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A Novel Alternative Algorithm for Solving Integer Linear Programming Problems Having Three Variables

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***Abstract**: In this study, a novel alternative method based on parameterization for solving Integer Linear Programming (ILP) problems having three variables is developed. This method, which is better than the cutting plane and branch boundary method, can be applied to pure integer linear programming problems with m linear inequality constraints, a linear objective function with three variables. Both easy to understand and to apply, the method provides an effective tool for solving three variable integer linear programming problems. The method proposed here is not only easy to understand and apply, it is also highly reliable, and there are no computational difficulties faced by other methods used to solve the three-variable pure integer linear programming problem. Numerical examples are provided to demonstrate the ease, effectiveness and reliability of the proposed algorithm.*

***Keywords**: Linear Integer Programming (LIP), linear Diophantine equations, optimal hyperplane, pure integer programming problems, optimal solution.*

1. Introduction

Linear Programming (LP) is a part of mathematical programming where the object function and constraints and decision variables are written as linear functions. Simplex algorithm which is an effective algorithm is used in solving linear programming problems. In real life, linear programming problems are often encountered, whose solution should be integers. Such problems can be formulated as integer optimization problems. In such problems, rounding the solutions obtained from the LP solution to the nearest integers can make the solution far from the original optimum value and even give inappropriate solutions. For this reason, Linear Integer Programming (LIP) is used for the solution of optimization problems where the objective function and constraints are linear. Optimization of LIP problems is more difficult both theoretically and practically than LP problems. Moreover, LIP problems have been impressive for many scientists.

Many researchers have proposed different algorithms for the solution of LIP problems. In [2], a parametric approach to the general integer programming problem was explored. This approach guarantees a feasible solution in a reasonable time frame. Further, such a technique can be used to provide quick lower bound information for an optimal search procedure. In [3] a new method was given, called variable reduction method for a classes of pure integer linear programming problems in single stage. The variable reduction method is based on simple mathematical concepts. In [4], a general integer cut to exclude the previous solution was proposed and an algorithm to identify all alternative optimal solutions of an ILP was presented. In [5], considering a general daily staff scheduling problem with hourly requirement models, it is formulated into an integer linear programming problem. In [6], a survey of methods and approaches is presented to solve linear integer problems, developed during the last 50 years. These problems belong to the class of NP-hard optimization problems. In [7], an algorithm was developed for solving large scale integer program relying on column generation method and the algorithm for solving Capital budgeting and scheduling type problems was implemented. In [8], a method was given to solve an ILP by describing whether an approximated integer solution to the RLP is an optimal solution to the ILP and moreover, a test was provided to examine the optimality of the approximated solution obtained from RLP to be an optimal solution of the ILP, using the concept of linear Diophantine equations. In [10] a new algorithm for solving Integer Programming (IP) problems was proposed that is based on ideas from algebraic geometry. In [11] a new procedure based on the conjugate gradient projection method is given for solving the integer linear programming problem when the objective function is a linear function and the set of constraints is in the form of linear inequality constraints. By constructing an increasing mapping satisfying certain properties an alternative method called fixed-point iteration is developed in [12], for integer programming. In [13] an evolutionary algorithm is introduced for the solution of pure integer linear programs. All the variables of the problem are fixed by the evolutionary system. In [14] an iterative alternative algorithm based on parametrization has been proposed in order to overcome the pure LIP problems having two variables. In practice, it is important to develop the algorithm given in [14] to solve the LIP problem having three and more than three variables.

The aim of this study is to develop an algorithm based on parameterization that solves LIP problems having three variables. While this algorithm can easily solve LIP problems having three variables, it can also offer alternative solutions to decision maker.

This paper is organized as follows: Section 2 presents required information. In Section 3, the proposed approach is handled. Section 4 and Section 5 consist of our numerical examples and conclusions, respectively.

2. Preliminaries

In this section, brief requried information are presented.

Definition 1 [9]. The mathematical formulation of an ILP problem is described below:

$$P_1 : \begin{cases} \max(\min) \sum_{j=1}^{n} c_j x_j, \\ \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_j \le b_i, \\ x_j \ge 0 \text{ and integer, where } i = 1, 2, \dots, m, \text{ and } j = 1, 2, \dots, n. \end{cases}$$

Definition 2. Consider the objective hyperplane

$$\sum c_j x_j = z,$$

where each $c_j \in Z$ which is a linear Diophantine equation in integers [1]

$$d = gcd(c_j,\ c_j \neq 0, j = 1, 2, \dots, n).$$

It has an integer solution if and only if $d|z$; also, if a linear Diophantine equation has an integer solution; then there will be infinitely many solution for this equation [1].

Theorem 1. $(x_1, x_2, \dots, x_n, z)$ is a solution of the problem if and only if $(x_1, x_2, \dots, x_n)$ satisfies all constraints of P_1.

Theorem 2. Let z be an integer. Let S denote the set of all feasible solutions to the ILP. If $S \cap \{x / cx = z\}$ is non empty, then the optimum solution to the ILP will lie on the hyperplane $cx = z$ [1].

Our algorithm is based on the parametrization of the Diophantine equation, which is formed by the equalization of the optimal value obtained from the LP solution to the objective function.

3. A proposed Algorithm for Solving the Integer Linear Programming Problem Having Three Variables

It is known that the maximum value *z* for LIP cannot be greater than the maximum value *z* of the relaxed LP.

Optimum value *z* for LIP $\le$ optimum value *z* for relaxed LP.

This means that the optimal value *z* of relaxed LP is an upper limit for LIP. As stated in Section 2 of this study, we first make a parameterization by equalizing the integer part of the optimum value obtained from the solution of the relaxed LP problem to the objective function. Thus, it is checked whether the largest (smallest) integer obtained from the relaxed LP problem in the maximization problem (minimization problem) is an optimum value for the LIP problem. In addition, in our algorithm, by starting from the optimal integer value obtained from the relaxed LP solution, it is checked whether each integer is the optimum value by decreasing (increasing) one unit for the maximization problem (for the minimization problem) in each loop. Thus, our algorithm guarantees an optimum integer solution.

The Algorithm for Solving an ILP Problem Having Three Variables is given below.

Step 0. Load the LP problem P_1.

Step 1. Solve the relaxed LP problem P_1 to find the optimal solution.

Step 2. If the optimal point (x_1 , x_2, x_3) is integer, it is the solution. Stop. Otherwise, go to Step 3.

Step 3. Assign the integer part of the optimal value Z of P_1 to the expression $\sum_{j=1}^{3} c_j x_j$.

Step 4. Select any two variables in $\sum_{j=1}^{3} c_j x_j = z$ objective function and express them parametrically. Specify the other third variable from $\sum_{j=1}^{3} c_j x_j = z$ objective function according to these two parametric variables.

Step 5. Reconstruct the constraints parametrically.

Step 6. Draw the restructured constraints obtained according to the two selected parametric variables. Get a solution area from these graphics.

Step 7. If there is at least one integer pair belonging to this solution area obtained, go to Step 8. Otherwise, decrease (increase) the optimal value z one unit and to find new optimal value go to Step 3.

Step 8. Calculate the third variable using integer pair(s) that belong to the solution area. If there is at least one integer pair that makes the third variable an integer – if yes, the determined (x_1 , x_2, x_3) is an optimal solution, Stop. If there is more than one integer triplet belonging to the solution area, they will be alternative solutions. Otherwise, decrease (increase) the optimal value z and to find new optimal value go to Step 3.

An algorithm having easy implementation to solve integer linear programming model having three variables is developed.

4. Numerical experiment

Example 1. Consider the following ILP problem having three variables constraints and an objective function in three variables.

Step 0.

$$P_1\text{: min } 2x_1 - 4x_2 + 3x_3,$$

subject to

$$5x_1 - 6x_2 + 2x_3 \geq 5,$$
$$-x_1 + 3x_2 + 5x_3 \geq 8,$$
$$2x_1 + 5x_2 - 4x_3 \leq 4,$$
$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0 \text{ and integers.}$$

Step 1. If the relaxed LP problem is solved, $P_1: (x_1 , x_2, x_3) = (1.81 , 1.11, 1.93)$ and $z = 3$ optimum value is obtained.

Step 2. There is no integer solution , go to Step 3.

Step 3. $z = 3$ assigned to $2x_1 - 4x_2 + 3x_3$.

Step 4. By assigning $x_2 = m,\ x_3 = n,\ x_1 = \frac{3+4m-3n}{2}$ is obtained.

Step 5. Considering the parametric variables, the constraints are reconstruct as follows:

$$-8m + 11n \leq 5,$$
$$2m + 13n \geq 19\,,$$
$$9m - 7n \leq 1.$$

Step 6. Restructured constraints can be considered as a system of inequality and plotted. Thus, the solution area is obtained.

Step 7. Since the solution set obtained is an empty set, the optimal *z* is increased by one unit: i.e., *z* is taken as 4; and go to Step 3.

If optimal *z* is taken as 4, it is seen that the reconstructed constraints do not have any integer pair belonging to the solution area. Optimal *z* is increased one unit; i.e., optimal *z* is taken as 5 and if you go to Step 3; thus (1, 2) integer solution is determined from $2x_1 - 4x_2 + 3x_3 = 5$,

$$-8m + 11n \leq 15,$$
to $2m + 13n \geq 21,$
$$-9m + 7n \geq 1.$$

Step 8. By assigning *m* = 1 and *n* = 2, $x_1 = 3/2$ is obtained from $x_1 = \frac{5+4m-3n}{2}$. Since $(3/2\,,\ 1,\ 2)$ is not an integer solution, the optimal value *z* is increased one unit; i.e., *z* is taken as 6; and go to Step 3.

Thus, the constraints

$$-8m + 11n \leq 20,$$
$$2m + 13n \geq 22,$$
$$-9m + 7n \geq 2,$$

are determined from $2x_1 - 4x_2 + 3x_3 = 6$. The solution area of this inequality system has integer pair solution (1, 2). By assigning *m*= 1 and *n* = 2, $x_1 = 2$ is obtained from $x_1 = \frac{6+4m-3n}{2}$.

It is clear that (2, 1, 2) produces an integer solution which gives the optimal value as 6. Summarized results of Example 1 using the proposed algorithm are given in Table 1.

Table1. Summarized results of Example 1

Optimal z	x_1	Reconstructed constraints	Is there an integer pair belonging to the solution area?	x_1	Is x_1 an integer?	Is there an integer pair belonging to the solution area?
3	$\frac{3+4m-3n}{2}$	$-8m + 11n \leq 5$, $2m + 13n \geq 19$, $9m - 7n \leq 1$	No	-	No	No
4	$\frac{4+4m-3n}{2}$	$-8m + 11n \leq 10$, $2m + 13n \geq 20$, $9m - 7n \leq 0$	No	-	No	No
5	$\frac{5+4m-3n}{2}$	$-8m + 11n \leq 15$, $2m + 13n \geq 21$, $-9m + 7n \geq 1$	(1, 2)	3/2	No	No
6	$\frac{6+4m-3n}{2}$	$-8m + 11n \leq 20$, $2m + 13n \geq 22$, $9m - 7n \geq 2$	(1, 2)	2	Yes	**(2, 1, 2) (Integer solution)**

Example 2. Solve the following LP problem:

Step 0.

$$P_1: \quad \max \ 3x_1 + x_2 + 3x_3,$$

subject to

$$-x_1 + 2x_2 + x_3 \leq 4,$$
$$4x_2 - 3x_3 \leq 2,$$
$$x_1 - 3x_2 + 2x_3 \leq 3,$$
$$x_1 \geq 0, \ x_2 \geq 0, \ x_3 \geq 0 \text{ and integers.}$$

Step 1. By solving the relaxed LP problem

$$P_1: (x_1 , \ x_2, \ x_3) = (5.33 , \ 3, \ 3.33),$$

and optimal value $z = 29$ is obtained.

Step 2. There is no integer solution, go to Step 3.

Step 3. $z = 29$ assigned to $3x_1 + x_2 + 3x_3$.

Step 4. By assigning $x_1 = m, \ x_3 = n, \ x_2 = 29 - 3m - 3n$ is obtained.

Step 5. Considering the parametric variables, the constraints are reconstruct as follows:

$$7m + 5n \geq 54,$$
$$12m + 15n \geq 114,$$
$$10m + 11n \leq 90.$$

Step 6. Restructured constraints can be considered as a system of inequality and plotted. Thus, the solution area is obtained.

Step 7. Since the solution set obtained is an empty set, the optimal *z* is decreased by one unit: i.e., *z* is taken as 28; and go to Step 3.

If the Optimal *z* 28 is taken, it is seen that the inequality system obtained has no integer pair(s) belonging to the solution set; *z* is decreased by one unit; i.e., optimal *z* is taken as 27; and go to Step 3. If optimal *z* is taken 26, 25, 24, respectively, by decreased one unit at a time and go to Step 3, respectively, it is seen that there are no integer pair in the solution area. If optimal *z* is taken as 23; and go to Step 3. Thus, the constraints

$$7m + 5n \geq 42,$$
$$12m + 15n \geq 90,$$
$$10m + 11n \leq 82$$

are determined from $3x_1 + x_2 + 3x_3 = 23$. The solution area of this inequality system has (5, 2) integer pair solution.

Step 8. By assigning *m* = 5 and *n* = 2, $x_2 = 2$ is obtained from

$$23 - 3m - 3n.$$

It is clear that (5, 2, 2) produces an integer solution which gives the optimal value as 23. Summarized results of Example 2 using the proposed algorithm are given in Table 2.

Table 2. Summarized results of Example 2

Optimal z	x_2	Reconstructed constraints	Is there an integer pair belonging to the solution area?	x_2	Is x_2 an integer?	Is there an integer pair belonging to the solution region?
29	$29 - 3m - 3n$	$7m + 5n \geq 54$, $12m + 15n \geq 114$, $10m + 11n \leq 90$	No	–	No	No
28	$28 - 3m - 3n$	$7m + 5n \geq 52$, $12m + 15n \geq 110$, $10m + 11n \leq 87$	No	–	No	No
27	$27 - 3m - 3n$	$7m + 5n \geq 50$, $12m + 15n \geq 106$, $10m + 11n \leq 84$	No	–	No	No
26	$26 - 3m - 3n$	$7m + 5n \geq 48$, $12m + 15n \geq 102$, $10m + 11n \leq 81$	No	–	No	No
25	$25 - 3m - 3n$	$7m + 5n \geq 46$, $12m + 15n \geq 98$, $10m + 11n \leq 78$	No	–	No	No
24	$24 - 3m - 3n$	$7m + 5n \geq 44$, $12m + 15n \geq 94$, $10m + 11n \leq 75$	No	–	No	No
23	$23 - 3m - 3n$	$7m + 5n \geq 42$, $12m + 15n \geq 90$, $10m + 11n \leq 82$	(5, 2)	–	2	**(5, 2, 2) (Integer solution)**

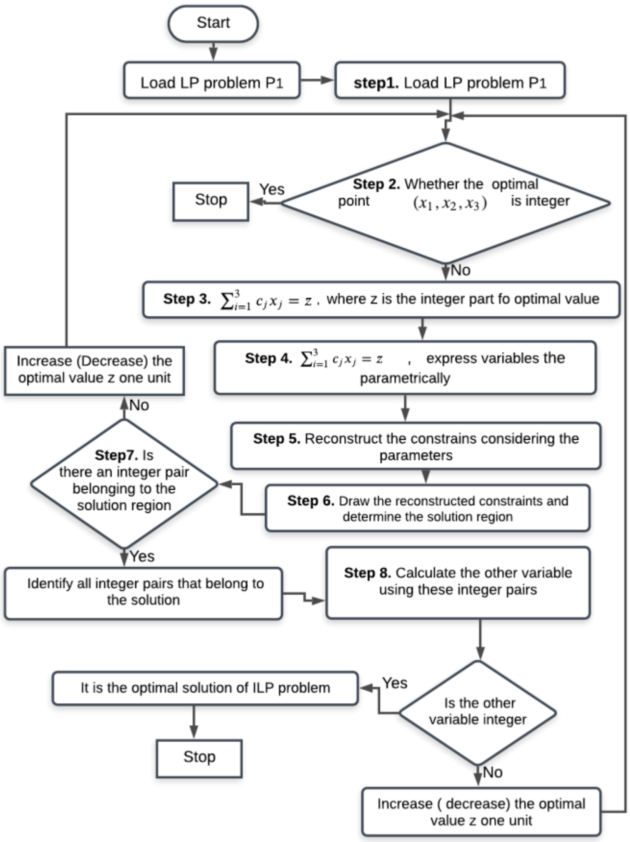

Fig. 1. Algorithm for Integer Linear Programming Model Having Three Variables

5. Conclusion

In this study, an alternative method based on parametrization is given and an algorithm is presented for solving the three variables integer linear programming problem using basic algebraic geometric knowledge. This method is very effectiv and reliable for solving integer linear programming problems having three variables. Numerical examples are given to demonstrate the simplicity, reliability and effectiveness of the method. The advantages of our method, as compared to other methods used in the solution of LIP problems, can be expressed in six steps as it is follow.

1. Branch-boundary method is a method that deals with only a small part of suitable solutions. Therefore, many possible optimal solutions can be excluded. Since our method examines all suitable solutions, no optimal solution is excluded.

2. While the number of constraints increases in the branch-boundary and cutting plane methods, a serious calculation load is encountered, the proposed method easily reaches the solution regardless of the number of constraints.

3. While the graphical method can only be applied to LIP problems having two variables, our algorithm can be applied to LIP problems having three variables.

4. In our method, calculations are very simple and easy.

5. In our method, the number of variables is reduced from three to two. The calculation to be made in this way is less.

6. In this proposed method, no new constraints are added.

In addition, our solution method can be developed for solving of integer linear programming problems having four and more variables and can be used in the solution of real life problems.

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