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A directional measure for marginal homogeneity in square contingency tables with ordered categories

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Summary

For square contingency tables with ordered categories, Iki, Tahata and Tomizawa (2012) considered a measure to represent the degree of departure from marginal homogeneity. However, the maximum value of this measure cannot distinguish two kinds of marginal inhomogeneity. The present paper proposes a measure which can distinguish two kinds of marginal inhomogeneity. In particular, the proposed measure is useful for representing the degree of departure from marginal homogeneity when the marginal cumulative logistic model holds.

Key words: Cumulative logistic model, Marginal homogeneity, Marginal odds, Square contingency table.

1. Introduction

Consider the data in Table 1, taken from Francom et al. (1989). These data describe the results of a randomized, double blind clinical trial comparing an active hypnotic drug with a placebo in patients who have insomnia problems. The response is the patient's reported time (in minutes) to fall asleep after going to bed. Patients responded before and following a two-week treatment period. The subjects receiving the two treatments are independent samples. For these data, we are interested in the difference in the time to fall asleep before and following a two-week treatment period, that is, whether the marginal probabilities of the row and column variables are the same.

Table 1. Time to falling asleep, by Treatment and Occasion (Francom et al., 1989), (a) Active, (b) Placebo

(a)	Follow-up				
Initial	< 20	20-30	30-60	> 60	Total
< 20	7	4	1	0	12
20-30	11	5	2	2	20
30-60	13	23	3	1	40
> 60	9	17	13	8	47
Total	40	49	19	11	119

(b)	Follow-up				
Initial	< 20	20-30	30-60	> 60	Total
< 20	7	4	2	1	14
20-30	14	5	1	0	20
30-60	6	9	18	2	35
> 60	4	11	14	22	51
Total	31	29	35	25	120

Consider an $R \times R$ square contingency table. Let p_{ij} denote the probability that an observation will fall in the ith row and jth column of the table ($i = 1, \dots, R$; $j = 1, \dots, R$), and let X and Y denote the row and column variables respectively. The marginal homogeneity (MH) model is defined by

$$p_{i\cdot} = p_{\cdot i} \quad (i = 1, \dots, R),$$

where $p_{i\cdot} = \sum_{t=1}^{R} p_{it}$ and $p_{\cdot i} = \sum_{s=1}^{R} p_{si}$ (see Stuart 1955; Bishop et al. 1975, p. 293; Tahata et al. 2007; Tahata and Yoshimoto 2015). This indicates that the row marginal distribution is identical to the column marginal distribution. This model is also expressed as

$$F_i^X = F_i^Y \quad (i = 1, \dots, R-1),$$

where $F_i^X = \sum_{k=1}^{i} p_{k\cdot}$ and $F_i^Y = \sum_{k=1}^{i} p_{\cdot k}$. Using the marginal logit, this model is expressed as

$$L_i^X = L_i^Y \quad (i = 1, \dots, R-1),$$

where

$$L_i^X = \log\left(\frac{F_i^X}{1 - F_i^X}\right), \quad L_i^Y = \log\left(\frac{F_i^Y}{1 - F_i^Y}\right).$$

This states that the log odds that X is i or below instead of $i+1$ or above is equal to the log odds that Y is i or below instead of $i+1$ or above for $i=1,\dots,R-1$. Further, the MH model is also expressed as

$$H_{1(i)} = H_{2(i)} \quad (i=1,\dots,R-1),$$

where

$$H_{1(i)} = \sum_{s=1}^{i}\sum_{t=i+1}^{R} p_{s\cdot}p_{\cdot t} = F_i^X(1-F_i^Y),$$
$$H_{2(i)} = \sum_{s=i+1}^{R}\sum_{t=1}^{i} p_{s\cdot}p_{\cdot t} = (1-F_i^X)F_i^Y.$$

This states that the probability that the row variable X selected at random from the row marginal distribution is in category i or below and the column variable Y selected independently at random from the column marginal distribution is in category $i+1$ or above is equal to the probability that such X is in category $i+1$ or above and such Y is in category i or below.

When the MH model does not hold, we are interested in measuring the degree of departure from the MH model. For square contingency tables with ordered categories, Iki et al. (2012) proposed the power-divergence measure $\Psi^{(\lambda)}$ to represent the degree of departure from MH (see Appendix for $\Psi^{(\lambda)}$). They also noted that, assuming $\{H_{1(i)} + H_{2(i)} \neq 0\}$, (i) the measure $\Psi^{(\lambda)}$ lies between 0 and 1, (ii) $\Psi^{(\lambda)} = 0$ if and only if the MH model holds, and (iii) $\Psi^{(\lambda)} = 1$ if and only if the degree of departure from MH is maximum, that is, $H_{1(i)} = 0$ (then $H_{2(i)} > 0$) or $H_{2(i)} = 0$ (then $H_{1(i)} > 0$) for all $i=1,\dots,R-1$.

Consider the artificial probabilities in Table 2, where we assume that the row and column categories have the same categories in Table 1, that is, X and Y represent baseline and follow-up respectively, and "(1)" is the best category and "(4)" is the worst category. In the case of Table 2a, this can be interpreted as follows: there are variations in symptoms at baseline, and only best symptoms at follow-up. In the case of Table 2b, it can be interpreted as follows: there are only worst symptoms at baseline, and variations in symptoms at follow-up. Then, for the probabilities in Tables 2a and 2b, the measure $\Psi^{(\lambda)} = 1$ with $\{H_{1(i)} = 0\}$. On the other hand, consider the artificial probabilities in Table 3. In the case of Table 3a, the interpretation is as follows: there are variations in symptoms at baseline, and only worst symptoms at follow-up. In the case of Table 3b, the interpretation is as follows: there are only best symptoms at baseline, and variations in symptoms at follow-up. Then, for the probabilities in Tables 3a and 3b, the measure $\Psi^{(\lambda)} = 1$ with $\{H_{2(i)} = 0\}$. Thus, the measure $\Psi^{(\lambda)}$ cannot distinguish two kinds of marginal inhomogeneity, where the marginal inhomogeneity is either of (i) $H_{1(i)} = 0$ (then $H_{2(i)} > 0$) for all $i = 1, \ldots, R-1$, or (ii) $H_{2(i)} = 0$ (then $H_{1(i)} > 0$) for all $i = 1, \ldots, R-1$. We are interested in proposing a measure which can take different values for these cases.

Table 2. Artificial probabilities satisfying $\Psi^{(\lambda)} = 1$ and $\Phi = 1$ with $\{H_{1(i)} = 0\}$

(a)	Y				
X	(1)	(2)	(3)	(4)	Total
(1)	0.3	0	0	0	0.3
(2)	0.1	0	0	0	0.1
(3)	0.4	0	0	0	0.4
(4)	0.2	0	0	0	0.2
Total	1	0	0	0	1

(b)	Y				
X	(1)	(2)	(3)	(4)	Total
(1)	0	0	0	0	0
(2)	0	0	0	0	0
(3)	0	0	0	0	0
(4)	0.2	0.4	0.1	0.3	1
Total	0.2	0.4	0.1	0.3	1

The purpose of this paper is to propose such a measure which can distinguish two kinds of marginal inhomogeneity for square contingency tables with ordered categories.

2. The measure

For an $R \times R$ square contingency table with ordered categories, let

$$\Gamma = \sum_{m=1}^{R-1} (H_{1(m)} + H_{2(m)}),$$

and let

$$H^*_{1(i)} = \frac{H_{1(i)}}{\Gamma}, \quad H^*_{2(i)} = \frac{H_{2(i)}}{\Gamma} \quad (i = 1, \ldots, R-1).$$

Table 3. Artificial probabilities satisfying $\Psi^{(\lambda)} = 1$ and $\Phi = -1$ with $\{H_{2(i)} = 0\}$

(a)	Y				
X	(1)	(2)	(3)	(4)	Total
(1)	0	0	0	0.2	0.2
(2)	0	0	0	0.3	0.3
(3)	0	0	0	0.4	0.4
(4)	0	0	0	0.1	0.1
Total	0	0	0	1	1

(b)	Y				
X	(1)	(2)	(3)	(4)	Total
(1)	0.1	0.4	0.3	0.2	1
(2)	0	0	0	0	0
(3)	0	0	0	0	0
(4)	0	0	0	0	0
Total	0.1	0.4	0.3	0.2	1

Assuming that $\{H_{1(i)} + H_{2(i)} \neq 0\}$, consider a measure defined by

$$\Phi = \frac{4}{\pi} \sum_{i=1}^{R-1} (H^*_{1(i)} + H^*_{2(i)}) \left(\theta_i - \frac{\pi}{4} \right),$$

where

$$\theta_i = \cos^{-1} \left(\frac{H^*_{1(i)}}{\sqrt{(H^*_{1(i)})^2 + (H^*_{2(i)})^2}} \right).$$

The range of θ_i is $0 \leqslant \theta_i \leqslant \frac{\pi}{2}$ $(i = 1, \ldots, R-1)$. Thus, the measure Φ lies between -1 and 1. The measure Φ has the characteristics that (i) $\Phi = -1$ if and only if $H_{2(i)} = 0$ (then $H_{1(i)} > 0$) for all $i = 1, \ldots, R-1$, and (ii) $\Phi = 1$ if and only if $H_{1(i)} = 0$ (then $H_{2(i)} > 0$) for all $i = 1, \ldots, R-1$. For example, consider the artificial probabilities in Tables 2 and 3 again. For Tables 2a and 2b, we see that the measure $\Phi = 1$, thus, it indicates that the degree of departure from MH toward improvement is maximum. For Tables 3a and 3b, we see that the measure $\Phi = -1$, thus, it indicates that the degree of departure from MH toward worsening is maximum.

In addition, $\Phi = 0$ indicates that the weighted average of $\{\theta_i - \frac{\pi}{4}\}$ equals zero. Thus when $\Phi = 0$, we shall refer to this structure as the average marginal homogeneity. We note that if the MH model holds then the average marginal homogeneity holds; however, the converse does not hold.

3. Relationships between the measure and a model

Consider $R \times R$ tables with ordered categories. The marginal cumulative logistic (ML) model is defined by

$$L_i^X = L_i^Y + \Delta \quad (i = 1, \dots, R-1);$$

see McCullagh (1977) and Agresti (2010, p. 231). This model states that the odds that X is i or below instead of $i+1$ or above, is $\exp(\Delta)$ times higher than the odds that Y is i or below instead of $i+1$ or above, for $i = 1, \dots, R-1$. A special case of the ML model with $\Delta = 0$ is the MH model. The ML model is also expressed as

$$H_{1(i)} = e^{\Delta} H_{2(i)} \quad (i = 1, \dots, R-1).$$

When the ML model holds, the measure Φ can be expressed simply (as a function of parameter Δ) as

$$\Phi = \frac{4}{\pi} \cos^{-1}\left(\frac{e^{\Delta}}{\sqrt{1+e^{2\Delta}}}\right) - 1.$$

Therefore, $\Phi = 0$ if and only if $\Delta = 0$, i.e., the MH model holds. As the value of Δ approaches ∞, the measure Φ approaches -1. As the value of Δ approaches $-\infty$, the measure Φ approaches 1. Thus, for comparisons between several tables, if it can be estimated that each model contains a structure of the ML model, then the measure Φ will be adequate for representing and comparing the degree of departure from the MH model toward two kinds of marginal inhomogeneity.

4. Approximate confidence interval for the measure

Let n_{ij} denote the observed frequency in the ith row and jth column of the table ($i = 1, \dots, R; j = 1, \dots, R$). Assuming that a multinomial distribution applies to the $R \times R$ table, we shall consider an approximate standard error and large-sample confidence interval for Φ using the delta method, descriptions of which are given by Bishop et al. (1975, Sec. 14.6). The sample version Φ, i.e., $\hat{\Phi}$, is given by Φ with $\{p_{ij}\}$ replaced by $\{\hat{p}_{ij}\}$, where $\hat{p}_{ij} = n_{ij}/n$ and $n = \sum\sum n_{ij}$.

Let $\hat{p}$ be the $R^2 \times 1$ vector

$$\hat{p} = (\hat{p}_{11}, \hat{p}_{12}, \ldots, \hat{p}_{1R}, \hat{p}_{21}, \ldots, \hat{p}_{RR})',$$

where "prime" denotes the transpose. Also, let the vector p be in terms of the p_{ij}'s in the same way as $\hat{p}$. Then $\sqrt{n}(\hat{p} - p)$ is asymptotically distributed as normal with mean zero and variance $\mathrm{diag}(p) - pp'$, where $\mathrm{diag}(p)$ denotes a diagonal matrix with the ith element of p as the ith diagonal element. We obtain

$$\hat{\Phi} = \Phi + \left(\frac{\partial \Phi}{\partial p'}\right)(\hat{p} - p) + o(\|\hat{p} - p\|),$$

where $\partial\Phi/\partial p'$ is a $1 \times R^2$ vector. Then $\sqrt{n}(\hat{\Phi} - \Phi)$ follows asymptotically (as $n \to \infty$) a normal distribution with mean zero and variance,

$$\sigma^2[\Phi] = \left(\frac{\partial \Phi}{\partial p'}\right)(\mathrm{diag}(p) - pp')\left(\frac{\partial \Phi}{\partial p'}\right)' = \sum_{k=1}^{R}\sum_{l=1}^{R} p_{kl}(D_{kl})^2,$$

where

$$D_{kl} = \frac{4}{\pi\Gamma}\sum_{i=1}^{R-1}\left\{(v_{kl(i)} + w_{kl(i)})\theta_i + \frac{(H_{1(i)} + H_{2(i)})(H_{1(i)}w_{kl(i)} - H_{2(i)}v_{kl(i)})}{(H_{1(i)})^2 + (H_{2(i)})^2}\right\}$$
$$- \frac{\Phi + 1}{\Gamma}\sum_{i=1}^{R-1}(v_{kl(i)} + w_{kl(i)}),$$

with

$$v_{kl(i)} = \sum_{s=1}^{i}\sum_{t=i+1}^{R}(I_{(s=k)}p_{\cdot t} + p_{s\cdot}I_{(t=l)}),$$
$$w_{kl(i)} = \sum_{s=i+1}^{R}\sum_{t=1}^{i}(I_{(s=k)}p_{\cdot t} + p_{s\cdot}I_{(t=l)}),$$

and $I_{(\cdot)}$ is the indicator function, $I_{(\cdot)} = 1$ if true, 0 if not.

Let $\hat{\sigma}^2[\Phi]$ denote $\sigma^2[\Phi]$ with $\{p_{ij}\}$ replaced by $\{\hat{p}_{ij}\}$. Thus, the square root of $\hat{\sigma}^2[\Phi]/n$ is an estimated standard error of $\hat{\Phi}$, and

$$\hat{\Phi} \pm z_{\alpha/2}\sqrt{\hat{\sigma}^2[\Phi]/n},$$

is an approximate $100(1 - \alpha)\%$ confidence interval for Φ, where $z_{\alpha/2}$ is the percentage point of the standard normal distribution corresponding to a two-tail probability of α.

5. Examples

Example 1: Consider the data in Table 1 again. We see from Table 5a that the estimated values of the measure Φ for the data in Tables 1a and 1b are 0.799 and 0.541 respectively. Since values in the confidence interval for Φ applied to the data in each of Tables 1a and 1b are positive, these would indicate that the structure of MH is not present, and the departure from MH is toward improvement in each table. When the degree of departure from MH in Tables 1a and 1b is compared using the confidence intervals for Φ, it would be greater for Table 1a than for Table 1b. Namely, the active drug is more effective than the placebo for patients with insomnia problems.

Example 2. Consider the data in Table 4, taken from Sugano et al. (2012). These data describe the results of a randomized, double blind clinical trial comparing esomeprazole with a placebo in patients having a history of peptic ulcer. The response is the modified LANZA score (Lanza et al., 1988). With the modified LANZA score, a score of 0 is the best evaluation and a score of $+4$ is the worst evaluation. Patients responded before and following a 24-week treatment period. The subjects receiving the two treatments are independent samples. From Table 5b, since the values in the confidence interval for Φ applied to the data in Table 4a are positive, this would indicate that the structure of MH is not present, and the departure from MH is toward improvement. Since the values in the confidence interval for Φ applied to the data in Table 4b are negative, this would indicate that the structure of MH is not present, and the degree of departure from MH is toward worsening.

6. Concluding remarks

The proposed measure Φ is useful for representing the degree of departure from the average marginal homogeneity toward two kinds of marginal inhomogeneity (i.e., improvement and worsening). The measure Φ can distinguish these two kinds of marginal inhomogeneity, while the measure in Iki et al. (2012) cannot distinguish them.

The measure Φ would be useful for comparing the degree of departure from the average marginal homogeneity in several tables. In particular, if the table contains a structure of the ML model, then Φ will be adequate for representing the degree of departure from MH toward two kinds of marginal inhomogeneity.

Table 4. Shift analysis of modified LANZA score after 24 weeks' treatment with esomeprazole 20 mg once daily and placebo, stratified by modified LANZA score at baseline (Sugano et al., 2012), (a) Esomeprazole 20 mg once daily, (b) Placebo

(a)	Study end					
Baseline	0	+1	+2	+3	+4	Total
0	78	1	9	1	3	92
+1	9	5	1	0	0	15
+2	26	6	10	1	1	44
+3	3	4	3	0	1	11
+4	1	0	1	0	2	4
Total	117	16	24	2	7	166

(b)	Study end					
Baseline	0	+1	+2	+3	+4	Total
0	41	8	12	0	29	90
+1	2	0	4	1	7	14
+2	19	4	14	1	11	49
+3	0	0	3	3	6	12
+4	0	0	0	0	0	0
Total	62	12	33	5	53	165

Table 5. Estimate of Φ, estimated approximate standard error for $\hat{\Phi}$, and approximate 95% confidence interval for Φ, applied to Tables 1 (a) and 4 (b)

(a) Applied data	Estimated measure	Standard error	Confidence interval
Table 1a	0.799	0.045	(0.712, 0.887)
Table 1b	0.541	0.072	(0.401, 0.681)

(b) Applied data	Estimated measure	Standard error	Confidence interval
Table 4a	0.366	0.103	(0.165, 0.568)
Table 4b	−0.615	0.069	(−0.750, −0.480)

The measure Φ should be applied to the ordinal data of square contingency tables with the same row and column classification because Φ is not invariant under arbitrary similar permutations of row and column categories.

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Appendix

The measure of departure from marginal homogeneity considered by Iki et al. (2012) is given as follows: assuming that $\{H_{1(i)} + H_{2(i)} \neq 0\}$, for $\lambda > -1$,

$$\Psi^{(\lambda)} = \frac{\lambda(\lambda+1)}{2^{\lambda}-1} I^{(\lambda)},$$

where

$$I^{(\lambda)} = \frac{1}{\lambda(\lambda+1)} \sum_{i=1}^{R-1} \left[H_{1(i)}^{*} \left\{ \left(\frac{H_{1(i)}^{*}}{Q_i^{*}} \right)^{\lambda} - 1 \right\} + H_{2(i)}^{*} \left\{ \left(\frac{H_{2(i)}^{*}}{Q_i^{*}} \right)^{\lambda} - 1 \right\} \right],$$

with

$$\Gamma = \sum_{m=1}^{R-1} (H_{1(m)} + H_{2(m)}), \quad H_{1(i)}^{*} = \frac{H_{1(i)}}{\Gamma}, \quad H_{2(i)}^{*} = \frac{H_{2(i)}}{\Gamma}, \quad Q_i^{*} = \frac{1}{2}(H_{1(i)}^{*} + H_{2(i)}^{*}),$$

and the value at $\lambda = 0$ is taken to be the limit as $\lambda \to 0$.