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THE RYJÁČEK CLOSURE AND A FORBIDDEN SUBGRAPH

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Abstract

The Ryjáček closure is a powerful tool in the study of Hamiltonian properties of claw-free graphs. Because of its usefulness, we may hope to use it in the classes of graphs defined by another forbidden subgraph. In this note, we give a negative answer to this hope, and show that the claw is the only forbidden subgraph that produces non-trivial results on Hamiltonicity by the use of the Ryjáček closure.

Keywords: closure, claw-free graph, Hamiltonian graph, perfect matching, traceable graph.

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1. Introduction

In this note we only consider finite simple graphs. For a given graph H, a graph G is said to be *H-free* if G does not contain an induced subgraph which is isomorphic to H. If G is H-free, we also say that H is *forbidden* in G. The complete bipartite graph $K_{1,3}$ is called the *claw*. In the study of Hamiltonian properties, the class of claw-free graphs often appears as a well-behaved class.

A vertex v in a graph G is said to be *locally-connected* if the neighborhood of v, denoted by $N_G(v)$, induces a connected graph in G. If every vertex in G is locally-connected, we say that G is a *locally-connected graph*. Oberly and Sumner [5] have proved that every connected and locally-connected claw-free graph of order at least three is Hamiltonian. Ryjáček [7] has generalized this result by introducing a powerful closure operation. Let G be a graph and let v be a vertex in G. *Local completion* at v is the operation of adding an edge to every pair of nonadjacent vertices in $N_G(v)$. Note that if G' is the graph obtained from G by local completion at v, then $N_{G'}(v) = N_G(v)$ and $N_{G'}(v)$ induces a complete graph in G'.

A vertex v in a graph G is said to be *eligible* if v is locally-connected and $N_G(v)$ does not induce a complete graph. Now consider a sequence of graphs $G_0, G_1, \ldots, G_n$ such that $G_0 = G$, and G_i is obtained from G_{i-1} by local completion at some eligible vertex of G_{i-1} ($1 \le i \le n$). If no vertex in G_n is eligible, then G_n is called the *Ryjáček closure* of G and denoted by $\mathrm{cl_R}(G)$. Ryjáček [7] proved that $\mathrm{cl_R}(G)$ is uniquely determined, regardless of the choice of an eligible vertex at each step. He also proved the following theorem.

Theorem 1 [7]**.** *Let G be a claw-free graph. Then $\mathrm{cl_R}(G)$ is also a claw-free graph, and G is Hamiltonian if and only if $\mathrm{cl_R}(G)$ is Hamiltonian.*

The Ryjáček closure has given a significant impact to the study of Hamiltonian properties of graphs. For example, if G is a connected and locally-connected graph, then it is easy to see that $\mathrm{cl_R}(G)$ is a complete graph. Therefore, Theorem 1 implies the result of Oberly–Sumner. Matthews and Sumner [4] have conjectured that every 4-connected claw-free graph is Hamiltonian. Later, Thomassen [9] has conjectured that every 4-connected line graph is Hamiltonian. Since every line graph is claw-free, Thomassen's conjecture was considered to be a partial solution to the conjecture by Matthews and Sumner. However, Ryjáček [7] has pointed out that if G is claw-free, then $\mathrm{cl_R}(G)$ is a line graph and that the conjectures by Thomassen and Matthews-Sumner are equivalent. Besides the above results, the Ryjáček closure has played an important role in many studies on Hamiltonian properties of claw-free graphs.

Since the Ryjáček closure is such a powerful operation for claw-free graphs, one may hope to apply it to the class of graphs defined by another forbidden subgraph. The purpose of this note is to give a negative answer to this hope and show that the claw is the only forbidden subgraph that gives non-trivial results through the Ryjáček closure.

In the next section, we study the relationship between the existence of a perfect matching and a forbidden subgraph in the class of connected and locally-connected graphs. Then we discuss the Ryjáček closure and Hamiltonian properties in Section 3. We give a conclusion in Section 4.

For graph-theoretic notation and definitions not explained in this paper, we refer the reader to [1]. For graphs G and H, the join of G and H is denoted by $G \vee H$, and for a positive integer n, we denote by nG the union of n disjoint copies of G. For $S \subset V(G)$, we denote by $G[S]$ the subgraph of G induced by S. We say that G is Hamiltonian if G contains a Hamiltonian cycle, and that G is traceable if G contains a Hamiltonian path.

2. Deficiency

For a graph G, the *deficiency* of G, denoted by $\operatorname{def}(G)$, is the number of vertices not saturated by a maximum matching. Thus, G has a perfect matching if and only if $\operatorname{def}(G) = 0$. A maximum matching of G is called a *near-perfect matching* if $\operatorname{def}(G) = 1$. Berge's Formula says $\operatorname{def}(G) = \max\{w_o(G - S) - |S| \colon S \subset V(G)\}$, where $w_o(G-S)$ is the number of odd components of $G-S$. Note $\operatorname{def}(G) \equiv |V(G)| \pmod 2$. Las Vergnas [3] and Sumner [8] have independently proved that a connected claw-free graph of even order has a perfect matching. Later Jünger, Pulleyblank and Reinelt [2] have proved that a claw-free graph of odd order has a near-perfect matching. Combining these two theorems, we obtain the following.

Theorem 2 [2, 3, 8]**.** *A connected claw-free graph G satisfies* $\operatorname{def}(G) \leq 1$.

In [6], Plummer and Saito have studied what forbidden subgraph forces a connected graph of sufficiently large order to have a perfect or a near-perfect matching, and proved that only $K_{1,2}$ and $K_{1,3}$ do.

Theorem 3 [6]**.** *Let H be a connected graph of order at least three. If there exists a positive integer n_0 such that every connected H-free graph G of order at least n_0 satisfies* $\operatorname{def}(G) \leq 1$, *then H is either $K_{1,2}$ or $K_{1,3}$.*

Now we consider the same problem in the class of connected and locally-connected graphs. Since this class is smaller than the class of connected graphs, a weaker assumption on a forbidden subgraph may guarantee a bounded deficiency. It means that a forbidden subgraph other than $K_{1,2}$ and $K_{1,3}$ may force a connected and locally-connected graph of sufficiently large order to have a perfect or a near-perfect matching. We first prove that this speculation is correct.

Theorem 4. *Let H be a connected graph of order at least three. If there exists a positive integer n_0 such that every connected, locally-connected H-free graph G of order at least n_0 satisfies* $\mathrm{def}(G) \leq 1$*, then H is isomorphic to $K_{1,2}$, $K_{1,3}$, K_3 or $K_2 \vee 2K_1$.*

Before we give a proof of the above theorem, we introduce one graph which we use in the proof. For an integer n with $n \geq 2$, let H^1, H^2 and H^3 be three disjoint copies of K_n. Then choose one vertex v_i in H^i, $1 \leq i \leq 3$, and add edges v_1v_2, v_1v_3 and v_2v_3. Let $H_0(n)$ be the resulting graph. Note that $H_0(n)$ is a connected graph with independence number three.

***Proof of Theorem* 4.** Using the integer n_0 in the statement of the theorem, we set $n = \max\{2n_0, 4\}$. Note that n is an even number. Let G_1 be a graph isomorphic to $K_2 \vee nK_1$. Then G_1 is a connected and locally-connected graph of order greater than n_0. On the other hand, since $n \geq 4$, $\mathrm{def}(G) = n - 2 \geq 2$. Therefore, G_1 is not H-free and it contains an induced subgraph which is isomorphic to H. Since $|V(H)| \geq 3$, $H \simeq K_{1,m}$ or $H \simeq K_2 \vee mK_1$ for some positive integer m.

Let G_2^0 be a copy of K_4 with $V(G_2^0) = \{v_1, v_2, v_3, v_4\}$. For each i, j with $1 \leq i < j \leq 4$, we introduce a new graph $H_{i,j}$ which is a copy of K_{n+1}. Then add edges $\{v_ix, v_jx \colon 1 \leq i < j \leq 4, x \in V(H_{i,j})\}$. Let G_2 be the resulting graph. Note that G_2 is a connected graph of even order greater than n_0.

For $v \in V(G_2)$, let G_v be the subgraph of G_2 induced by $N_{G_2}(v)$. If $v \in V(H_{i,j})$ for some i, j with $1 \leq i < j \leq 4$, then $G_v \simeq K_{n+2}$. If $v = v_i$ for some i with $1 \leq i \leq 4$, then $N_G(v_i) = \bigcup_{j \in J} V(H_{i,j}) \cup \{v_j \colon j \in J\}$, where $J = \{1,2,3,4\} \setminus \{i\}$, and G_v is isomorphic to $H_0(n+2)$. Therefore, G_2 is a locally-connected graph. Since $G_2 - \{v_1, v_2, v_3, v_4\}$ has six odd components, $\mathrm{def}(G_2) \geq 2$. Therefore, G_2 is not H-free and there is an induced subgraph G' of G_2 which is isomorphic to $K_{1,m}$ or $K_2 \vee mK_1$ for some positive integer m. Since $\alpha(G_v) \leq 3$ for every $v \in V(G_2)$, we have $m \leq 3$. Therefore, H is one of $K_{1,2}$, $K_{1,3}$, K_3, $K_2 \vee 2K_1$ and $K_2 \vee 3K_1$.

Assume $H \simeq K_2 \vee 3K_1$. Then G' contains three independent vertices a_1, a_2, a_3 which have two common neighbors b_1 and b_2. Since $N_{G_2}(b_1)$ and $N_{G_2}(b_2)$ have three independent vertices, $\{b_1, b_2\} \subset \{v_1, v_2, v_3, v_4\}$. By symmetry, we may assume $b_1 = v_1$ and $b_2 = v_2$. However, $N_{G_2}(v_1) \cap N_{G_2}(v_2) = \{v_3, v_4\} \cup V(H_{1,2})$, which does not contain three independent vertices. This is a contradiction and hence $H \not\simeq K_2 \vee 3K_1$. ∎

Theorem 4 gives us four candidates as a forbidden subgraph which guarantees the existence of a perfect or a near-perfect matching in the class of connected and locally-connected graphs. For $K_{1,2}$ and $K_{1,3}$, Theorem 2 says that they force the existence of a perfect or a near-perfect matching and that the assumption of local connectedness is not necessary. Moreover, the class of connected $K_{1,2}$-free graphs coincides with the class of complete graphs, in which the problem is trivial.

Next consider K_3-free graphs. Since the neighborhood of every vertex in a K_3-free graph is independent, the degree of a locally-connected vertex in a K_3-free graph is one. In particular, the class of connected and locally-connected K_3-free graphs consists only of K_2. Thus, the problem is again trivial.

The remaining class is the class of $(K_2 \vee 2K_1)$-free graphs. We will prove that the problem is again trivial. But we first make a discussion in a general setting, aiming at a topic in the next section.

Let G be a graph and let H be a (not necessarily connected) graph of order at least three. Then we say that G is *locally H-free* if $N_G(v)$ induces an H-free graph for every $v \in V(G)$. The following is a trivial but useful observation.

Theorem 5. *Let H be a graph of order at least two and let G be a graph. Then G is locally H-free if and only if G is $(H \vee K_1)$-free.*

Recall that a locally-connected vertex is called eligible if its neighborhood does not induce a complete graph in G.

By Theorem 5, G is $(K_2 \vee 2K_1)$-free if and only if G is locally $K_{1,2}$-free. However, since every component of a $K_{1,2}$-free graph is a complete graph, we have the following corollaries.

Corollary 6. *No vertex in a $(K_2 \vee 2K_1)$-free graph is eligible.*

Corollary 7. *A connected and locally-connected $(K_2 \vee 2K_1)$-free graph is a complete graph.*

As a conclusion of the above discussion, we see that the claw is the only forbidden subgraph that forces a connected and locally-connected graph to have a perfect or a near-perfect matching in a non-trivial manner. Moreover, in the class of claw-free graphs, the assumption of local connectedness is redundant.

3. The Ryjáček Closure

Given a connected graph H of order at least three, let $S(H)$ be a statement on H stated in the set of finite graphs. Also let $S_A^d(H)$ be the following specific statement.

- $S_A^d(H)$ ="There exists a positive integer n_0 such that every connected and locally-connected H-free graph satisfies $\mathrm{def}(G) \le 1$."

Suppose $S(H)$ implies $S_A^d(H)$ and $S(H)$ is a true statement. Then $S_A^d(H)$ is also a true statement and hence by Theorem 4, $H \in \{K_{1,2}, K_{1,3}, K_3, K_2 \vee 2K_1\}$. Therefore, we have the following immediate corollary.

Corollary 8. *Let H be a connected graph of order at least three, and let $S(H)$ be a statement on H stated in the set of finite graphs. If $S(H)$ implies $S_A^d(H)$ and $S(H)$ is a true statement, then H is isomorphic to $K_{1,2}$, $K_{1,3}$, K_3 or $K_2 \vee 2K_1$.*

Again, let H be a finite connected graph of order at least three. Consider the following five statements.

- $S_A^t(H)$ = "There exists a positive integer n_0 such that every connected and locally-connected graph of order at least n_0 is traceable."
- $S_A^h(H)$ = "There exists a positive integer n_0 such that every connected and locally-connected graph of order at least n_0 is Hamiltonian."
- $S_C^d(H)$ = "There exists a positive integer n_0 such that for every connected graph G of order at least n_0, $\mathrm{def}(G) \leq 1$ if and only if $\mathrm{def}(\mathrm{cl_R}(G)) \leq 1$."
- $S_C^t(H)$ = "There exists a positive integer n_0 such that for every connected graph G of order at least n_0, G is traceable if and only if $\mathrm{cl_R}(G)$ is traceable."
- $S_C^h(H)$ = "There exists a positive integer n_0 such that for every connected graph G of order at least n_0, G is Hamiltonian if and only if $\mathrm{cl_R}(G)$ is Hamiltonian."

Then we have the following implications.

- $S_A^h(H) \Longrightarrow S_A^t(H) \Longrightarrow S_A^d(H)$,
- $S_C^h(H) \Longrightarrow S_A^h(H)$, $S_C^t(H) \Longrightarrow S_A^t(H)$, and $S_C^d(H) \Longrightarrow S_A^d(H)$.

Therefore, by Corollary 8, if $S_C^h(H)$ holds, then $H \in \{K_{1,2}, K_{1,3}, K_3, K_2 \vee 2K_1\}$. This means that if we try to prove the existence of a Hamiltonian cycle in the class of H-free graphs through the Ryjáček closure, H must be one of the four graphs above. However, as we have seen in the previous section, if $H \in \{K_{1,2}, K_3, K_2 \vee 2K_1\}$, then an H-free graph G does not contain an eligible vertex. Therefore, $\mathrm{cl_R}(G) = G$ and the Ryjáček closure does not work.

4. Conclusion

In this note, we have studied the relationship among various Hamiltonian properties, forbidden subgraphs and the Ryjáček closure. The main result is summarized as follows.

- Let H be a connected graph of order at least three. Then the statement $S_C^h(H)$ holds if and only if $H \in \{K_{1,2}, K_{1,3}, K_3, K_2 \vee 2K_1\}$.
- If $H \in \{K_{1,2}, K_3, K_2 \vee 2K_1\}$ and G is an H-free graph, then $\mathrm{cl_R}(G) = G$.

Therefore, if we study the existence of a Hamiltonian cycle in a class of graphs defined by a forbidden subgraph, the class of claw-free graphs is the only one that gives a non-trivial result by the use of the Ryjáček closure. We have also shown that this is the case even if we instead consider the existence of Hamiltonian path and a perfect/near-perfect matching.

We have also proved that a sufficiently large connected and locally-connected H-free graph G is Hamiltonian if and only if $H \in \{K_{1,2}, K_{1,3}, K_3, K_2 \vee 2K_1\}$. And even if we consider a weaker statement that a sufficiently large connected and locally-connected graph G satisfies $\mathrm{def}(G) \leq 1$, the same set of forbidden subgraphs appears. It says that in the class of connected and locally-connected graphs, the difference between the existence of a Hamiltonian cycle and that of a perfect/near-perfect matching cannot be recognized through a forbidden subgraph. On the other hand, the assumption of local connectedness is affected. While not every connected claw-free graph is Hamiltonian, every connected claw-free graph G satisfies $\mathrm{def}(G) \leq 1$.

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