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On a new approach to the analysis of variance for experiments with orthogonal block structure. III. Experiments in row-column designs

Tadeusz Caliński, Agnieszka Łacka, Idzi Siatkowski

Department of Mathematical and Statistical Methods,
Poznań University of Life Sciences, Wojska Polskiego 28, 60-637 Poznań, Poland,
e-mail: calinski@up.poznan.pl, agnieszka.lacka@up.poznan.pl,
idzi.siatkowski@up.poznan.pl

Summary

The main estimation and hypothesis testing procedures are presented for experiments conducted in row-column designs of a certain desirable type. It is shown that, under appropriate randomization, these experiments have the convenient orthogonal block structure. Due to this property, the analysis of experimental data can be performed in a comparatively simple way. Relevant simplifying procedures are indicated. The main advantage of the presented methodology concerns the analysis of variance and related hypothesis testing procedures. Under the adopted approach one can perform these analytical methods directly, not by combining results from analyses based on some stratum submodels. Practical application of the presented theory is illustrated by four examples of real experiments in the relevant row-column designs. The present paper is the third in the projected series of publications concerning the analysis of experiments with orthogonal block structure.

Key words: analysis of variance, estimation, hypothesis testing, orthogonal block structure, randomization-derived model, row-column designs

1. Introduction

It is well known that experiments with orthogonal block structure (OBS) can be analyzed in a comparatively simple way. This desirable property was originally indicated by Nelder (1965) and then considered by others, particularly Houtman and Speed (1983). Recently the advantage of the OBS property has been reconsidered by Caliński and Siatkowski (2017, 2018), with special attention paid to the analysis of variance (ANOVA). A new approach to the application of ANOVA in experiments with the OBS property has been suggested. This approach provides an advantage over the classical procedure of first conducting the analyses based on stratum submodels and then combining the results obtained from them. In the new approach the ANOVA can be performed directly.

Because practical application of the suggested analytical approach depends on the structure of the experimental design, it was found desirable to present the relevant methodology for different classes of designs separately. Thus, a set of research papers focused on practical applications of the new approach has been planned. The first paper (Caliński and Siatkowski, 2017) is devoted to experiments conducted in proper block designs. The second (Caliński and Siatkowski, 2018) is devoted to experiments in nested block designs. Of course, in both papers, the subject matter is confined to experiments with the OBS property. In both of these papers the exact definition of this property is recalled. The present paper, as the third in this series of publications, is devoted to experiments conducted in row-column designs inducing the OBS property.

Row-column designs are used frequently in various research areas, particularly when the researcher is interested in the elimination of local heterogeneity in two directions, as indicated, for example, by Singh and Dey (1978). The application of row-column designs has a long history, starting from the idea of Latin squares, with combinatorial properties attributed to Euler (1782), as noted by Hinkelmann and Kempthorne (2008, p. 374), who also indicated the first applications of Latin square designs in agricultural experiments. The statistical properties of row-column designs have been considered in many publications, as reviewed by Siatkowski (2004).

The purpose of the present paper is to show how the OBS property of an experiment conducted in a row-column design provides the possibility of performing the analysis of experimental data with a comparatively simple methodology (as shown in the first two papers of the present series). Following this, in Section 2 the randomization-derived mixed model, from which the described methodology follows, is indicated. The theoretical background of the derived analysis is presented in Section 3. In Section 4 some simplifications of the adopted analytical methods are presented. In Section 5 attention is drawn to some consequences resulting from the use of estimated stratum variances. Examples illustrating the application of the derived analytical methods, ANOVA in particular, are presented in Section

6. Relevant concluding remarks concerning the advantages of the proposed approach are given in Section 7. Finally, several appendices with helpful derivations of the applied methods are provided.

2. A randomization-derived model

Consider an experiment carried out in a row-column design with v treatments allocated in $n = rc$ units (plots) which are grouped, perpendicularly, into r rows of c units and c columns of r units. Such a row-column design is said to induce the OBS property.

Suppose that independent randomizations of rows and of columns have been implemented in the experiment according to the usual procedure (following Nelder, 1965). The randomization-derived model can then be written as

$$\boldsymbol{y} = \boldsymbol{X}_1\boldsymbol{\tau} + \boldsymbol{X}_R\boldsymbol{\rho} + \boldsymbol{X}_C\boldsymbol{\gamma} + \boldsymbol{\eta} + \boldsymbol{e}, \tag{1}$$

where $\boldsymbol{y} = [\boldsymbol{y}_1', \boldsymbol{y}_2', ..., \boldsymbol{y}_r']'$ is an $n \times 1$ vector of data concerning yield (or another variable trait) observed on $n = rc$ plots of the experiment, $\boldsymbol{y}_h = [y_{1h}, y_{2h}, ..., y_{ch}]'$ represents the yields observed on c units of the row h $(= 1, 2, ..., r)$,

$$\boldsymbol{X}_1 = [\boldsymbol{X}_{11}' : \boldsymbol{X}_{12}' : \cdots : \boldsymbol{X}_{1r}']', \ \boldsymbol{X}_R = \boldsymbol{I}_r \otimes \boldsymbol{1}_c \ , \ \boldsymbol{X}_C = \boldsymbol{1}_r \otimes \boldsymbol{I}_c \tag{2}$$

are the known design matrices, and $\boldsymbol{\tau} = [\tau_1, \tau_2, ..., \tau_v]'$ represents the unobservable treatment parameters (their fixed effects), $\boldsymbol{\rho} = [\rho_1, \rho_2, ..., \rho_r]'$ stands for the row random effects, $\boldsymbol{\gamma} = [\gamma_1, \gamma_2, ..., \gamma_c]'$ stands for the column random effects, while the $n \times 1$ vectors $\boldsymbol{\eta}$ and $\boldsymbol{e}$ stand for the unit error and technical error random variables, all of these random variables being unobservable.

The first design matrix in (2) provides the vector $\boldsymbol{X}_1'\boldsymbol{1}_n = \boldsymbol{r} = [r_1, ..., r_v]'$ of treatment replications. It also supplies the diagonal matrix $\boldsymbol{X}_1'\boldsymbol{X}_1 = \boldsymbol{r}^\delta = \text{diag}[r_1, r_2, ..., r_v]$, used in this form or as its inverse denoted by $\boldsymbol{r}^{-\delta}$.

It may be interesting to note that, on account of (1) and (2), for any row (i.e., any h) one can write

$$\boldsymbol{y}_h = \boldsymbol{X}_{1h}\boldsymbol{\tau} + \boldsymbol{1}_c\rho_h + \boldsymbol{\gamma} + \boldsymbol{\eta}_h + \boldsymbol{e}_h.$$

This provides the product $\boldsymbol{X}_{1h}'[\boldsymbol{1}_c : \boldsymbol{I}_c] = [\boldsymbol{r}_h : \boldsymbol{X}_{1h}']$, in which $\boldsymbol{r}_h$ is the vector of treatment replications in the hth row, and $\boldsymbol{X}_{1h}'$ describes the layout of treatments in that row, i.e., in the hth component of the design. Thus, the whole design can be described by the $v \times (rc)$ incidence matrix

$$\boldsymbol{N} = \boldsymbol{X}_1' = [\boldsymbol{X}_{11}' : \boldsymbol{X}_{12}' : \cdots : \boldsymbol{X}_{1r}'], \tag{3}$$

with rows corresponding to the treatments and columns corresponding to the units (plots) of the row-column design, the units being ordered as described in relation to formula (1).

Because both the rows of the design are of equal size and its columns are of equal size (not necessarily the same size as the rows), an experiment in such a row-column design has, under the randomization-derived model (1), the OBS property (see Example 2 in Houtman and Speed, 1983, Section 2.3). This allows the considered model to be resolved into four simple stratum submodels, in accordance with the stratification of the experimental units. Using Nelder's (1965) notation, this stratification ("block-structure") can be represented by the relation

Units (plots) → (Rows × Columns) → Total experimental area.

Thus, the observed vector $\boldsymbol{y}$ can be decomposed as

$$\boldsymbol{y} = \boldsymbol{y}_1 + \boldsymbol{y}_2 + \boldsymbol{y}_3 + \boldsymbol{y}_4, \quad \text{where}$$
$$\boldsymbol{y}_1 = \boldsymbol{\phi}_1\boldsymbol{y}, \quad \boldsymbol{y}_2 = \boldsymbol{\phi}_2\boldsymbol{y}, \quad \boldsymbol{y}_3 = \boldsymbol{\phi}_3\boldsymbol{y}, \quad \boldsymbol{y}_4 = \boldsymbol{\phi}_4\boldsymbol{y}.$$

This allows the expectation vector and the covariance (dispersion) matrix of $\boldsymbol{y}$ to be written as

$$\mathrm{E}(\boldsymbol{y}) = \boldsymbol{\phi}_1\boldsymbol{X}_1\boldsymbol{\tau} + \boldsymbol{\phi}_2\boldsymbol{X}_1\boldsymbol{\tau} + \boldsymbol{\phi}_3\boldsymbol{X}_1\boldsymbol{\tau} + \boldsymbol{\phi}_4\boldsymbol{X}_1\boldsymbol{\tau} = \boldsymbol{X}_1\boldsymbol{\tau}, \tag{4}$$

$$\mathrm{D}(\boldsymbol{y}) \equiv \boldsymbol{V} = \sigma_1^2\boldsymbol{\phi}_1 + \sigma_2^2\boldsymbol{\phi}_2 + \sigma_3^2\boldsymbol{\phi}_3 + \sigma_4^2\boldsymbol{\phi}_4, \tag{5}$$

where the matrices

$$\boldsymbol{\phi}_1 = \boldsymbol{I}_n - c^{-1}\boldsymbol{X}_R\boldsymbol{X}_R' - r^{-1}\boldsymbol{X}_C\boldsymbol{X}_C' + n^{-1}\mathbf{1}_n\mathbf{1}_n',$$
$$\boldsymbol{\phi}_2 = c^{-1}\boldsymbol{X}_R\boldsymbol{X}_R' - n^{-1}\mathbf{1}_n\mathbf{1}_n', \ \boldsymbol{\phi}_3 = r^{-1}\boldsymbol{X}_C\boldsymbol{X}_C' - n^{-1}\mathbf{1}_n\mathbf{1}_n', \quad \text{and}$$
$$\boldsymbol{\phi}_4 = n^{-1}\mathbf{1}_n\mathbf{1}_n'$$

are symmetric, idempotent and pairwise orthogonal, summing to the identity matrix, and the scalars $\sigma_1^2, \sigma_2^2, \sigma_3^2$ and σ_4^2 represent the relevant unknown stratum variances (defined as in Houtman and Speed, 1983, Section 2.3).

Such an experiment, in a row-column design, can be extended by the use of a nested row-column design (as introduced by Singh and Dey, 1979, and described, for example, in John, 1987, Chapter 5; I. Mejza and S. Mejza, 1994; Kozłowska, Łacka and Skorupska, 2012). To see this, consider an experiment in a row-column design replicated with the same treatments in b blocks, each composed of r_0 rows and c_0 columns. Such a nested row-column design also induces the OBS property.

Suppose that independent randomizations of blocks, as well as of rows and of columns within the blocks, have been implemented in the experiment according to the usual procedure (suggested by Nelder, 1965). The randomization-derived model can then be written as

$$\boldsymbol{y} = \boldsymbol{X}_1\boldsymbol{\tau} + \boldsymbol{X}_B\boldsymbol{\beta} + \boldsymbol{X}_{R(B)}\boldsymbol{\rho} + \boldsymbol{X}_{C(B)}\boldsymbol{\gamma} + \boldsymbol{\eta} + \boldsymbol{e}, \tag{6}$$

where $\boldsymbol{y} = [\boldsymbol{y}_1', \boldsymbol{y}_2', ..., \boldsymbol{y}_b']'$ is an $n \times 1$ vector of data concerning yield (or another variable trait) observed on $n = br_0c_0$ plots of the experiment, $\boldsymbol{y}_g = [y_{1g}, y_{2g}, ..., y_{n_0g}]'$ representing the yields observed on $n_0 = r_0c_0$ units (plots) of the block g $(= 1, 2, ..., b)$ of the experiment, grouped (perpendicularly) into r_0 rows and c_0 columns. Furthermore,

$$\boldsymbol{X}_1 = [\boldsymbol{X}_{11}' : \boldsymbol{X}_{12}' : \cdots : \boldsymbol{X}_{1b}']', \quad \boldsymbol{X}_B = \boldsymbol{I}_b \otimes \boldsymbol{1}_{n_0},$$

$$\boldsymbol{X}_{R(B)} = \boldsymbol{I}_b \otimes \boldsymbol{I}_{r_0} \otimes \boldsymbol{1}_{c_0}, \quad \boldsymbol{X}_{C(B)} = \boldsymbol{I}_b \otimes \boldsymbol{1}_{r_0} \otimes \boldsymbol{I}_{c_0}$$

are the known design matrices, and $\boldsymbol{\tau} = [\tau_1, \tau_2, ..., \tau_v]'$ represents the unobservable treatment parameters (fixed effects), $\boldsymbol{\beta} = [\beta_1, \beta_2, ..., \beta_b]'$ stands for the block random effects, $\boldsymbol{\rho} = [\boldsymbol{\rho}_1', ..., \boldsymbol{\rho}_b']'$, with $\boldsymbol{\rho}_g = [\rho_{1(g)}, ..., \rho_{r_0(g)}]'$, stands for the row random effects, $\boldsymbol{\gamma} = [\boldsymbol{\gamma}_1', ..., \boldsymbol{\gamma}_b']'$, with $\boldsymbol{\gamma}_g = [\gamma_{1(g)}, ..., \gamma_{c_0(g)}]'$, stands for the column random effects, while the $n \times 1$ vectors $\boldsymbol{\eta}$ and $\boldsymbol{e}$ stand for the unit error and technical error random variables, all of these random variables being unobservable.

Suppose that each of the data vectors $\{\boldsymbol{y}_1, \boldsymbol{y}_2, ..., \boldsymbol{y}_b\}$ introduced in the description of formula (6) is ordered as described in relation to formula (1), i.e., according to the design rows. Then, taking into account all $r = br_0$ rows simultaneously, the resulting combined row-column design, denoted by $\mathcal{D}^*$, can be described by the $v \times (bn_0)$ incidence matrix

$$\boldsymbol{N} = [\boldsymbol{X}_{11}' : \boldsymbol{X}_{12}' : \cdots : \boldsymbol{X}_{1b}'] = [\boldsymbol{N}_1 : \boldsymbol{N}_2 : \cdots : \boldsymbol{N}_b],$$

with $\boldsymbol{N}_g = \boldsymbol{X}_{1g}'$ as the $v \times n_0$ incidence matrix describing, as in formula (3), the gth component design, denoted by $\mathcal{D}_g$ $(g = 1, 2, ..., b)$. Note that $\boldsymbol{N}_g\boldsymbol{1}_{n_0} = \boldsymbol{r}_g$, the vector of treatment replications in $\mathcal{D}_g$. Furthermore, note that the design, denoted by $\mathcal{D}$, according to which the v treatments are assigned to the b blocks is described by the $v \times b$ incidence matrix

$$\boldsymbol{M} = \boldsymbol{X}_1'\boldsymbol{X}_B = [\boldsymbol{X}_{11}' : \boldsymbol{X}_{12}' : \cdots : \boldsymbol{X}_{1b}'](\boldsymbol{I}_b \otimes \boldsymbol{1}_{n_0}) = [\boldsymbol{r}_1 : \boldsymbol{r}_2 : \cdots : \boldsymbol{r}_b].$$

Because $\mathcal{D}^*$ has all rows of equal size and all columns of equal size, and also $\mathcal{D}$ has all blocks of equal size, an experiment in a nested row-column design of this type has, under the randomization-derived model (6), the OBS property. This allows the model (6) to be resolved into five simple stratum submodels, in accordance with the stratification of the experimental units. Following the notation used by Nelder (1965), this can be represented by the relation

Units (plots) $\rightarrow$ (Rows $\times$ Columns) $\rightarrow$ Blocks $\rightarrow$ Total experimental area.

Thus, the data vector $\boldsymbol{y}$ can be decomposed as

$$\boldsymbol{y} = \boldsymbol{y}_1 + \boldsymbol{y}_2 + \boldsymbol{y}_3 + \boldsymbol{y}_4 + \boldsymbol{y}_5,$$

where

$$\boldsymbol{y}_1 = \boldsymbol{\phi}_1\boldsymbol{y}, \quad \boldsymbol{y}_2 = \boldsymbol{\phi}_2\boldsymbol{y}, \quad \boldsymbol{y}_3 = \boldsymbol{\phi}_3\boldsymbol{y}, \quad \boldsymbol{y}_4 = \boldsymbol{\phi}_4\boldsymbol{y}, \quad \boldsymbol{y}_5 = \boldsymbol{\phi}_5\boldsymbol{y}.$$

This allows the expectation vector and the covariance (dispersion) matrix of $\boldsymbol{y}$ to be written as

$$\mathrm{E}(\boldsymbol{y}) = \boldsymbol{\phi}_1\boldsymbol{X}_1\boldsymbol{\tau} + \boldsymbol{\phi}_2\boldsymbol{X}_1\boldsymbol{\tau} + \boldsymbol{\phi}_3\boldsymbol{X}_1\boldsymbol{\tau} + \boldsymbol{\phi}_4\boldsymbol{X}_1\boldsymbol{\tau} + \boldsymbol{\phi}_5\boldsymbol{X}_1\boldsymbol{\tau} = \boldsymbol{X}_1\boldsymbol{\tau}, \quad (7)$$

$$\mathrm{D}(\boldsymbol{y}) \equiv \boldsymbol{V} = \sigma_1^2\boldsymbol{\phi}_1 + \sigma_2^2\boldsymbol{\phi}_2 + \sigma_3^2\boldsymbol{\phi}_3 + \sigma_4^2\boldsymbol{\phi}_4 + \sigma_5^2\boldsymbol{\phi}_5, \quad (8)$$

where the matrices

$$\boldsymbol{\phi}_1 = \boldsymbol{I}_n - c_0^{-1}\boldsymbol{X}_{R(B)}\boldsymbol{X}_{R(B)}' - r_0^{-1}\boldsymbol{X}_{C(B)}\boldsymbol{X}_{C(B)}' + n_0^{-1}\boldsymbol{X}_B\boldsymbol{X}_B',$$

$$\boldsymbol{\phi}_2 = c_0^{-1}\boldsymbol{X}_{R(B)}\boldsymbol{X}_{R(B)}' - n_0^{-1}\boldsymbol{X}_B\boldsymbol{X}_B',$$

$$\boldsymbol{\phi}_3 = r_0^{-1}\boldsymbol{X}_{C(B)}\boldsymbol{X}_{C(B)}' - n_0^{-1}\boldsymbol{X}_B\boldsymbol{X}_B',$$

$$\boldsymbol{\phi}_4 = n_0^{-1}\boldsymbol{X}_B\boldsymbol{X}_B' - n^{-1}\boldsymbol{1}_n\boldsymbol{1}_n' \quad \text{and} \quad \boldsymbol{\phi}_5 = n^{-1}\boldsymbol{1}_n\boldsymbol{1}_n'$$

are symmetric, idempotent and pairwise orthogonal, summing to the identity matrix, and the scalars $\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2$ and σ_5^2 represent the relevant unknown stratum variances (defined similarly as in Houtman and Speed, 1983, Section 2.3, with some extension).

It is worth noting that if $b = 1$, the matrices $\{\boldsymbol{\phi}_\alpha\}$ in (8) are reduced to those in (5).

3. Theoretical background of the analysis

When analyzing data from an experiment modeled by (1) or (6), attention is usually paid to estimates and tests concerning the parameters $\boldsymbol{\tau} = [\tau_1, \tau_2, ..., \tau_v]'$, or rather the treatment main effects, defined as

$$(\boldsymbol{I}_v - n^{-1}\boldsymbol{1}_v\boldsymbol{r}')\boldsymbol{\tau} = [\tau_1 - \tau_., \tau_2 - \tau_., ..., \tau_v - \tau_.]', \quad \text{where} \quad \tau_. = n^{-1}\sum_{i=1}^{v}(r_i\tau_i),$$

and also their linear functions. In connection with this, we note first (referring, for example, to Rao and Kleffe, 1988, Section 1.3) that, taking the orthogonal ($\boldsymbol{V}^{-1}$-orthogonal) projector

$$\boldsymbol{P}_{X_1(V^{-1})} = \boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}^{-1}, \tag{9}$$

one can decompose the analyzed data vector $\boldsymbol{y}$ into two uncorrelated parts, as

$$\boldsymbol{y} = \boldsymbol{P}_{X_1(V^{-1})}\boldsymbol{y} + (\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}. \tag{10}$$

The interesting role of the two parts on the right in (10) can easily be seen.

Under the model (1), with properties (4) and (5), the first term of the partition in (10) provides the best linear unbiased estimator (BLUE) of $\boldsymbol{X}_1\boldsymbol{\tau}$ in (4), which can be expressed as

$$\widehat{\boldsymbol{X}_1\boldsymbol{\tau}} = \boldsymbol{P}_{X_1(V^{-1})}\boldsymbol{y}, \tag{11}$$

as follows from Rao (1974, Theorem 3.2). With regard to the second term in (10), it can be seen as the residual vector, giving the residual sum of squares in the form

$$\begin{aligned} \|(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2_{V^{-1}} &= \boldsymbol{y}'(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})'\boldsymbol{V}^{-1}(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y} \\ &= \boldsymbol{y}'[\boldsymbol{V}^{-1} - \boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}^{-1}]\boldsymbol{y} \\ &= \boldsymbol{y}'\boldsymbol{V}^{-1}(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}, \end{aligned} \tag{12}$$

with the residual degrees of freedom given by $\text{rank}(\boldsymbol{V} : \boldsymbol{X}_1) - \text{rank}(\boldsymbol{X}_1) = n - v$. See Rao (1974, Theorem 3.4) and formula (3.17) there.

For convenience note that, when using the projector (9) in the applications under consideration, the variance σ_4^2 in the involved matrix $\boldsymbol{V}$, defined in (5), can be replaced by 1. This is evident from the application of a relevant spectral decomposition, as indicated in Caliński and Siatkowski (2018, Section 3).

Also note that, since $\boldsymbol{\tau} = \boldsymbol{r}^{-\delta}\boldsymbol{X}_1'\boldsymbol{X}_1\boldsymbol{\tau}$, the BLUE of $\boldsymbol{\tau}$ can be obtained, by (9) and (11), as

$$\hat{\boldsymbol{\tau}} = (\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{y}. \tag{13}$$

Its covariance (dispersion) matrix then takes the form

$$\begin{aligned}\mathrm{D}(\hat{\boldsymbol{\tau}}) &= (\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}^{-1}\mathrm{D}(\boldsymbol{y})\boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\\ &= (\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}.\end{aligned} \tag{14}$$

The results given in (12)$-$(14) can be checked by referring to Theorem 3.1 in Rao (1971). For this one has to show that the equality

$$\begin{bmatrix} \boldsymbol{V} & \boldsymbol{X}_1 \\ \boldsymbol{X}_1' & \mathbf{O} \end{bmatrix}^{-} = \begin{bmatrix} \boldsymbol{V}^{-1}(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})}) & \boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1} \\ (\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}^{-1} & -(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1} \end{bmatrix}$$

holds, which can easily be checked.

With these results, the concept for testing the hypothesis

$$H_0 : (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')\boldsymbol{\tau} = \mathbf{0} \tag{15}$$

can be considered. First one has to check whether the hypothesis (15) is consistent. For this, note that the BLUE of $\boldsymbol{\tau}_* = (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')\boldsymbol{\tau}$ is $\hat{\boldsymbol{\tau}}_* = (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')\hat{\boldsymbol{\tau}}$, with $\hat{\boldsymbol{\tau}}$ as given in (13). Its dispersion matrix is, by (14), of the form

$$\mathrm{D}(\hat{\boldsymbol{\tau}}_*) = (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}(\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}_v'), \tag{16}$$

with rank $v - 1$. It appears that as a g-inverse of $\mathrm{D}(\hat{\boldsymbol{\tau}}_*)$ one can take $[\mathrm{D}(\hat{\boldsymbol{\tau}}_*)]^- = \boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1$. Hence,

$$\mathrm{D}(\hat{\boldsymbol{\tau}}_*)[\mathrm{D}(\hat{\boldsymbol{\tau}}_*)]^-\hat{\boldsymbol{\tau}}_* = \hat{\boldsymbol{\tau}}_* \tag{17}$$

(see Appendix 1). The equality (17) indicates that H_0 in (15) is consistent; see formula (3.2.8) in Rao (1971).
Assuming now that $\boldsymbol{y} \sim N_n(\boldsymbol{X}_1\boldsymbol{\tau}, \boldsymbol{V})$ and, hence, that $\hat{\boldsymbol{\tau}}_* \sim N_v[\boldsymbol{\tau}_*, \mathrm{D}(\hat{\boldsymbol{\tau}}_*)]$, where $\boldsymbol{\tau}_*$ is as defined above, and $\mathrm{D}(\hat{\boldsymbol{\tau}}_*)$ is as in (16), one can test the hypothesis H_0 using the statistic

$$F = \frac{n-v}{v-1}\,\frac{\mathrm{SS}_V}{\mathrm{SS}_R} = \frac{n-v}{v-1}\,\frac{\hat{\boldsymbol{\tau}}_*'\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1\hat{\boldsymbol{\tau}}_*}{\boldsymbol{y}'\boldsymbol{V}^{-1}(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}}, \tag{18}$$

as follows from Theorem 3.2 in Rao (1971). Note, however, that the sums of squares in (18) can equivalently be written (see Appendix 2) as

$$\begin{aligned}&\mathrm{SS}_V\\&= \boldsymbol{y}'\boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}(\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}_v')\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{y},\end{aligned} \tag{19}$$

$$\mathrm{SS}_R = \boldsymbol{y}'[\boldsymbol{V}^{-1} - \boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}^{-1}]\boldsymbol{y}. \tag{20}$$

Referring now to Theorems 9.2.1 and 9.4.1 in Rao and Mitra (1971), one can show that, independently,

$$\mathrm{SS}_V \sim \chi^2(v-1, \delta), \quad \text{with} \quad \delta = \boldsymbol{\tau}_*'\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1\boldsymbol{\tau}_*, \tag{21}$$

$$\mathrm{SS}_R \sim \chi^2(n-v, 0). \tag{22}$$

Evidently, the distribution in (21) is central if H_0 is true, whereas that in (22) is central whether H_0 is true or not. These results imply that the statistic (18) has a noncentral F distribution with $v-1$ and $n-v$ d.f., and with the noncentrality parameter δ as in (21). Thus, the distribution is central if H_0 is true.

It should be noted, however, that the above estimation and hypothesis testing procedures are applicable directly if the stratum variances σ_1^2, σ_2^2, σ_3^2 and σ_4^2 are known. In practice they are usually unknown and have to be estimated. For this, it will be helpful to return to formula (12), writing it as

$$\begin{aligned}&\|(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2_{V^{-1}}\\&\quad= \boldsymbol{y}'(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})'\boldsymbol{V}^{-1}(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\\&\quad= \sigma_1^{-2}\boldsymbol{y}'(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})'\boldsymbol{\phi}_1(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\\&\qquad+ \sigma_2^{-2}\boldsymbol{y}'(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})'\boldsymbol{\phi}_2(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\\&\qquad+ \sigma_3^{-2}\boldsymbol{y}'(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})'\boldsymbol{\phi}_3(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y},\end{aligned} \tag{23}$$

which follows from the form of $\mathrm{D}(\boldsymbol{y}) \equiv \boldsymbol{V}$, given in (5). This form also implies, on account of the relation $\boldsymbol{\phi}_4 = n^{-1}\mathbf{1}_n\mathbf{1}_n' = n^{-1}\mathbf{1}_n\mathbf{1}_v'\boldsymbol{X}_1'$, that

$$\boldsymbol{\phi}_4(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})}) = \sigma_4^2\boldsymbol{\phi}_4\boldsymbol{V}^{-1}(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})}) = \mathbf{O}.$$

Now, from (23), one can write

$$\begin{aligned}\mathrm{E}\{\|(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2_{V^{-1}}\} &= \sigma_1^{-2}\mathrm{E}\{\|\boldsymbol{\phi}_1(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2\}\\&\quad+ \sigma_2^{-2}\mathrm{E}\{\|\boldsymbol{\phi}_2(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2\}\\&\quad+ \sigma_3^{-2}\mathrm{E}\{\|\boldsymbol{\phi}_3(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2\}\\&= d_1' + d_2' + d_3' = n - v,\end{aligned} \tag{24}$$

because, as can be shown,

$$\mathrm{E}\{\|\boldsymbol{\phi}_1(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2\} = \sigma_1^2 d_1', \tag{25}$$

where $d_1' = \mathrm{tr}[\boldsymbol{\phi}_1(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})]$,

$$\mathrm{E}\{\|\boldsymbol{\phi}_2(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2\} = \sigma_2^2 d_2', \tag{26}$$

where $d_2' = \mathrm{tr}[\boldsymbol{\phi}_2(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})]$,

$$\mathrm{E}\{\|\boldsymbol{\phi}_3(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2\} = \sigma_3^2 d_3', \tag{27}$$

where $d_3' = \mathrm{tr}[\boldsymbol{\phi}_3(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})]$.

Thus, it is natural to use as estimators of σ_1^2, σ_2^2 and σ_3^2 the solutions of the equations

$$\|\boldsymbol{\phi}_1(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2 = \sigma_1^2 d_1', \tag{28}$$

$$\|\boldsymbol{\phi}_2(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2 = \sigma_2^2 d_2', \tag{29}$$

$$\|\boldsymbol{\phi}_3(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2 = \sigma_3^2 d_3', \tag{30}$$

respectively, as suggested by Nelder (1968, Section 3). This approach was also advocated by Houtman and Speed (1983, Section 4.5) and applied recently by Caliński and Siatkowski (2018).

For completeness, following Caliński and Siatkowski (2018, Section 3), note that the equations (28), (29) and (30), with the formulae (25), (26) and (27), imply – on account of (24) – that

$$\begin{aligned}\hat{\sigma}_1^{-2}\|\boldsymbol{\phi}_1(\boldsymbol{I}_n - \boldsymbol{P}_{X(V^{-1})})\boldsymbol{y}\|^2 + \hat{\sigma}_2^{-2}\|\boldsymbol{\phi}_2(\boldsymbol{I}_n - \boldsymbol{P}_{X(V^{-1})})\boldsymbol{y}\|^2 \\ + \hat{\sigma}_3^{-2}\|\boldsymbol{\phi}_3(\boldsymbol{I}_n - \boldsymbol{P}_{X(V^{-1})})\boldsymbol{y}\|^2 \\ = d_1' + d_2' + d_3' = n - v.\end{aligned} \tag{31}$$

Now, returning to SS_R as given in (12), note that, on account of the relations $\boldsymbol{V}^{-1} = \sigma_1^{-2}\boldsymbol{\phi}_1 + \sigma_2^{-2}\boldsymbol{\phi}_2 + \sigma_3^{-2}\boldsymbol{\phi}_3 + \sigma_4^{-2}\boldsymbol{\phi}_4$ and $\boldsymbol{\phi}_4(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})}) = \mathbf{O}$, it can be written in the form

$$\begin{aligned}\mathrm{SS}_R = \sigma_1^{-2}\|\boldsymbol{\phi}_1(\boldsymbol{I}_n - \boldsymbol{P}_{X(V^{-1})})\boldsymbol{y}\|^2 + \sigma_2^{-2}\|\boldsymbol{\phi}_2(\boldsymbol{I}_n - \boldsymbol{P}_{X(V^{-1})})\boldsymbol{y}\|^2 \\ + \sigma_3^{-2}\|\boldsymbol{\phi}_3(\boldsymbol{I}_n - \boldsymbol{P}_{X(V^{-1})})\boldsymbol{y}\|^2.\end{aligned} \tag{32}$$

A comparison of formulae (31) and (32) shows that, if the stratum variances are estimated by solutions of the equations (28), (29) and (30), the result

$$\begin{aligned}\widehat{\mathrm{SS}}_R = \hat{\sigma}_1^{-2}\|\boldsymbol{\phi}_1(\boldsymbol{I}_n - \boldsymbol{P}_{X(V^{-1})})\boldsymbol{y}\|^2 + \hat{\sigma}_2^{-2}\|\boldsymbol{\phi}_2(\boldsymbol{I}_n - \boldsymbol{P}_{X(V^{-1})})\boldsymbol{y}\|^2 \\ + \hat{\sigma}_3^{-2}\|\boldsymbol{\phi}_3(\boldsymbol{I}_n - \boldsymbol{P}_{X(V^{-1})})\boldsymbol{y}\|^2 = n - v\end{aligned} \tag{33}$$

then follows. By (33), the statistic F in (18) is reduced to the form

$$\widehat{F} = \frac{n-v}{v-1}\,\frac{\widehat{\mathrm{SS}}_V}{n-v} = \frac{\widehat{\mathrm{SS}}_V}{v-1}, \tag{34}$$

where $\widehat{\mathrm{SS}}_V$ is as in (19), but with σ_1^2, σ_2^2 and σ_3^2 there replaced by their estimates.

However, the χ^2 distribution of SS_V, indicated in (21), is valid exactly only if the true stratum variances are used in the applied matrix $\boldsymbol{V}^{-1} = \sigma_1^{-2}\boldsymbol{\phi}_1 + \sigma_2^{-2}\boldsymbol{\phi}_2 + \sigma_3^{-2}\boldsymbol{\phi}_3 + \sigma_4^{-2}\boldsymbol{\phi}_4$, resulting from (5). As for the component $\sigma_4^{-2}\boldsymbol{\phi}_4$, it does not in fact play any role in the application of formula (19) given for SS_V (as will be shown in the next section). Thus, when using in $\boldsymbol{V}^{-1}$ the estimates of σ_1^2, σ_2^2 and σ_3^2 obtained from (28), (29) and (30) respectively, the distribution (21) can be regarded as approximate only.

Proceeding now to the analysis of data modeled by (6), i.e., data from an experiment conducted in a nested row-column design, one can follow the methodology presented for model (1), taking into account the following changes.

In fact, the modifications are related to the difference in the form of the covariance (dispersion) matrix. For model (1) this matrix is presented in formula (5), and for model (6) in formula (8). One has to be aware of the differences in the number and structure of the matrices $\{\boldsymbol{\phi}_\alpha\}$ there.

Remembering these differences, one can perform the analysis of data from an experiment in a nested row-column design following the methods based on formulae (9)$-$(22) above. The main difference in the methodology begins with the problem of estimating the relevant stratum variances.

Due to the difference between (5) and (8), formula (23) is to be replaced by

$$\begin{aligned}
&\|(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2_{V^{-1}} \\
&\quad = \boldsymbol{y}'(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})'\boldsymbol{V}^{-1}(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y} \\
&\quad = \sigma_1^{-2}\boldsymbol{y}'(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})'\boldsymbol{\phi}_1(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y} \\
&\qquad + \sigma_2^{-2}\boldsymbol{y}'(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})'\boldsymbol{\phi}_2(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y} \\
&\qquad + \sigma_3^{-2}\boldsymbol{y}'(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})'\boldsymbol{\phi}_3(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}, \\
&\qquad + \sigma_4^{-2}\boldsymbol{y}'(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})'\boldsymbol{\phi}_4(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}.
\end{aligned} \tag{35}$$

As before, the supposed 5th component does not appear, because $\boldsymbol{\phi}_5 = n^{-1}\mathbf{1}_n\mathbf{1}_n'$, which implies the equality $\boldsymbol{\phi}_5(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})}) = \mathbf{O}$, as for (23).

Relevant extensions of formulae (24)−(30) can easily be seen, noting that formula (24) now takes the form

$$\begin{aligned}\mathrm{E}\{\|(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2_{V^{-1}}\} &= \sigma_1^{-2}\mathrm{E}\{\|\boldsymbol{\phi}_1(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2\}\\ &\quad + \sigma_2^{-2}\mathrm{E}\{\|\boldsymbol{\phi}_2(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2\}\\ &\quad + \sigma_3^{-2}\mathrm{E}\{\|\boldsymbol{\phi}_3(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2\}\\ &\quad + \sigma_4^{-2}\mathrm{E}\{\|\boldsymbol{\phi}_4(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2\}\\ &= d_1' + d_2' + d_3' + d_4' = n - v, \end{aligned} \tag{36}$$

with d_1', d_2', d_3' as defined in (25), (26), (27), respectively, and with

$$d_4' = \mathrm{tr}[\boldsymbol{\phi}_4(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})].$$

Thus, the required estimates of the stratum variances σ_1^2, σ_2^2, σ_3^2 and σ_4^2 can be obtained by solving the equations

$$\|\boldsymbol{\phi}_1(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2 = \sigma_1^2 d_1', \tag{37}$$

$$\|\boldsymbol{\phi}_2(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2 = \sigma_2^2 d_2', \tag{38}$$

$$\|\boldsymbol{\phi}_3(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2 = \sigma_3^2 d_3', \tag{39}$$

$$\|\boldsymbol{\phi}_4(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2 = \sigma_4^2 d_4', \tag{40}$$

resulting from the same approach as that leading to equations (28)−(30), presented for a simple (not nested) row-column design.

Now, as given earlier in (31)−(33) for a simple row-column design, the residual sum of squares, SS_R, when the stratum variances σ_1^2, σ_2^2, σ_3^2 and σ_4^2 are replaced by their estimates, takes the form

$$\begin{aligned}\widehat{\mathrm{SS}}_R &= \hat\sigma_1^{-2}\|\boldsymbol{\phi}_1(\boldsymbol{I}_n - \boldsymbol{P}_{X(V^{-1})})\boldsymbol{y}\|^2 + \hat\sigma_2^{-2}\|\boldsymbol{\phi}_2(\boldsymbol{I}_n - \boldsymbol{P}_{X(V^{-1})})\boldsymbol{y}\|^2\\ &\quad + \hat\sigma_3^{-2}\|\boldsymbol{\phi}_3(\boldsymbol{I}_n - \boldsymbol{P}_{X(V^{-1})})\boldsymbol{y}\|^2 + \hat\sigma_4^{-2}\|\boldsymbol{\phi}_4(\boldsymbol{I}_n - \boldsymbol{P}_{X(V^{-1})})\boldsymbol{y}\|^2\\ &= n - v. \end{aligned} \tag{41}$$

Hence, the relevant test statistic F takes the same form as that given in (34).

Finally, the comments following formula (34) are also applicable, with appropriate adjustments, to the analysis of data from an experiment conducted in a nested row-column design.

4. Some simplifying reformulations

According to certain remarks made in the previous section, the component $\sigma_4^{-2}\boldsymbol{\phi}_4$ in $\boldsymbol{V}^{-1} = \sigma_1^{-2}\boldsymbol{\phi}_1 + \sigma_2^{-2}\boldsymbol{\phi}_2 + \sigma_3^{-2}\boldsymbol{\phi}_3 + \sigma_4^{-2}\boldsymbol{\phi}_4$ seems to play no role in the formulae applicable in the considered analysis of experimental data. This suggests that some reformulation in the methodology presented in Section 3 would simplify the analysis without causing any changes to its results.

A desirable simplification can be obtained when (following Houtman and Speed, 1983, p. 1071) the dispersion matrix $\boldsymbol{V}$ in the form given in (5) is replaced by the matrix

$$\boldsymbol{V}_* = \sigma_1^2\boldsymbol{\phi}_1 + \sigma_2^2\boldsymbol{\phi}_2 + \sigma_3^2\boldsymbol{\phi}_3 + \sigma_{4(*)}^2\boldsymbol{\phi}_4, \quad \text{with} \;\; \sigma_{4(*)}^2 = -\sigma_1^2 + \sigma_2^2 + \sigma_3^2,$$

which equivalently can be written as

$$\begin{aligned}\boldsymbol{V}_* = {} & \sigma_1^2(\boldsymbol{\phi}_1 - \boldsymbol{\phi}_4) + \sigma_2^2(\boldsymbol{\phi}_2 + \boldsymbol{\phi}_4) + \sigma_3^2(\boldsymbol{\phi}_3 + \boldsymbol{\phi}_4) \\ = {} & \sigma_1^2(\boldsymbol{I}_n - c^{-1}\boldsymbol{X}_R\boldsymbol{X}_R' - r^{-1}\boldsymbol{X}_C\boldsymbol{X}_C') + \sigma_2^2 c^{-1}\boldsymbol{X}_R\boldsymbol{X}_R' + \sigma_3^2 r^{-1}\boldsymbol{X}_C\boldsymbol{X}_C',\end{aligned}$$

on account of the definition of the stratum components in (5).

The inverted matrix $\boldsymbol{V}_*^{-1}$ can be obtained (equivalently) either directly, as

$$\begin{aligned}\boldsymbol{V}_*^{-1} = {} & [\sigma_1^2(\boldsymbol{I}_n - c^{-1}\boldsymbol{X}_R\boldsymbol{X}_R' - r^{-1}\boldsymbol{X}_C\boldsymbol{X}_C') \\ + {} & \sigma_2^2 c^{-1}\boldsymbol{X}_R\boldsymbol{X}_R' + \sigma_3^2 r^{-1}\boldsymbol{X}_C\boldsymbol{X}_C']^{-1},\end{aligned}$$

or as

$$\boldsymbol{V}_*^{-1} = \sigma_1^{-2}\boldsymbol{\phi}_1 + \sigma_2^{-2}\boldsymbol{\phi}_2 + \sigma_3^{-2}\boldsymbol{\phi}_3 + \sigma_{4(*)}^{-2}\boldsymbol{\phi}_4,$$

with $\sigma_{4(*)}^{-2} = 1/(-\sigma_1^2 + \sigma_2^2 + \sigma_3^2)$.

For the former it will be helpful to note that $\boldsymbol{V}_*$ can also be written as

$$\boldsymbol{V}_* = \sigma_1^2\boldsymbol{I}_n + (\sigma_2^2 - \sigma_1^2)c^{-1}\boldsymbol{X}_R\boldsymbol{X}_R' + (\sigma_3^2 - \sigma_1^2)r^{-1}\boldsymbol{X}_C\boldsymbol{X}_C',$$

where $c^{-1}\boldsymbol{X}_R\boldsymbol{X}_R' = \boldsymbol{I}_r \otimes c^{-1}\mathbf{1}_c\mathbf{1}_c'$ and $r^{-1}\boldsymbol{X}_C\boldsymbol{X}_C' = r^{-1}\mathbf{1}_r\mathbf{1}_r' \otimes \boldsymbol{I}_c$. This should simplify the computation of $\boldsymbol{V}_*^{-1}$.

The relations between the matrices $\boldsymbol{V}$ and $\boldsymbol{V}_*$, and their inverses, are given by the equalities

$$\begin{aligned}&\boldsymbol{V} = \boldsymbol{V}_* + (\sigma_4^2 - \sigma_{4(*)}^2)n^{-1}\mathbf{1}_n\mathbf{1}_n' \quad \text{and} \\ &\boldsymbol{V}^{-1} = \boldsymbol{V}_*^{-1} + (\sigma_4^{-2} - \sigma_{4(*)}^{-2})n^{-1}\mathbf{1}_n\mathbf{1}_n'.\end{aligned} \tag{42}$$

From (42) it follows (see Appendix 3) that

$$(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1} = (\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1} + (\sigma_4^2 - \sigma_{4(*)}^2)n^{-1}\mathbf{1}_v\mathbf{1}_v'. \tag{43}$$

Applying the equality (43), it can be shown (see again Appendix 3) that the BLUE of $\boldsymbol{\tau}_* = (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')\boldsymbol{\tau}$ following from (13), i.e.,

$$\hat{\boldsymbol{\tau}}_* = (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')\hat{\boldsymbol{\tau}} = (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{y},$$

can equivalently be written as

$$\hat{\boldsymbol{\tau}}_* = (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{y}_*, \tag{44}$$

where $\boldsymbol{y}_* = (\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n')\boldsymbol{y}$, for which
$\mathrm{E}(\boldsymbol{y}_*) = (\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n')\boldsymbol{X}_1\boldsymbol{\tau} = \boldsymbol{X}_1(\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')\boldsymbol{\tau} = \boldsymbol{X}_1\boldsymbol{\tau}_*$ and
$\mathrm{D}(\boldsymbol{y}_*) = (\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n')\boldsymbol{V}_*(\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n')$.
The dispersion matrix of $\hat{\boldsymbol{\tau}}_*$, given in (16), can on account of (43) be presented as

$$\mathrm{D}(\hat{\boldsymbol{\tau}}_*) = (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}(\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}_v'). \tag{45}$$

Furthermore, the formulae of SS_V and SS_R, given in (19) for treatments (varieties) and in (20) for residuals, can equivalently be written (see Appendices 3 and 4) as

$$\mathrm{SS}_V = \hat{\boldsymbol{\tau}}_*'\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1\hat{\boldsymbol{\tau}}_* = \boldsymbol{y}_*'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{y}_*, \tag{46}$$

$$\mathrm{SS}_R = \boldsymbol{y}_*'[\boldsymbol{V}_*^{-1} - \boldsymbol{V}_*^{-1}\boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}]\boldsymbol{y}_*, \tag{47}$$

with $\boldsymbol{y}_* = (\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n')\boldsymbol{y}$, as defined in relation to (44). The formulae (46) and (47) provide the sum

$$\mathrm{SS}_V + \mathrm{SS}_R = \boldsymbol{y}_*'\boldsymbol{V}_*^{-1}\boldsymbol{y}_* = \mathrm{SS}_T \quad \text{(say)}, \tag{48}$$

which can be called the total sum of squares. Referring again to Rao and Mitra (1971, Theorem 9.2.1), it can be shown that

$$\mathrm{SS}_T \ \sim\ \chi^2(n-1,\delta), \quad \text{with} \quad \delta = \boldsymbol{\tau}_*'\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1\boldsymbol{\tau}_*$$

equivalent to δ as given in (21). These results can be summarized in the form of an ANOVA table, as presented in Table 1.

The presentation of ANOVA results in Table 1 corresponds well with the formula (18) of the relevant F statistic.

Suppose now that after rejecting the hypothesis (15) one is interested in testing the hypothesis $H_{0,\mathrm{L}} : \boldsymbol{U}_{\mathrm{L}}'\boldsymbol{\tau} = \mathbf{0}$, where $\boldsymbol{U}_{\mathrm{L}}'\mathbf{1}_v = \mathbf{0}$. Note that this

Table 1. Analysis of variance for an experiment in a row-column design with orthogonal block structure

Source of variation	Degrees of freedom	Sum of squares	Expected mean square
Treatments	$v-1$	SS_V	$1+\delta/(v-1)$
Residuals	$n-v$	SS_R	1
Total	$n-1$	SS_T	—

hypothesis, concerning a set of contrasts among treatment parameters, can also be written as

$$H_{0,\mathrm{L}} : \boldsymbol{U}'_{\mathrm{L}}\boldsymbol{\tau}_* = \boldsymbol{0}, \quad \text{where} \quad \boldsymbol{\tau}_* = (\boldsymbol{I}_v - n^{-1}\boldsymbol{1}_v\boldsymbol{r}')\boldsymbol{\tau}. \tag{49}$$

This shows that $H_{0,\mathrm{L}}$ is implied by H_0, given in (15). To find the relevant sum of squares, first note that the BLUE of $\boldsymbol{U}'_{\mathrm{L}}\boldsymbol{\tau}_*$ is, on account of (44), of the form

$$\boldsymbol{U}'_{\mathrm{L}}\hat{\boldsymbol{\tau}}_* = \boldsymbol{U}'_{\mathrm{L}}\hat{\boldsymbol{\tau}} = \boldsymbol{U}'_{\mathrm{L}}(\boldsymbol{X}'_1\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}'_1\boldsymbol{V}_*^{-1}\boldsymbol{y}_*. \tag{50}$$

Its dispersion matrix is, on account of (45), of the form

$$\mathrm{D}(\boldsymbol{U}'_{\mathrm{L}}\hat{\boldsymbol{\tau}}_*) = \boldsymbol{U}'_{\mathrm{L}}(\boldsymbol{X}'_1\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\mathrm{L}}. \tag{51}$$

Note that, applying Lemma 2.2.6(c) from Rao and Mitra (1971), one can write

$$\boldsymbol{U}'_{\mathrm{L}}(\boldsymbol{X}'_1\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\mathrm{L}}[\boldsymbol{U}'_{\mathrm{L}}(\boldsymbol{X}'_1\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\mathrm{L}}]^{-}\boldsymbol{U}'_{\mathrm{L}} = \boldsymbol{U}'_{\mathrm{L}},$$

which, with (51), gives the equality $\mathrm{D}(\boldsymbol{U}'_{\mathrm{L}}\hat{\boldsymbol{\tau}}_*)[\mathrm{D}(\boldsymbol{U}'_{\mathrm{L}}\hat{\boldsymbol{\tau}}_*)]^{-}\boldsymbol{U}'_{\mathrm{L}}\hat{\boldsymbol{\tau}}_* = \boldsymbol{U}'_{\mathrm{L}}\hat{\boldsymbol{\tau}}_*$. This shows that the hypothesis in (49) is consistent. The relevant sum of squares can then be obtained (following Theorem 3.2 of Rao, 1971) in the form

$$\begin{aligned}\mathrm{SS}(\boldsymbol{U}_{\mathrm{L}}) &= \hat{\boldsymbol{\tau}}'_*\boldsymbol{U}_{\mathrm{L}}[\mathrm{D}(\boldsymbol{U}'_{\mathrm{L}}\hat{\boldsymbol{\tau}}_*)]^{-}\boldsymbol{U}'_{\mathrm{L}}\hat{\boldsymbol{\tau}}_* \\ &= \hat{\boldsymbol{\tau}}'_*\boldsymbol{U}_{\mathrm{L}}[\boldsymbol{U}'_{\mathrm{L}}(\boldsymbol{X}'_1\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\mathrm{L}}]^{-}\boldsymbol{U}'_{\mathrm{L}}\hat{\boldsymbol{\tau}}_*,\end{aligned} \tag{52}$$

with the d.f. equal to $\mathrm{rank}(\boldsymbol{U}_{\mathrm{L}})$, i.e., equal to $\mathrm{rank}[\mathrm{D}(\boldsymbol{U}'_{\mathrm{L}}\hat{\boldsymbol{\tau}}_*)]$. Note that $\boldsymbol{U}'_{\mathrm{L}}\hat{\boldsymbol{\tau}}_*$ is given in (50), and $[\boldsymbol{U}'_{\mathrm{L}}(\boldsymbol{X}'_1\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\mathrm{L}}]^{-}$ follows from (51). Also note, referring to Lemma 2.2.6(d) in Rao and Mitra (1971), that $\boldsymbol{U}_{\mathrm{L}}[\boldsymbol{U}'_{\mathrm{L}}(\boldsymbol{X}'_1\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\mathrm{L}}]^{-}\boldsymbol{U}'_{\mathrm{L}}$ is invariant for any choice of the g-inverse appearing, and is of rank equal to the rank of $\boldsymbol{U}_{\mathrm{L}}$. Of course, if the columns of $\boldsymbol{U}_{\mathrm{L}}$ are linearly independent, then

$$[\boldsymbol{U}_{\mathrm{L}}'(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\mathrm{L}}]^{-} \quad \text{becomes} \quad [\boldsymbol{U}_{\mathrm{L}}'(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\mathrm{L}}]^{-1}.$$

Now, following the assumption $\boldsymbol{y} \sim N_n(\boldsymbol{X}_1\boldsymbol{\tau}, \boldsymbol{V})$, adopted in Section 3, one may also assume that $\boldsymbol{U}_{\mathrm{L}}'\hat{\boldsymbol{\tau}}_* \sim N[\boldsymbol{U}_{\mathrm{L}}'\boldsymbol{\tau}_*, \mathrm{D}(\boldsymbol{U}_{\mathrm{L}}'\hat{\boldsymbol{\tau}}_*)]$. Using this, and applying Theorem 9.2.3 from Rao and Mitra (1971), it can be shown that

$$\mathrm{SS}(\boldsymbol{U}_{\mathrm{L}}) \sim \chi^2[\mathrm{rank}(\boldsymbol{U}_{\mathrm{L}}), \delta_L], \quad \text{with} \quad \delta_{\mathrm{L}} = \boldsymbol{\tau}_*'\boldsymbol{U}_{\mathrm{L}}[\mathrm{D}(\boldsymbol{U}_{\mathrm{L}}'\hat{\boldsymbol{\tau}}_*)]^{-}\boldsymbol{U}_{\mathrm{L}}'\boldsymbol{\tau}_*,$$

this distribution being central, i.e., with $\delta_{\mathrm{L}} = 0$, if $H_{0,\mathrm{L}}$ is true.

If there are several sets of contrasts for which individual hypothesis testing is of interest, then for each of them the sum of squares presented in (52) can be used accordingly. In some situations a relevant partition of the treatment sum of squares, given in (46), may be of interest in the application of ANOVA. The question then arises of what kind of conditions the chosen sets of contrasts have to satisfy. It can be shown (as in Caliński and Siatkowski, 2018, pp. 158-159) that for two such sets of contrasts, e.g. $\boldsymbol{U}_{\mathrm{A}}'\boldsymbol{\tau}_*$ and $\boldsymbol{U}_{\mathrm{B}}'\boldsymbol{\tau}_*$, the equality

$$\mathrm{SS}(\boldsymbol{U}_{\mathrm{A}}) + \mathrm{SS}(\boldsymbol{U}_{\mathrm{B}}) = \mathrm{SS}_V \tag{53}$$

holds, for any vector $\hat{\boldsymbol{\tau}}_* = (\boldsymbol{I}_v - n^{-1}\boldsymbol{1}_v\boldsymbol{r}')\hat{\boldsymbol{\tau}}$, if and only if

$$\begin{aligned}&(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\mathrm{A}}[\boldsymbol{U}_{\mathrm{A}}'(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\mathrm{A}}]^{-}\boldsymbol{U}_{\mathrm{A}}'\\ &+ (\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\mathrm{B}}[\boldsymbol{U}_{\mathrm{B}}'(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\mathrm{B}}]^{-}\boldsymbol{U}_{\mathrm{B}}' = \boldsymbol{I}_v - n^{-1}\boldsymbol{1}_v\boldsymbol{r}'.\end{aligned} \tag{54}$$

This, in turn, implies (on account of Lemma 2.2.6 in Rao and Mitra, 1971) that

$$\boldsymbol{U}_{\mathrm{B}}'(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\mathrm{A}} = \mathbf{O}. \tag{55}$$

These results can be extended for any number of considered sets of contrasts used in a partition of the type (53). The condition (55) can then be written as

$$\boldsymbol{U}_{\mathrm{L}}'(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\mathrm{L}^*} = \mathbf{O} \quad \text{for} \quad \mathrm{L} \neq \mathrm{L}^*. \tag{56}$$

It may be interesting to note that for some layouts of treatments in rows and columns of the design, the condition (56) can be reduced to $\boldsymbol{U}_{\mathrm{L}}'\boldsymbol{U}_{\mathrm{L}^*} = \mathbf{O}$.

Proceeding now to the analysis of data from an experiment in a nested row-column design, we first note that the relevant covariance (dispersion) matrix is of the form (8), i.e.,

$$\mathrm{D}(\boldsymbol{y}) \equiv \boldsymbol{V} = \sigma_1^2\boldsymbol{\phi}_1 + \sigma_2^2\boldsymbol{\phi}_2 + \sigma_3^2\boldsymbol{\phi}_3 + \sigma_4^2\boldsymbol{\phi}_4 + \sigma_5^2\boldsymbol{\phi}_5,$$

extended in comparison with the form given in (5). A desirable simplification of the analysis can be obtained when this matrix $\boldsymbol{V}$ is replaced by

$$\begin{aligned}\boldsymbol{V}_* = &\ \sigma_1^2\boldsymbol{\phi}_1 + \sigma_2^2\boldsymbol{\phi}_2 + \sigma_3^2\boldsymbol{\phi}_3 + \sigma_4^2\boldsymbol{\phi}_4 + \sigma_4^2\boldsymbol{\phi}_5\\ = &\ \sigma_1^2\boldsymbol{\phi}_1 + \sigma_2^2\boldsymbol{\phi}_2 + \sigma_3^2\boldsymbol{\phi}_3 + \sigma_4^2(\boldsymbol{\phi}_4 + \boldsymbol{\phi}_5),\end{aligned}$$

where $\boldsymbol{\phi}_4 + \boldsymbol{\phi}_5 = n_0^{-1}\boldsymbol{X}_B\boldsymbol{X}_B' = \boldsymbol{I}_b \otimes n_0^{-1}\mathbf{1}_{n_0}\mathbf{1}_{n_0}'$. Hence, the inverted matrix $\boldsymbol{V}_*^{-1}$ is now obtainable as

$$\begin{aligned}\boldsymbol{V}_*^{-1} = &\ \sigma_1^{-2}\boldsymbol{\phi}_1 + \sigma_2^{-2}\boldsymbol{\phi}_2 + \sigma_3^{-2}\boldsymbol{\phi}_3 + \sigma_4^{-2}\boldsymbol{\phi}_4 + \sigma_4^{-2}\boldsymbol{\phi}_5\\ = &\ \sigma_1^{-2}\boldsymbol{\phi}_1 + \sigma_2^{-2}\boldsymbol{\phi}_2 + \sigma_3^{-2}\boldsymbol{\phi}_3 + \sigma_4^{-2}(\boldsymbol{\phi}_4 + \boldsymbol{\phi}_5),\end{aligned}$$

with $\boldsymbol{\phi}_4 + \boldsymbol{\phi}_5$ as given above.

Now the relations between the matrices $\boldsymbol{V}$ and $\boldsymbol{V}_*$, and their inverses, can be written as

$$\boldsymbol{V} = \boldsymbol{V}_* + (\sigma_5^2 - \sigma_4^2)n^{-1}\mathbf{1}_n\mathbf{1}_n' \quad \text{and} \quad \boldsymbol{V}^{-1} = \boldsymbol{V}_*^{-1} + (\sigma_5^{-2} - \sigma_4^{-2})n^{-1}\mathbf{1}_n\mathbf{1}_n'.$$

Taking account of these relations, the simplified analysis can be performed similarly as presented in this section for the simple (not nested) row-column design.

5. Application with estimated stratum variances

The hypothesis testing procedures presented in Section 4 are fully applicable if the involved stratum variances are known. As already mentioned in Section 3, in practical applications these variances are usually unknown and have to be estimated. This can be done by solving the equations indicated. However, with these estimates the residual sum of squares, SS_R, is reduced to $n - v$, the corresponding d.f., as shown in formula (33). This leads to a corresponding reduction of the F statistic (18) to that presented in (34). The estimated treatment sum of squares appearing there, $\widehat{\mathrm{SS}}_V$, can be written as

$$\widehat{\mathrm{SS}}_V = \boldsymbol{y}_*'\hat{\boldsymbol{V}}_*^{-1}\boldsymbol{y}_* - (n - v) \equiv \widehat{\mathrm{SS}}_T - n + v. \tag{57}$$

In the case of known (true) values of the relevant variances, the quadratic form $\mathrm{SS}_T = \boldsymbol{y}'_* \boldsymbol{V}_*^{-1} \boldsymbol{y}_*$ is distributed as $\chi^2(n-1, \delta)$. If the hypothesis H_0 given in (15) is true, then $\delta = 0$ and the distribution is central. However, the indicated distribution of SS_T is fully applicable only if the true stratum variances appearing in $\boldsymbol{V}_*^{-1}$ are used. Because now the matrix $\boldsymbol{V}_*^{-1}$ considered in Section 4 is replaced by its estimate, the estimated total sum of squares $\widehat{\mathrm{SS}}_T$, appearing in (57), does not have an exact χ^2 distribution with $n-1$ d.f. That distribution can, however, be considered as an approximation of the real distribution of $\widehat{\mathrm{SS}}_T$. This approximation will be the closer the larger is the number n, i.e., the size of the experiment.

With this approximation, the estimated mean square $\widehat{\mathrm{MS}}_V = \widehat{\mathrm{SS}}_V/(v-1)$, denoted by $\widehat{F}$ in (34), may be treated in a practical application as having (under H_0) approximately the distribution of $\chi^2(v-1, 0)/(v-1)$, as follows from the relation in (57). Thus, referring the test statistic (34) to the $\chi^2(v-1, 0)/(v-1)$ distribution, one will obtain an approximate test of the hypothesis H_0. This means that when calculating the relevant P values (i.e., the critical levels of significance) for testing H_0, or hypotheses implied by H_0, one has to consider them as approximate. The results obtained by Volaufova (2009) seem to suggest that the above ANOVA type F test approximation will in most cases provide reasonably accurate P values.

Finally, see also the comments in Johnson, Kotz and Balakrishnan (1995, p. 338) whereby, if in the F statistic as in (18) the d.f. $n-v$ is large, then the natural approximation to be used is that this F statistic is distributed as $\chi^2(v-1, 0)/(v-1)$. In fact, according to those comments the distribution of the statistic (34) corresponds to the F distribution with the second d.f. tending to infinity; see formula (27.27) there.

6. Examples

The analytical methods considered in the previous sections will now be illustrated using data from four experiments conducted in different row-column designs inducing the OBS property. The first three of these experiments (Examples 1, 2 and 3) were conducted in simple row-column designs, whereas the fourth (Example 4) was conducted in a nested row-column design. All required computations were performed using R (R Core Team, 2017).

Example 1. Gawęcki and Wagner (1984, Section 10.3.6) analyzed data from an experiment conducted in a Latin square design, in which 5 randomly selected male rats (columns) were treated with 5 different diets (A, B, C, D, E) applied in different orders during 5 two-week periods (rows). Using appropriate methods, the individual dynamic activity of the consumed diets was measured. The results of the experiment are presented in Table 2.

Table 2. The layout of treatments (diets) and their observed effects from the experiment in a Latin square design analyzed in Example 1

Rows	Columns				
	Treatment labels / Observations				
	1	2	3	4	5
1	A 25.4	B 20.4	C 29.4	D 25.4	E 51.1
2	E 55.3	A 18.5	B 28.1	C 27.6	D 27.1
3	D 21.7	E 47.2	A 23.1	B 19.7	C 27.5
4	C 28.2	D 30.8	E 48.5	A 20.2	B 25.4
5	B 23.5	C 28.4	D 34.5	E 50.7	A 25.1

For this experiment the incidence matrix $\boldsymbol{N}$, defined in (3), has the form

$$\boldsymbol{N} = \boldsymbol{X}_1' = \begin{bmatrix} 1&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&1 \\ 0&1&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&1&1&0&0&0&0 \\ 0&0&1&0&0&0&0&0&1&0&0&0&0&0&1&1&0&0&0&0&0&1&0&0&0 \\ 0&0&0&1&0&0&0&0&0&1&1&0&0&0&0&0&1&0&0&0&0&0&1&0&0 \\ 0&0&0&0&1&1&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&1&0 \end{bmatrix}.$$

The results of the ANOVA obtained by the methods described in Sections 3 and 4 are presented in Table 3.

Table 3. Analysis of variance for an experiment in a Latin square design – Example 1

Source of variation	Degrees of freedom	Sum of squares	Mean square	$\hat{F}$	P value
Treatments	4	284.256	71.064	71.064	< 0.0001
Residuals	20	20	1	—	—
Total	24	304.256	—	—	—

The results in Table 3 were obtained with the use of the empirical estimates (i.e., based on $\hat{\sigma}_1^2 = 9.307267$, $\hat{\sigma}_2^2 = 14.4386$ and $\hat{\sigma}_3^2 = 13.5246$)

$$\tilde{\boldsymbol{\tau}} = [22.46,\ 23.42,\ 28.22,\ 27.90,\ 50.56]'$$

and

$$\tilde{\boldsymbol{\tau}}_* = [-8.052,\ -7.092,\ -2.292,\ -2.612,\ 20.048]',$$

the former obtainable by the use of formula (13), the latter either from the relation $\tilde{\boldsymbol{\tau}}_* = (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')\tilde{\boldsymbol{\tau}}$, or directly by formula (44). Using with $\tilde{\boldsymbol{\tau}}_*$ the formula (46) with $\boldsymbol{V}_*^{-1}$ replaced by its estimate, the estimated sum of squares $\widehat{\mathrm{SS}}_V$ is obtained. Similarly, using formula (52) in the same way, the relevant components of $\widehat{\mathrm{SS}}_V$ can be obtained. Evidently, as follows from (33), the estimated residual sum of squares $\widehat{\mathrm{SS}}_R$ is reduced to $n - v$, its d.f. The term "empirical estimates", used above, is taken from Rao and Kleffe (1988, p. 274).

Example 2. Siatkowski (2004, Section 5.3) analyzed an experiment conducted in a row-column design with 6 rows and 6 columns, in which the yield of 9 sunflower varieties was examined (Table 4).

Table 4. The layout of varieties and their yields from the experiment in a row-column design analyzed in Example 2

Rows	Columns											
	Variety labels	Observations										
	1		2		3		4		5		6	
1	B	17.5	D	19.5	E	19.6	A	15.3	F	19.1	C	19.2
2	A	15.4	C	19.0	H	16.5	B	17.1	I	15.1	G	19.3
3	G	19.3	E	19.6	D	19.3	I	15.0	H	16.7	F	19.0
4	H	16.5	B	17.6	F	18.9	C	18.9	E	19.7	I	14.9
5	D	19.4	F	19.0	I	14.9	G	19.1	C	19.1	A	15.5
6	E	19.7	A	15.6	G	19.2	H	16.6	B	17.3	D	19.4

For this row-column design the incidence matrix $\boldsymbol{N}$ is of the form

$$
\boldsymbol{N} = \boldsymbol{X}_1' = [\boldsymbol{X}_{11}' : \boldsymbol{X}_{12}' : \cdots : \boldsymbol{X}_{16}'] =
$$
$$
\begin{bmatrix}
0&0&0&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0\\
1&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&1\\
0&0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&0&0&0&1&0&0&1&0&0&0&0&0&0&0&0&0
\end{bmatrix}.
$$

The results of the ANOVA obtained by the methods described in Sections 3 and 4 are presented in Table 5.

Table 5. Analysis of variance for an experiment in a row-column design – Example 2

Source of variation	Degrees of freedom	Sum of squares	Mean square	$\hat{F}$	P value
Treatments	8	10394.84	1299.355	1299.355	< 0.0001
Residuals	27	27	1	—	—
Total	35	10421.84	—	—	—

The results in Table 5 were obtained with the use of the empirical estimates (i.e., based on $\hat{\sigma}_1^2 = 0.00969$, $\hat{\sigma}_2^2 = 0.00875$ and $\hat{\sigma}_3^2 = 0.03641$)

$$\tilde{\boldsymbol{\tau}} = [15.449, 17.368, 19.039, 19.393, 19.637, 18.983, 19.249, 16.593, 14.990]'$$

and

$$\tilde{\boldsymbol{\tau}}_* = [-2.407, -0.488, 1.184, 1.537, 1.781, 1.128, 1.394, -1.262, -2.866]',$$

following the same approach as in Example 1.

Example 3. Przybyłowski and Walkowiak (1977, Section 6.1) analyzed an experiment carried out in a Youden Square design with 3 replications. This was a vase experiment, the purpose of which was to examine the effect of 7 types of fertilization on the growth of a certain variety of maize. Plant lengths, measured 7 days after germination, are shown in Table 6.

Table 6. The layout of treatments (fertilizations) and their observed plant length effects from the experiment in a row-column design analyzed in Example 3

Rows	Columns						
	Variety labels / Observations						
	1	2	3	4	5	6	7
1	G 6	A 3	B 3	C 4	D 2	E 5	F 9
2	F 4	G 9	A 2	B 1	C 1	D 1	E 8
3	D 2	E 6	F 1	G 7	A 1	B 1	C 2

For this row-column design the incidence matrix $\boldsymbol{N}$ has the form

$$\boldsymbol{N} = \boldsymbol{X}_1' = [\boldsymbol{X}_{11}' : \boldsymbol{X}_{12}' : \boldsymbol{X}_{13}'] = \begin{bmatrix} 0\,1\,0\,0\,0\,0\,0\,0\,0\,1\,0\,0\,0\,0\,0\,0\,0\,0\,1\,0\,0 \\ 0\,0\,1\,0\,0\,0\,0\,0\,0\,0\,1\,0\,0\,0\,0\,0\,0\,0\,0\,1\,0 \\ 0\,0\,0\,1\,0\,0\,0\,0\,0\,0\,0\,1\,0\,0\,0\,0\,0\,0\,0\,0\,1 \\ 0\,0\,0\,0\,1\,0\,0\,0\,0\,0\,0\,0\,1\,0\,1\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,1\,0\,0\,0\,0\,0\,0\,0\,1\,0\,1\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,1\,1\,0\,0\,0\,0\,0\,0\,0\,0\,1\,0\,0\,0\,0 \\ 1\,0\,0\,0\,0\,0\,0\,0\,1\,0\,0\,0\,0\,0\,0\,0\,0\,1\,0\,0\,0 \end{bmatrix}.$$

The results of the ANOVA obtained by the methods described in Sections 3 and 4 are presented in Table 7.

Table 7. Analysis of variance for an experiment in a row-column design – Example 3

Source of variation	Degrees of freedom	Sum of squares	Mean square	$\hat{F}$	P value
Treatments	6	29.8486	4.97477	4.97477	0.00634
Residuals	14	14	1	—	—
Total	20	43.84862	—	—	—

The results in Table 7 were obtained with the use of the empirical estimates (i.e., based on $\hat{\sigma}_1^2 = 2.857143$, $\hat{\sigma}_2^2 = 5.142857$ and $\hat{\sigma}_3^2 = 4.448980$, obtained after 13 iteration cycles)

$\tilde{\boldsymbol{\tau}} = [2.086379, 1.853821, 2.146179, 1.940199, 6.102990, 4.594684, 7.275748]'$
and
$\tilde{\boldsymbol{\tau}}_* = [-1.62791, -1.86047, -1.56811, -1.77409, 2.3887, 0.8804, 3.56146]'$,
following the same approach as in Example 1.

Example 4. Kozłowska et al. (2011) analyzed data from a plant protection experiment. The field experiment was carried out in Winna Góra at the Field Experiment Station of the Institute of Plant Protection – National Research Institute in Poznań, Poland. Its purpose was to examine whether, under conditions typical for spring cereal cultivation, it is possible to reduce the dose of herbicidal active ingredient (a.i.) methyl tribenuron relative to the recommended dose, without a significant difference in the impact on the yield of spring wheat of the Banti variety. The experiment under consideration is a near-factorial experiment carried out in a nested row-column design with $b = 3$ blocks, each containing $n_0 = 16$ experimental units grouped into $r_0 = c_0 = 4$ rows and columns, according to a scheme based on the theoretical plan:

Block 1	Block 2	Block 3
1 1 2 3	2 3 4 5	4 5 1 1
1 2 3 1	3 4 5 2	5 1 1 4
2 3 1 1	4 5 2 3	1 1 4 5
3 1 1 2	5 2 3 4	1 4 5 1

The study included $v = 5$ treatments: combinations of two experimental factors, A and B, and the control treatment (untreated units). The first experimental factor (factor A) is the date of the herbicidal procedure. This factor occurs at two levels, i.e. A_1 (term 1) – the beginning, and A_2 (term 2) – the end of the cereal tillering phase. The herbicide was used in two doses (experimental factor B): B_1 – the recommended dose (15 g/ha), and B_2 – the reduced dose, equal to 2/3 of the recommended dose (10 g/ha). Such a design has the incidence matrix $\boldsymbol{N} = [\boldsymbol{N}_1 : \boldsymbol{N}_2 : \boldsymbol{N}_3]$ where:

$$\boldsymbol{N}_1 = \begin{bmatrix} 1&1&0&0&1&0&0&1&0&0&1&1&0&1&1&0 \\ 0&0&1&0&0&1&0&0&1&0&0&0&0&0&0&1 \\ 0&0&0&1&0&0&1&0&0&1&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix} \begin{array}{lcc} i & \mathrm{A} & \mathrm{B} \\ 1(\text{control}) & - & - \\ 2 & A_1 & B_1 \\ 3 & A_1 & B_2 \\ 4 & A_2 & B_1 \\ 5 & A_2 & B_2 \end{array}$$

$$\boldsymbol{N}_2 = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&1&0&0&1&0&0&1&0&0 \\ 0&1&0&0&1&0&0&0&0&0&0&1&0&0&1&0 \\ 0&0&1&0&0&1&0&0&1&0&0&0&0&0&0&1 \\ 0&0&0&1&0&0&1&0&0&1&0&0&1&0&0&0 \end{bmatrix} \begin{array}{lcc} i & \mathrm{A} & \mathrm{B} \\ 1(\text{control}) & - & - \\ 2 & A_1 & B_1 \\ 3 & A_1 & B_2 \\ 4 & A_2 & B_1 \\ 5 & A_2 & B_2 \end{array}$$

$$\boldsymbol{N}_3 = \begin{bmatrix} 0&0&1&1&0&1&1&0&1&1&0&0&1&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&1&0&0&1&0&0&1&0&0 \\ 0&1&0&0&1&0&0&0&0&0&0&1&0&0&1&0 \end{bmatrix} \begin{matrix} i & \mathrm{A} & \mathrm{B} \\ 1(\text{control}) & - & - \\ 2 & A_1 & B_1 \\ 3 & A_1 & B_2 \\ 4 & A_2 & B_1 \\ 5 & A_2 & B_2 \end{matrix}$$

Table 8. Experimental observations of the field plot yield of the treatments analyzed in Example 4

Treatment	Block	Row	Column	Observ.	Treatment	Block	Row	Column	Observ.
1	1	1	1	2,55	4	2	3	1	4,98
1	1	1	2	2,93	5	2	3	2	4,2
2	1	1	3	2,67	2	2	3	3	3,89
3	1	1	4	2,69	3	2	3	4	4,37
1	1	2	1	2,8	5	2	4	1	3,81
2	1	2	2	3,59	2	2	4	2	4,68
3	1	2	3	2,66	3	2	4	3	4,1
1	1	2	4	2,33	4	2	4	4	4,49
2	1	3	1	2,75	4	3	1	1	3,13
3	1	3	2	3,52	5	3	1	2	3,42
1	1	3	3	2,87	1	3	1	3	2,79
1	1	3	4	3,11	1	3	1	4	2,39
3	1	4	1	3,44	5	3	2	1	3,65
1	1	4	2	2,5	1	3	2	2	2,39
1	1	4	3	3,26	1	3	2	3	3,05
2	1	4	4	2,98	4	3	2	4	2,91
2	2	1	1	3,66	1	3	3	1	2,87
3	2	1	2	3,74	1	3	3	2	3,14
4	2	1	3	3,77	4	3	3	3	2,69
5	2	1	4	4,74	5	3	3	4	3,57
3	2	2	1	4,64	1	3	4	1	2,57
4	2	2	2	4,57	4	3	4	2	2,83
5	2	2	3	5,11	5	3	4	3	3,33
2	2	2	4	4,47	1	3	4	4	2,05

Now we may proceed to the general ANOVA, based on the theory presented in Sections 3 and 4. The results in Table 9 were obtained following the same approach as in Example 1, with the use of the empirical estimates $\tilde{\boldsymbol{\tau}}$ and $\tilde{\boldsymbol{\tau}}_*$, based on $\hat{\sigma}_1^2 = 0.1655973$, $\hat{\sigma}_2^2 = 0.1903188$, $\hat{\sigma}_3^2 = 0.07988542$, $\hat{\sigma}_4^2 = 7.843859$ (obtained after 6 iteration cycles):

$$\tilde{\boldsymbol{\tau}} = [3.118,\ 3.359,\ 3.417,\ 3.506,\ 3.814]'$$

and

$$\tilde{\boldsymbol{\tau}}_* = [-0.271,\ -0.030,\ 0.029,\ 0.118,\ 0.425]'.$$

Because of the purpose of the experiment, the researcher was interested in estimating and testing the set of contrasts that can be presented as basic

Table 9. Analysis of variance for an experiment in a nested row-column design – Example 4

Source of variation	Degrees of freedom	Sum of squares	Mean square	$\hat{F}$	P value
Treatments	4	13.09749	3.274372	3.274372	< 0.01980923
Residuals	43	43	1	—	—
Total	47	56.09749	—	—	—

contrasts (see Definition 3.4.1 in Caliński and Kageyama, 2000)

$$\{\boldsymbol{c}_i'\boldsymbol{\tau} \equiv \boldsymbol{c}_i'\boldsymbol{\tau}_*, i = 1, 2, 3, 4\},$$

determined by the following vectors:

$$\begin{aligned}
\boldsymbol{c}_1' &= (\sqrt{6}/3)\left[\; 4, \quad -1, \quad -1, \quad -1, \quad -1 \;\right], \\
\boldsymbol{c}_2' &= \sqrt{2}\left[\; 0, \quad -1, \quad -1, \quad 1, \quad 1 \;\right], \\
\boldsymbol{c}_3' &= 2\left[\; 0, \quad -1, \quad 1, \quad 0, \quad 0 \;\right], \\
\boldsymbol{c}_4' &= 2\left[\; 0, \quad 0, \quad 0, \quad -1, \quad 1 \;\right].
\end{aligned}$$

For each of these four basic contrasts the BLUE is obtainable by formula (50), with $\boldsymbol{U}_{\mathrm{L}}'$ replaced by $\boldsymbol{c}_i'$, and the relevant sum of squares, $\mathrm{SS}(\boldsymbol{c}_i)$, follows from (52) with the same replacement. It may be noted that for the set of basic contrasts so defined:

$$\mathrm{SS}(\boldsymbol{c}_1) + \mathrm{SS}(\boldsymbol{c}_2) + \mathrm{SS}(\boldsymbol{c}_3) + \mathrm{SS}(\boldsymbol{c}_4) = \mathrm{SS}_V,$$

which can also be seen by analyzing Table 10, presenting the ANOVA for the above basic contrasts. The empirical estimates of the considered basic contrasts are: $\widetilde{\boldsymbol{c}_1'\boldsymbol{\tau}} = -1.32701$, $\widetilde{\boldsymbol{c}_2'\boldsymbol{\tau}} = 0.7691924$, $\widetilde{\boldsymbol{c}_3'\boldsymbol{\tau}} = 0.1175$ and $\widetilde{\boldsymbol{c}_4'\boldsymbol{\tau}} = 0.615$.

The presented results and conclusions regarding the basic contrasts coincide with the results of the analysis under the bottom stratum submodel presented by Kozłowska et al. (2011). However, use of the direct analysis allowed the rejection of the general hypothesis (Table 9), which was not possible when stratum analyses were used.

Table 10. Analysis of variance for the contrasts considered in Example 4

Source	Degrees of freedom	Sum of squares	Mean square	$\hat{F}$	P value
Treatments	4	13.0975	3.2744	3.2744	0.0198
$\boldsymbol{c}_1'\boldsymbol{\tau}$	1	8.0316	8.0316	8.0316	0.007
$\boldsymbol{c}_2'\boldsymbol{\tau}$	1	2.6985	2.6985	2.6985	0.108
$\boldsymbol{c}_3'\boldsymbol{\tau}$	1	0.0834	0.0834	0.0834	0.774
$\boldsymbol{c}_4'\boldsymbol{\tau}$	1	2.284	2.284	2.284	0.138
Residuals	43	43	1		
Total	47	56.0975			

7. Concluding remarks

This paper is the third in a planned series concerning a new approach to the analysis of experiments with the OBS property. The first paper in this series (Caliński and Siatkowski, 2017) concerns experiments conducted in proper block designs. The second (Caliński and Siatkowski, 2018) is devoted to experiments in nested block designs. Here the new approach is applied to experiments in row-column designs (simple or nested) that induce the OBS property.

Just as in the first two works, it appears that when the unknown stratum variances within the covariance (dispersion) matrix $\boldsymbol{V}$, given in (5) or (8), are replaced by their estimates, obtained from the estimation procedure suggested by Nelder (1968), the residual sum of squares SS_R is reduced to its d.f., that is, its expectation. This result is obtainable due to the proposed new approach to the analysis of experimental data.

The indicated result, presented in Section 3, follows from the use of a covariance matrix $\boldsymbol{V}$ not in the form $\boldsymbol{V} = \sigma_1^2\boldsymbol{T}$ (say), appearing in the general Gauss−Markov model, as usually indicated in the literature (for example, in Siatkowski, 2004, Section 4.2), but in its original form $\boldsymbol{V} = \sigma_1^2\boldsymbol{\phi}_1 + \sigma_2^2\boldsymbol{\phi}_2 + \sigma_3^2\boldsymbol{\phi}_3 + \sigma_4^2\boldsymbol{\phi}_4$, or that in (8) for nested row-column designs. This ensures that $\mathrm{E}(\mathrm{SS}_R) = n-v$, as follows from (12) and (24). As a consequence of this procedure, the test statistic (18) is reduced to the form (34), i.e., to the estimated treatment mean square, $\widehat{\mathrm{MS}}_V = \widehat{\mathrm{SS}}_V/(v-1)$. This can be seen as an advantage, particularly with regard to the approximation of the relevant distribution, indicated at the end of Section 5.

Another feature of the proposed approach concerns some simplification of the main analytical procedures, as presented in Section 4. One of the resulting advantages is the reduction of the number of stratum variances involved, which substantially simplifies the computations.

However, as can be seen from the examples analyzed in Section 6, the main advantage of the proposed approach is the fact that the ANOVA results are obtainable directly, not by first performing some partial analyses under relevant stratum submodels and then combining their results (as suggested in most of the relevant literature).

The advantages indicated here are similar to those presented in the first two papers in this series, referring to different classes of designs inducing the OBS property.

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Appendix

Appendix 1

For formula (17) one has first to show that $\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1$ can be taken as a g-inverse of the matrix $\mathrm{D}(\hat{\boldsymbol{\tau}}_*)$ given in (16), i.e., that the equality

$$\begin{aligned}(\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}(\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}_v')\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{I}_v \\ - n^{-1}\mathbf{1}_v\boldsymbol{r}')(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}(\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}_v') \\ = (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}(\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}_v')\end{aligned}$$

holds. For this, it is sufficient to consider the equalities

$$\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}') = (\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}_v')\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1 \quad \text{and}$$
$$(\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}_v')(\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}_v') = \boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}_v'.$$

The second equality is obvious. To prove the first, one may use the equalities

$$\boldsymbol{X}_1(\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}') = (\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n')\boldsymbol{X}_1 \quad \text{and}$$
$$\boldsymbol{V}^{-1}(\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n') = (\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n')\boldsymbol{V}^{-1},$$

which can easily be checked remembering that $\boldsymbol{X}_1\mathbf{1}_v = \mathbf{1}_n$ and $\mathbf{1}_n'\boldsymbol{X}_1 = \boldsymbol{r}'$, and also recalling the properties of the matrices $\boldsymbol{\phi}_1$, $\boldsymbol{\phi}_2$, $\boldsymbol{\phi}_3$ and $\boldsymbol{\phi}_4$ appearing in formula (5).

Now, with $\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1$ as a g-inverse of $\mathrm{D}(\hat{\boldsymbol{\tau}}_*)$, the equality (17) follows, which can easily be checked noting that $(\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')\hat{\boldsymbol{\tau}}_* = \hat{\boldsymbol{\tau}}_*$.

Appendix 2

For formula (18) note that the sum of squares SS_V can, on account of (13) and the relation $\hat{\boldsymbol{\tau}}_* = (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')\hat{\boldsymbol{\tau}}$, be written as

$$\begin{aligned}\mathrm{SS}_V = \boldsymbol{y}'\boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}(\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}_v')\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{I}_v \\ - n^{-1}\mathbf{1}_v\boldsymbol{r}')(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{y},\end{aligned}$$

which, by the equalities

$$\begin{aligned}\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1} = \boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}_v' \quad \text{and} \\ (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1} = (\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}(\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}_v')\end{aligned}$$

(see Appendix 1), can be reduced to the form in (19). As to the sum of squares SS_R, its formula (20) follows directly from (18) on account of (12).

Appendix 3

For the formulae in (42) note that, using the well-known formula

$$(\boldsymbol{A} + \boldsymbol{BCD})^{-1} = \boldsymbol{A}^{-1} - \boldsymbol{A}^{-1}\boldsymbol{B}(\boldsymbol{C}^{-1} + \boldsymbol{D}\boldsymbol{A}^{-1}\boldsymbol{B})^{-1}\boldsymbol{D}\boldsymbol{A}^{-1},$$

one can write

$$\begin{aligned}\boldsymbol{V}^{-1} = \boldsymbol{V}_*^{-1} - \boldsymbol{V}_*^{-1}n^{-1}\mathbf{1}_n\mathbf{1}_n'[(\sigma_4^2 - \sigma_{4(*)}^2)^{-1}\boldsymbol{I}_n \\ + n^{-1}\mathbf{1}_n\mathbf{1}_n'\boldsymbol{V}_*^{-1}n^{-1}\mathbf{1}_n\mathbf{1}_n']^{-1}n^{-1}\mathbf{1}_n\mathbf{1}_n'\boldsymbol{V}_*^{-1},\end{aligned}$$

from which

$$\begin{aligned}\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1 &= \boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1 - \boldsymbol{X}_1'\boldsymbol{V}_*^{-1}n^{-1}\mathbf{1}_n\mathbf{1}_n'[(\sigma_4^2 - \sigma_{4(*)}^2)^{-1}\boldsymbol{I}_n \\ &\quad + n^{-1}\mathbf{1}_n\mathbf{1}_n'\boldsymbol{V}_*^{-1}n^{-1}\mathbf{1}_n\mathbf{1}_n']^{-1}n^{-1}\mathbf{1}_n\mathbf{1}_n'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1 \\ &= \boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1 - \boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1\mathbf{1}_v n^{-1}\mathbf{1}_n'[(\sigma_4^2 - \sigma_{4(*)}^2)^{-1}\boldsymbol{I}_n \\ &\quad + n^{-1}\mathbf{1}_n\mathbf{1}_v'\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1\mathbf{1}_v n^{-1}\mathbf{1}_n']^{-1}n^{-1}\mathbf{1}_n\mathbf{1}_v'\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1 \\ &= [(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1} + n^{-1}\mathbf{1}_v\mathbf{1}_n'(\sigma_4^2 - \sigma_{4(*)}^2)n^{-1}\mathbf{1}_n\mathbf{1}_v']^{-1} \\ &= [(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1} + (\sigma_4^2 - \sigma_{4(*)}^2)n^{-1}\mathbf{1}_v\mathbf{1}_v']^{-1}.\end{aligned}$$

Taking the inverse of this, we obtain the formula (43). From (42) it also follows that

$$\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1 = \boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1 + (\sigma_4^{-2} - \sigma_{4(*)}^{-2})n^{-1}\boldsymbol{r}\boldsymbol{r}',$$

due to the relation $\mathbf{1}_n'\boldsymbol{X}_1 = \boldsymbol{r}'$. Furthermore, using these results, the equality (44) can be proved, proceeding as follows:

$$\begin{aligned}
\hat{\boldsymbol{\tau}}_* &= (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{y} \\
&= (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{y} \\
&= (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')(\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{y} \\
&= (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}(\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}_v')\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{y} \\
&= (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'(\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n')\boldsymbol{V}^{-1}\boldsymbol{y} \\
&= (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'(\boldsymbol{I}_v - n^{-1}\mathbf{1}_n\mathbf{1}_n')[\boldsymbol{V}_*^{-1} \\
&\quad + (\sigma_4^{-2} - \sigma_{4(*)}^{-2})n^{-1}\mathbf{1}_n\mathbf{1}_n']\boldsymbol{y} \\
&= (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}(\boldsymbol{I}_v - n^{-1}\mathbf{1}_n\mathbf{1}_n')\boldsymbol{y},
\end{aligned}$$

because $(\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}_v')\boldsymbol{X}_1' = \boldsymbol{X}_1'(\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n')$ and $(\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n')\boldsymbol{V}_*^{-1} = \boldsymbol{V}_*^{-1}(\boldsymbol{I}_v - n^{-1}\mathbf{1}_n\mathbf{1}_n')$, as can easily be checked (see also Appendix 1).

Appendix 4

Formulae (46) and (47) are to be shown to be equivalent to formulae (19) and (20) respectively. To prove this, it may be helpful first to note the following equalities, which can easily be checked (see also Appendices 1 and 3):

$$\begin{aligned}
(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}(\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}_v') &= (\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}(\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}_v'), \\
\boldsymbol{X}_1(\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}') &= (\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n')\boldsymbol{X}_1, \\
(\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}_v')\boldsymbol{X}_1' &= \boldsymbol{X}_1'(\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n'), \\
\boldsymbol{V}^{-1}(\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n') &= \boldsymbol{V}_*^{-1}(\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n'), \\
(\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n')\boldsymbol{V}^{-1} &= (\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n')\boldsymbol{V}_*^{-1}, \\
\boldsymbol{V}_*^{-1}(\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n') &= (\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n')\boldsymbol{V}_*^{-1}.
\end{aligned}$$

With these observations, it is easy to proceed as follows, beginning with (19):

$$\begin{aligned}
\mathrm{SS}_V &= \boldsymbol{y}'\boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}(\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}_v')\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{y} \\
&= \boldsymbol{y}'\boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}(\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}_v')\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{y} \\
&= \boldsymbol{y}'\boldsymbol{V}^{-1}(\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n')\boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'(\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n')\boldsymbol{V}^{-1}\boldsymbol{y} \\
&= \boldsymbol{y}'(\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n')\boldsymbol{V}_*^{-1}\boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}(\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n')\boldsymbol{y},
\end{aligned}$$

which, with $\boldsymbol{y}_* = (\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}'_n)\boldsymbol{y}$, is equivalent to the formula (46).

Now, considering formula (20), first note (recalling Appendix 1) that

$$\begin{aligned}
&[\boldsymbol{V}^{-1} - \boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{X}'_1\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}'_1\boldsymbol{V}^{-1}]n^{-1}\mathbf{1}_n\mathbf{1}'_n \\
&= \boldsymbol{V}^{-1}n^{-1}\mathbf{1}_n\mathbf{1}'_n - \boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{X}'_1\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}'_1\boldsymbol{V}^{-1}n^{-1}\mathbf{1}_n\mathbf{1}'_n \\
&= \boldsymbol{V}^{-1}n^{-1}\mathbf{1}_n\mathbf{1}'_n - \boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{X}'_1\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}'_1\boldsymbol{V}^{-1}\boldsymbol{X}_1\mathbf{1}_v n^{-1}\mathbf{1}'_n \\
&= \boldsymbol{V}^{-1}n^{-1}\mathbf{1}_n\mathbf{1}'_n - \boldsymbol{V}^{-1}\boldsymbol{X}_1\mathbf{1}_v n^{-1}\mathbf{1}'_n = \boldsymbol{V}^{-1}n^{-1}\mathbf{1}_n\mathbf{1}'_n - \boldsymbol{V}^{-1}n^{-1}\mathbf{1}_n\mathbf{1}'_n = \mathbf{O}.
\end{aligned}$$

Using this result, formula (20) can be written as

$$\begin{aligned}
\mathrm{SS}_R \;&=\; \boldsymbol{y}'[\boldsymbol{V}^{-1} - \boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{X}'_1\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}'_1\boldsymbol{V}^{-1}]\boldsymbol{y} \\
&= \boldsymbol{y}'[\boldsymbol{V}^{-1} - \boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{X}'_1\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}'_1\boldsymbol{V}^{-1}](\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}'_n)\boldsymbol{y} \\
&= \boldsymbol{y}'[\boldsymbol{V}^{-1} - \boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{X}'_1\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}'_1\boldsymbol{V}^{-1}](\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}'_n)(\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}'_n)\boldsymbol{y} \\
&= \boldsymbol{y}'[\boldsymbol{V}^{-1}(\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}'_n) - \boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{X}'_1\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}'_1\boldsymbol{V}^{-1}(\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}'_n)]\boldsymbol{y}_* \\
&= \boldsymbol{y}'[(\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}'_n)\boldsymbol{V}^{-1} - \boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{X}'_1\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}'_1(\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}'_n)\boldsymbol{V}^{-1}]\boldsymbol{y}_* \\
&= \boldsymbol{y}'[(\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}'_n)\boldsymbol{V}_*^{-1} - \boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{X}'_1\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}'_1(\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}'_n)\boldsymbol{V}_*^{-1}]\boldsymbol{y}_* \\
&= \boldsymbol{y}'[(\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}'_n)\boldsymbol{V}_*^{-1} - \boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{X}'_1\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}(\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}'_v)\boldsymbol{X}'_1\boldsymbol{V}_*^{-1}]\boldsymbol{y}_* \\
&= \boldsymbol{y}'[(\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}'_n)\boldsymbol{V}_*^{-1} - \boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{X}'_1\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}(\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}'_v)\boldsymbol{X}'_1\boldsymbol{V}_*^{-1}]\boldsymbol{y}_* \\
&= \boldsymbol{y}'[(\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}'_n)\boldsymbol{V}_*^{-1} - \boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')(\boldsymbol{X}'_1\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}'_1\boldsymbol{V}_*^{-1}]\boldsymbol{y}_* \\
&= \boldsymbol{y}'[(\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}'_n)\boldsymbol{V}_*^{-1} - \boldsymbol{V}^{-1}(\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}'_n)\boldsymbol{X}_1(\boldsymbol{X}'_1\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}'_1\boldsymbol{V}_*^{-1}]\boldsymbol{y}_* \\
&= \boldsymbol{y}'[(\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}'_n)\boldsymbol{V}_*^{-1} - (\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}'_n)\boldsymbol{V}_*^{-1}\boldsymbol{X}_1(\boldsymbol{X}'_1\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}'_1\boldsymbol{V}_*^{-1}]\boldsymbol{y}_* \\
&= \boldsymbol{y}'_*[\boldsymbol{V}_*^{-1} - \boldsymbol{V}_*^{-1}\boldsymbol{X}_1(\boldsymbol{X}'_1\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}'_1\boldsymbol{V}_*^{-1}]\boldsymbol{y}_*,
\end{aligned}$$

which is identical to (47).