{1,\overrightarrow{p}}(\Omega) \text{ and a. e. in } \Omega \quad \forall k > 0.$$

Step4. Passing to the limit. Now, let $\varphi \in K_\psi \cap L^\infty(\Omega)$, we take $v = u_n - T_k(u_n - \varphi)$ as test function in approximate problem (4.1), we obtain

$$\sum_{i=1}^{N} \int_\Omega a_i(x, u_n, \nabla u_n) \partial_i T_k(u_n - \varphi) dx + \sum_{i=1}^{N} \int_\Omega \phi_i^n(u_n) \partial_i T_k(u_n - \varphi) dx$$

$$\leq \int_\Omega f_n T_k(u_n - \varphi) dx,$$

which implies that,

$$\sum_{i=1}^{N} \int_\Omega a_i(x, T_{k+\|\varphi\|_\infty}(u_n), \nabla T_{k+\|\varphi\|_\infty}(u_n)) \partial_i T_k(u_n - \varphi) dx + \sum_{i=1}^{N} \int_\Omega \phi_i(T_{k+\|\varphi\|_\infty}(u_n)) \partial_i T_k(u_n - \varphi) dx$$

$$\leq \int_\Omega f_n T_k(u_n - \varphi) dx.$$

Since $T_k(u_n) \to T_k(u)$ strongly in $W_0^{1,\overrightarrow{p}}(\Omega)$ and a. e. in $\Omega \quad \forall k > 0$, we have
$a_i(x, T_{k+\|\varphi\|_\infty}(u_n), \nabla T_{k+\|\varphi\|_\infty}(u_n)) \rightharpoonup a_i(x, T_{k+\|\varphi\|_\infty}(u), \nabla T_{k+\|\varphi\|_\infty}(u))$ weakly in $L^{p_i'}(\Omega)$,
$\phi_i(T_{k+\|\varphi\|_\infty}(u_n)) \to \phi_i(T_{k+\|\varphi\|_\infty}(u))$ strongly in $L^{p_i'}(\Omega)$ and $\partial_i T_k(u_n - \varphi) \to \partial_i T_k(u - \varphi)$ strongly in $L^{p_i}(\Omega)$ we can pass to limit in

$$\begin{cases} u_n \in K_\psi. \\ \displaystyle\sum_{i=1}^{N} \int_\Omega a_i(x, u_n, \nabla u_n) \partial_i T_k(u_n - \varphi) dx + \sum_{i=1}^{N} \int_\Omega \phi_i^n(u_n) \partial_i T_k(u_n - \varphi) dx \\ \qquad\qquad\qquad\qquad\qquad\qquad \leq \displaystyle\int_\Omega f_n T_k(u_n - \varphi) dx, \\ \forall \varphi \in K_\psi \cap L^\infty(\Omega) \text{ and } \forall k > 0, \end{cases}$$

this completes the proof of theorem 4.1.

5. Appendix

In this section, we will show that the operator $B_n = A + \Phi_n$ is coercive and pseudo-monotone .

Proof of Lemma 4.1. We consider the operator $\Phi_n : K_\psi \to W^{-1,\overrightarrow{p'}}(\Omega)$ defined by $<\Phi_n u, v>= \sum_{i=1}^{N} \int_\Omega \phi_i^n(u)\partial_i v dx$. By Hölder's inequality, we have, for all $u, v \in X$,

$$
\begin{aligned}
|<\Phi_n u, v>| &\le \sum_{i=1}^{N}\Big(\int_\Omega |\phi_i(T_n(u))|^{p_i'}dx\Big)^{\frac{1}{p_i'}}\Big(\int_\Omega |\partial_i v|^{p_i}dx\Big)^{\frac{1}{p_i}}\\
&\le \sum_{i=1}^{N}\Big(\int_\Omega \sup_{|s|\le n}|\phi_i(s)|^{p_i'}dx\Big)^{\frac{1}{p_i'}}\Big(\int_\Omega |\partial_i v|^{p_i}dx\Big)^{\frac{1}{p_i}}\\
&\le \sum_{i=1}^{N}\Big(\int_\Omega (\sup_{|s|\le n}|\phi_i(s)|+1)^{p_i'}dx\Big)^{\frac{1}{p_i'}}\Big(\int_\Omega |\partial_i v|^{p_i}dx\Big)^{\frac{1}{p_i}}\\
&\le \sum_{i=1}^{N}(\sup_{|s|\le n}|\phi_i(s)|+1)\Big(\int_\Omega 1dx\Big)^{\frac{1}{p_i'}}\Big(\int_\Omega |\partial_i v|^{p_i}dx\Big)^{\frac{1}{p_i}}\\
&\le \max_{1\le i\le N}(\sup_{|s|\le n}|\phi_i(s)|+1)(meas(\Omega)+1)^{\frac{1}{p'_-}}\sum_{i=1}^{N}\Big(\int_\Omega |\partial_i v|^{p_i}dx\Big)^{\frac{1}{p_i}}\\
&\le C(n)\|v\|_{W_0^{1,\overrightarrow{p}}(\Omega)},
\end{aligned}
$$

which implies that $\dfrac{|<\Phi_n u, v>|}{\|v\|_{1,\overrightarrow{p}}} \le C(n)$. Moreover, let $v_0 \in K_\psi$, thanks to Hölder's inequality and (3.2), we have

$$
\begin{aligned}
|<Av, v_0>| &\le \sum_{i=1}^{N}\Big(\int_\Omega |a_i(x,v,\nabla v)|^{p_i'}dx\Big)^{\frac{1}{p_i'}}\Big(\int_\Omega |\partial_i v_0|^{p_i}dx\Big)^{\frac{1}{p_i}}\\
&\le \beta\sum_{i=1}^{N}\Big(\int_\Omega (j_i(x)^{p_i'}+|v|^{p_i}+|\partial_i v|^{p_i})dx\Big)^{\frac{1}{p_i'}}\Big(\int_\Omega |\partial_i v_0|^{p_i}dx\Big)^{\frac{1}{p_i}}\\
&\le \beta\sum_{i=1}^{N}\Big(C_1+\int_\Omega |\partial_i v|^{p_i}+\int_\Omega |\partial_i v|^{p_i})dx\Big)^{\frac{1}{p_i'}}\Big(\int_\Omega |\partial_i v_0|^{p_i}dx\Big)^{\frac{1}{p_i}}\\
&\le \beta\sum_{i=1}^{N}C_1^{\frac{1}{p_i'}}\Big(1+\frac{2}{C_1}\sum_{i=1}^{N}\int_\Omega |\partial_i v|^{p_i})dx\Big)^{\frac{1}{p_i'}}\Big(\int_\Omega |\partial_i v_0|^{p_i}dx\Big)^{\frac{1}{p_i}}\\
&\le \beta C_2\sum_{i=1}^{N}\Big(1+\frac{2}{C_1}\sum_{i=1}^{N}\int_\Omega |\partial_i v|^{p_i})dx\Big)^{\frac{1}{p'_-}}\Big(\int_\Omega |\partial_i v_0|^{p_i}dx\Big)^{\frac{1}{p_i}}\\
&\le \beta C_2\sum_{i=1}^{N}\Big(1+C_3\Big(\sum_{i=1}^{N}\int_\Omega |\partial_i v|^{p_i})dx\Big)^{\frac{1}{p'_-}}\Big)\Big(\int_\Omega |\partial_i v_0|^{p_i}dx\Big)^{\frac{1}{p_i}}\\
&\le \beta C_2\Big(1+C_3\Big(\sum_{i=1}^{N}\int_\Omega |\partial_i v|^{p_i})dx\Big)^{\frac{1}{p'_-}}\Big)\sum_{i=1}^{N}\Big(\int_\Omega |\partial_i v_0|^{p_i}dx\Big)^{\frac{1}{p_i}}\\
&\le \beta C_2\Big(1+C_3\Big(\sum_{i=1}^{N}\int_\Omega |\partial_i v|^{p_i})dx\Big)^{\frac{1}{p'_-}}\Big)\|v_0\|_{W_0^{1,\overrightarrow{p}}(\Omega)}.
\end{aligned}
$$

Hence

$$
\frac{|<Av, v-v_0>|}{\|v\|_{W_0^{1,\overrightarrow{p}}(\Omega)}} \ge \alpha\frac{\sum_{i=1}^{N}\int_\Omega |\partial_i v|^{p_i}dx}{\|v\|_{W_0^{1,\overrightarrow{p}}(\Omega)}} - \frac{\beta C_2\|v_0\|_{W_0^{1,\overrightarrow{p}}(\Omega)}}{\|v\|_{W_0^{1,\overrightarrow{p}}(\Omega)}}
$$

$$-\frac{\beta C_2 C_3}{\|v\|_{W_0^{1,\overrightarrow{p}}(\Omega)}}\Big(\sum_{i=1}^{N}\int_\Omega |\partial_i v|^{p_i}dx\Big)^{\frac{1}{p'_-}}\|v_0\|_{W_0^{1,\overrightarrow{p}}(\Omega)}.$$

Then

$$\frac{|< Av, v-v_0 >|}{\|v\|_{W_0^{1,\overrightarrow{p}}(\Omega)}} \geq$$

$$\alpha\frac{\sum_{i=1}^{N}\int_\Omega |\partial_i v|^{p_i}dx}{\|v\|_{W_0^{1,\overrightarrow{p}}(\Omega)}}\Big[1-\frac{\beta}{\alpha}C_2C_3\Big(\sum_{i=1}^{N}\int_\Omega |\partial_i v|^{p_i}dx\Big)^{\frac{1}{p'_-}-1}\|v_0\|_{W_0^{1,\overrightarrow{p}}(\Omega)}\Big]-\frac{\beta C_2\|v_0\|_{W_0^{1,\overrightarrow{p}}(\Omega)}}{\|v\|_{W_0^{1,\overrightarrow{p}}(\Omega)}}. \quad (5.1)$$

Additionally, by Jensen's inequality, we have

$$\begin{aligned}\|v\|_{1,\overrightarrow{p}}^{p_-^+} &= \Big(\sum_{i=1}^{N}(\int_\Omega |\partial_i v|^{p_i}dx)^{\frac{1}{p_i}}\Big)^{p_-^+}\\ &\leq \Big(\sum_{i=1}^{N}(\int_\Omega |\partial_i v|^{p_i}dx)^{\frac{1}{p_-^+}}\Big)^{p_-^+}\\ &\leq C\sum_{i=1}^{N}\int_\Omega |\partial_i v|^{p_i}dx,\end{aligned}$$

where

$$p_-^+ = \begin{cases} p^- & \text{if } \|\partial_i v\|_{L^{p_i}(\Omega)} \geq 1\\ p^+ & \text{if } \|\partial_i v\|_{L^{p_i}(\Omega)} < 1.\end{cases}$$

Then $\dfrac{\sum_{i=1}^{N}\int_\Omega |\partial_i v|^{p_i}dx}{\|v\|_{W_0^{1,\overrightarrow{p}}(\Omega)}} \to +\infty$ and $\sum_{i=1}^{N}\int_\Omega |\partial_i v|^{p_i}dx \to +\infty$ as $\|v\|_{W_0^{1,\overrightarrow{p}}(\Omega)} \to +\infty$.

Using (5.1), we get $\dfrac{|< Av, v-v_0 >|}{\|v\|_{W_0^{1,\overrightarrow{p}}(\Omega)}} \to +\infty$ as $\|v\|_{1,\overrightarrow{p}} \to +\infty$.

Since $\dfrac{< \Phi_n v, v >}{\|v\|_{W_0^{1,\overrightarrow{p}}(\Omega)}}$ and $\dfrac{< \Phi_n v, v_0 >}{\|v\|_{W_0^{1,\overrightarrow{p}}(\Omega)}}$ are bounded, then we have $\dfrac{< B_n v, v-v_0 >}{\|v\|_{W_0^{1,\overrightarrow{p}}(\Omega)}} = \dfrac{< Av, v-v_0 >}{\|v\|_{W_0^{1,\overrightarrow{p}}(\Omega)}} + \dfrac{< \Phi_n v, v-v_0 >}{\|v\|_{W_0^{1,\overrightarrow{p}}(\Omega)}} \to +\infty$ as $\|v\|_{W_0^{1,\overrightarrow{p}}(\Omega)} \to +\infty$. We deduce that the operator $B_n = A + \Phi_n$ is coercive. It remains to prove that the operator B_n is pseudo-monotone. Let $(u_k)_k$ be a sequence in $W_0^{1,\overrightarrow{p}}(\Omega)$ such that

$$\begin{cases} u_k \rightharpoonup u \quad \text{weakly in } W_0^{1,\overrightarrow{p}}(\Omega)\\ B_n u_k \rightharpoonup \chi \quad \text{weakly in } W^{-1,\overrightarrow{p}'}(\Omega)\\ \limsup_{k\to+\infty} < B_n u_k, u_k > \ \leq \ < \chi, u >.\end{cases}$$

We will prove that $\chi = B_n u$ and $< B_n u_k, u_k > \to < \chi, u >$ as $k \to +\infty$. Since $W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^{p^-}(\Omega)$, then $u_k \to u$ strongly in $L^{p^-}(\Omega)$ and a.e. in Ω for a subsequence denoted again $(u_k)_k$. Since $(u_k)_k$ is bounded in $W_0^{1,\overrightarrow{p}}(\Omega)$, by (3.2), we have $(a_i(x, u_k, \nabla u_k))_k$ is bounded in $L^{p'_i}(\Omega)$. Then there exists a function $\varphi_i \in L^{p'_i}(\Omega)$ such that

$$a_i(x, u_k, \nabla u_k) \rightharpoonup \varphi_i \text{ as } k \to +\infty. \quad (5.2)$$

What is more, since $(\phi_i^n(u_k))_k$ is bounded in $L^{p'_i}(\Omega)$ and $\phi_i^n(u_k) \to \phi_i^n(u)$ a.e. in Ω, we have

$$\phi_i^n(u_k) \to \phi_i^n(u) \text{ strongly in } L^{p'_i}(\Omega) \text{ as } k \to +\infty. \quad (5.3)$$

For all $v \in W_0^{1,\overrightarrow{p}}(\Omega)$, using (5.2) and (5.3), we obtain

$$\begin{aligned} <\chi, v> &= \lim_{k\to+\infty} < B_n u_k, v > \\ &= \lim_{k\to+\infty} \sum_{i=1}^{N} \int_\Omega a_i(x,u_k,\nabla u_k)\partial_i v dx + \lim_{k\to+\infty} \sum_{i=1}^{N} \int_\Omega \phi_i^n(u_k)\partial_i v dx \\ &= \sum_{i=1}^{N} \int_\Omega \varphi_i \partial_i v dx + \sum_{i=1}^{N} \int_\Omega \phi_i^n(u)\partial_i v dx. \end{aligned}$$

Hence, we have

$$\begin{aligned} \limsup_{k\to+\infty} < B_n u_k, u_k > &= \limsup_{k\to+\infty} \Big[\sum_{i=1}^{N} \int_\Omega a_i(x,u_k,\nabla u_k)\partial_i u_k dx + \sum_{i=1}^{N} \int_\Omega \phi_i^n(u_k)\partial_i u_k dx \Big] \\ &= \limsup_{k\to+\infty} \sum_{i=1}^{N} \int_\Omega a_i(x,u_k,\nabla u_k)\partial_i u_k dx + \sum_{i=1}^{N} \int_\Omega \phi_i^n(u)\partial_i u dx \\ &\leq \ < \chi, u > \\ &= \sum_{i=1}^{N} \int_\Omega \varphi_i \partial_i u dx + \sum_{i=1}^{N} \int_\Omega \phi_i^n(u)\partial_i u dx \end{aligned}$$

which implies that

$$\limsup_{k\to+\infty} \sum_{i=1}^{N} \int_\Omega a_i(x,u_k,\nabla u_k)\partial_i u_k dx \leq \sum_{i=1}^{N} \int_\Omega \varphi_i \partial_i u dx. \tag{5.4}$$

By (3.3), we have $\sum_{i=1}^{N} \int_\Omega \Big(a_i(x,u_k,\nabla u_k) - a_i(x,u_k,\nabla u) \Big)(\partial_i u_k - \partial_i u) dx > 0.$

Then

$$\begin{aligned} \sum_{i=1}^{N} \int_\Omega a_i(x,u_k,\nabla u_k)\partial_i u_k dx \geq & -\sum_{i=1}^{N} \int_\Omega a_i(x,u_k,\nabla u)\partial_i u dx + \sum_{i=1}^{N} \int_\Omega a_i(x,u_k,\nabla u_k)\partial_i u dx. \\ & + \sum_{i=1}^{N} \int_\Omega a_i(x,u_k,\nabla u)\partial_i u_k dx. \end{aligned}$$

Using (5.2), we get

$$\liminf_{k\to+\infty} \sum_{i=1}^{N} \int_\Omega a_i(x,u_k,\nabla u_k)\partial_i u_k dx \geq \sum_{i=1}^{N} \int_\Omega \varphi_i \partial_i u dx. \tag{5.5}$$

Combining (5.4) and (5.5), we obtain

$$\lim_{k\to+\infty} \sum_{i=1}^{N} \int_\Omega a_i(x,u_k,\nabla u_k)\partial_i u_k dx = \sum_{i=1}^{N} \int_\Omega \varphi_i \partial_i u dx. \tag{5.6}$$

$$\begin{aligned} \lim_{k\to+\infty} < B_n u_k, u_k > &= \lim_{k\to+\infty} \sum_{i=1}^{N} \int_\Omega a_i(x,u_k,\nabla u_k)\partial_i u_k dx + \lim_{k\to+\infty} \sum_{i=1}^{N} \int_\Omega \phi_i^n(u_k)\partial_i u_k dx \\ &= \sum_{i=1}^{N} \int_\Omega \varphi_i \partial_i u dx + \sum_{i=1}^{N} \int_\Omega \phi_i^n(u)\partial_i u dx \\ &= < \chi, u > . \end{aligned}$$

In addition to this, since $a_i(x,u_k,\nabla u)$ converges to $a_i(x,u,\nabla u)$ strongly in $L^{p_i'}(\Omega)$, by (5.6), we obtain

$$\sum_{i=1}^{N} \int_\Omega \Big(a_i(x,u_k,\nabla u_k) - a_i(x,u_k,\nabla u) \Big)(\partial_i u_k - \partial_i u) dx = 0.$$

Using lemma 3.2, we get u_k converges to u strongly in $W_0^{1,\overrightarrow{p}}(\Omega)$ and a. e. in Ω, then $a_i(x,u_k,\nabla u)$ converges to $a_i(x,u,\nabla u)$ weakly in $L^{p_i'}(\Omega)$ and $\phi_i^n(u_k)$ converges to $\phi_i^n(u)$ strongly in $L^{p_i'}(\Omega)$. Then for all $v \in W_0^{1,\overrightarrow{p}}(\Omega)$, we have

$$
\begin{aligned}
<\chi, v> &= \lim_{k\to+\infty} < B_n u_k, v > \\
&= \lim_{k\to+\infty} \sum_{i=1}^{N} \int_\Omega a_i(x, u_k, \nabla u_k)\partial_i v dx + \lim_{k\to+\infty} \sum_{i=1}^{N} \int_\Omega \phi_i(u_k)\partial_i v dx \\
&= \sum_{i=1}^{N} \int_\Omega a_i(x, u, \nabla u)\partial_i v dx + \sum_{i=1}^{N} \int_\Omega \phi_i(u)\partial_i v dx \\
&=< B_n u, v >
\end{aligned}
$$

which implies that $B_n u = \chi$.

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