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On a new approach to the analysis of variance for experiments with orthogonal block structure.

II. Experiments in nested block designs

Tadeusz Caliński, Idzi Siatkowski

Department of Mathematical and Statistical Methods, Poznań University of Life Sciences, Wojska Polskiego 28, 60-637 Poznań, Poland,
e-mail: calinski@up.poznan.pl, idzi.siatkowski@up.poznan.pl

Summary

The main estimation and hypothesis testing procedures are presented for experiments conducted in nested block designs of a certain type. It is shown that, under appropriate randomization, these experiments have the convenient orthogonal block structure. Due to this property, the analysis of experimental data can be performed in a comparatively simple way. Certain simplifying procedures are indicated. The main advantage of the presented methodology concerns the analysis of variance and related hypothesis testing procedures. Under the adopted approach one can perform these analytical methods directly, not by combining the results from analyses based on stratum submodels. The application of the presented theory is illustrated by three examples of real experiments in relevant nested block designs. The present paper is the second in the planned series concerning the analysis of experiments with orthogonal block structure.

Key words: analysis of variance, estimation, hypothesis testing, nested block designs, orthogonal block structure, randomization-derived model

1. Introduction

The concept of orthogonal block structure, as a desirable property, was originally considered for a wide class of designs by Nelder (1965) and then formalized by Houtman and Speed (1983). After them, the following definition can be adopted.

Definition 1.1 (from Section 2.2 in Houtman and Speed, 1983). An experiment is said to have the orthogonal block structure (OBS) if the covariance (dispersion) matrix of the random variables observed on the experimental units (plots), $\boldsymbol{y} = [y_1, y_2, ..., y_n]'$, has a representation of the form

$$\mathrm{D}(\boldsymbol{y}) = \sigma_1^2\boldsymbol{\phi}_1 + \sigma_2^2\boldsymbol{\phi}_2 + \cdots + \sigma_t^2\boldsymbol{\phi}_t,$$

where the $\{\boldsymbol{\phi}_\alpha\}$, $\alpha = 1, 2, ..., t$, are known symmetric, idempotent and pairwise orthogonal matrices, summing to the identity matrix, the last usually being of the form $\boldsymbol{\phi}_t = n^{-1}\mathbf{1}_n\mathbf{1}_n'$.

It appears that experiments having the OBS property can be analyzed in a comparatively simple way. In particular, the analysis of variance (ANOVA) can be performed directly, avoiding the classical procedure of first conducting the analyses based on stratum submodels and then combining the informations obtained from them, as originally suggested by Yates (1939, 1940) and recently discussed by Kala (2017).

Because of this feature, it may be interesting to show the analytical advantage of various experiments having the OBS property. To indicate the underlying theory and relevant methodological procedures, it will be helpful to do this for different classes of designs separately. Thus, a set of research papers focused on practical applications has been projected. The present paper, as the second in this series, is devoted to experiments conducted in nested block designs inducing the OBS property.

Nested block (NB) designs are often used in practice, particularly in agricultural and industrial experimentation, when several sources of local variation are present − more than can be controlled by ordinary blocking of experimental units. The statistical properties of NB designs have been considered in many papers, as reviewed by Bailey (1999). Of special interest are those NB designs which induce the OBS property.

The purpose of the present paper is to show how the OBS property of an experiment in an NB design provides the possibility of performing the analysis of experimental data with a comparatively simple methodology. Similarly as in the first paper of the present series (Caliński and Siatkowski, 2017), in Section 2 the randomization-derived mixed model, from which the described methodology follows, is indicated. The theoretical background of the derived analysis is presented in Section 3. In Section 4 some simplifications of the proposed analytical methods are suggested. In Section 5 attention is drawn to some consequences resulting from the use of estimated stratum variances. Examples illustrating the application of the derived analytical methods, ANOVA in particular, are presented in Section 6. Some concluding remarks concerning the advantages of the proposed new approach are given in Section 7. Finally, several appendices with helpful derivations of the applied methods are provided.

2. A randomization-derived model

Consider an experiment carried out in an NB design with v treatments (crop varieties in particular) allocated in $b = ab_0$ blocks, each of k units (plots), grouped into a superblocks composed of b_0 blocks each. Such an NB design is said to induce the OBS property (in the above sense).

Suppose that independent randomizations of superblocks, of blocks within the superblocks and of plots within the blocks have been implemented in the experiment according to the usual procedure (as described, for example, in Caliński and Kageyama, 2000, Section 5.2.1, following Nelder, 1965). The randomization-derived model can then be written as

$$\boldsymbol{y} = \boldsymbol{X}_1\boldsymbol{\tau} + \boldsymbol{X}_A\boldsymbol{\alpha} + \boldsymbol{X}_B\boldsymbol{\beta} + \boldsymbol{\eta} + \boldsymbol{e}, \tag{1}$$

where $\boldsymbol{y} = [\boldsymbol{y}'_1, \boldsymbol{y}'_2, ..., \boldsymbol{y}'_a]'$ is an $n \times 1$ vector of data concerning yield (or an another variable trait) observed on $n = ab_0k$ plots of the experiment, $\boldsymbol{y}_h = [y_{1h}, y_{2h}, ..., y_{n_0h}]'$ representing the yields observed on $n_0 = kb_0$ units of the superblock h $(= 1, 2, ..., a)$,

$$\boldsymbol{X}_1 = [\boldsymbol{X}'_{11} : \boldsymbol{X}'_{12} : \cdots : \boldsymbol{X}'_{1a}]',$$

$$\boldsymbol{X}_A = \boldsymbol{I}_a \otimes \boldsymbol{1}_{n_0},\ \ \boldsymbol{X}_B = \text{diag}[\boldsymbol{X}_{B1} : \boldsymbol{X}_{B2} : \cdots : \boldsymbol{X}_{Ba}]$$

are the known design matrices, and $\boldsymbol{\tau} = [\tau_1, \tau_2, ..., \tau_v]'$ represents the unobservable treatment parameters (their fixed effects), $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, ..., \alpha_a]'$ stands for the superblock random effects, $\boldsymbol{\beta} = [\boldsymbol{\beta}'_1, \boldsymbol{\beta}'_2, ..., \boldsymbol{\beta}'_a]'$, with $\boldsymbol{\beta}_h = [\beta_{1(h)}, \beta_{2(h)}, ..., \beta_{b_0(h)}]'$, stands for the block random effects, while the $n \times 1$ vectors $\boldsymbol{\eta}$ and $\boldsymbol{e}$ stand for the unit error and technical error random variables, all of these random variables being unobservable.

The whole block design, denoted by $\mathcal{D}^*$, can be described by the $v \times b$ incidence matrix

$$\boldsymbol{N} = \boldsymbol{X}'_1\boldsymbol{X}_B = [\boldsymbol{N}_1 : \boldsymbol{N}_2 : \cdots : \boldsymbol{N}_a],$$

with $\boldsymbol{N}_h = \boldsymbol{X}'_{1h}\boldsymbol{X}_{Bh}$ as the $v \times b_0$ incidence matrix describing the hth component design, denoted by $\mathcal{D}_h$, where $\boldsymbol{N}'_h\boldsymbol{1}_v = k\boldsymbol{1}_{b_0}$ and $\boldsymbol{N}_h\boldsymbol{1}_{b_0} = \boldsymbol{r}_h$, the vector of treatment replications in $\mathcal{D}_h, h = 1, 2, ..., a$. Furthermore, note that the design, denoted by $\mathcal{D}$, by which the v treatments are assigned to the a superblocks is described by the $v \times a$ incidence matrix

$$\boldsymbol{M} = \boldsymbol{X}_1'\boldsymbol{X}_A = [\boldsymbol{r}_1 : \boldsymbol{r}_2 : \cdots : \boldsymbol{r}_a].$$

Because both $\mathcal{D}^*$ and $\mathcal{D}$ are proper, an experiment in such an NB design has the OBS property (see Lemma 5.4.1 in Caliński and Kageyama, 2000). This allows the model to be resolved into four simple stratum submodels, in accordance with the stratification of the experimental units. Using Nelder's (1965) notation, this stratification ("block-structure") can be represented by the relation

Units (plots) → Blocks → Superblocks → Total area.

Thus, the observed vector $\boldsymbol{y}$ can be decomposed as

$$\boldsymbol{y} = \boldsymbol{y}_1 + \boldsymbol{y}_2 + \boldsymbol{y}_3 + \boldsymbol{y}_4,$$

$$\boldsymbol{y}_1 = \boldsymbol{\phi}_1\boldsymbol{y},\ \ \boldsymbol{y}_2 = \boldsymbol{\phi}_2\boldsymbol{y},\ \ \boldsymbol{y}_3 = \boldsymbol{\phi}_3\boldsymbol{y},\ \ \boldsymbol{y}_4 = \boldsymbol{\phi}_4\boldsymbol{y},$$

which allows the expectation vector and the covariance (dispersion) matrix of $\boldsymbol{y}$ to be written as

$$\mathrm{E}(\boldsymbol{y}) = \boldsymbol{\phi}_1\boldsymbol{X}_1\boldsymbol{\tau} + \boldsymbol{\phi}_2\boldsymbol{X}_1\boldsymbol{\tau} + \boldsymbol{\phi}_3\boldsymbol{X}_1\boldsymbol{\tau} + \boldsymbol{\phi}_4\boldsymbol{X}_1\boldsymbol{\tau} = \boldsymbol{X}_1\boldsymbol{\tau}, \tag{2}$$

$$\mathrm{D}(\boldsymbol{y}) \equiv \boldsymbol{V} = \sigma_1^2\boldsymbol{\phi}_1 + \sigma_2^2\boldsymbol{\phi}_2 + \sigma_3^2\boldsymbol{\phi}_3 + \sigma_4^2\boldsymbol{\phi}_4, \tag{3}$$

where the matrices

$$\boldsymbol{\phi}_1 = \boldsymbol{I}_n - k^{-1}\boldsymbol{X}_B\boldsymbol{X}_B',\ \ \boldsymbol{\phi}_2 = k^{-1}\boldsymbol{X}_B\boldsymbol{X}_B' - n_0^{-1}\boldsymbol{X}_A\boldsymbol{X}_A',$$

$$\boldsymbol{\phi}_3 = n_0^{-1}\boldsymbol{X}_A\boldsymbol{X}_A' - n^{-1}\boldsymbol{1}_n\boldsymbol{1}_n' \ \text{ and } \ \boldsymbol{\phi}_4 = n^{-1}\boldsymbol{1}_n\boldsymbol{1}_n'$$

are symmetric, idempotent and pairwise orthogonal, summing to the identity matrix, and the scalars $\sigma_1^2, \sigma_2^2, \sigma_3^2$ and σ_4^2 represent the relevant unknown stratum variances (defined as in Caliński and Kageyama, 2000, Section 5.4).

3. Theoretical background of the analysis

When analyzing data from an experiment modelled by (1), a variety trial in particular, attention is usually paid to estimates and tests concerning the parameters $\boldsymbol{\tau} = [\tau_1, \tau_2, ..., \tau_v]'$, or rather the treatment (variety) main effects, defined as

$$(\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')\boldsymbol{\tau} = [\tau_1 - \tau_., \tau_2 - \tau_., ..., \tau_v - \tau_.]', \text{ where } \tau_. = n^{-1}\sum_{i=1}^{v}(r_i\tau_i),$$

and also their linear functions. In connection with this, first note (referring, for example, to Caliński and Kageyama, 2000, Section A.2) that, taking the orthogonal ($\boldsymbol{V}^{-1}$-orthogonal) projector

$$\boldsymbol{P}_{X_1(V^{-1})} = \boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}^{-1}, \tag{4}$$

one can decompose the vector $\boldsymbol{y}$ in (1) into two uncorrelated parts, as

$$\boldsymbol{y} = \boldsymbol{P}_{X_1(V^{-1})}\boldsymbol{y} + (\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}. \tag{5}$$

The role of the two parts on the right in (5) can easily be seen.

Under the model (1), with properties (2) and (3), the first term of the partition in (5) provides the best linear unbiased estimator (BLUE) of $\boldsymbol{X}_1\boldsymbol{\tau}$ in (2), which can be expressed as

$$\widehat{\boldsymbol{X}_1\boldsymbol{\tau}} = \boldsymbol{P}_{X_1(V^{-1})}\boldsymbol{y}, \tag{6}$$

as follows from Rao (1974, Theorem 3.2). With regard to the second term in (5), it can be seen as the residual vector, giving the residual sum of squares in the form

$$\begin{aligned}\|(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2_{V^{-1}} &= \boldsymbol{y}'(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})'\boldsymbol{V}^{-1}(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y} \\ &= \boldsymbol{y}'[\boldsymbol{V}^{-1} - \boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}^{-1}]\boldsymbol{y} \\ &= \boldsymbol{y}'\boldsymbol{V}^{-1}(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}, \end{aligned} \tag{7}$$

with the residual degrees of freedom given by $\text{rank}(\boldsymbol{V} : \boldsymbol{X}_1) - \text{rank}(\boldsymbol{X}_1) = n - v$. See Rao (1974, Theorem 3.4) and formula (3.17) there. For convenience note that, when using the projector (4) in the considered applications, the variance σ_4^2 in the involved matrix $\boldsymbol{V}$, defined in (3), can be replaced by 1. This is evident from the formula

$$\boldsymbol{P}_{X_1(V^{-1})} = \boldsymbol{X}_1\sum_{i=1}^{v-1}\varepsilon_i^{-1}\boldsymbol{s}_i\boldsymbol{s}_i'\boldsymbol{X}_1'\boldsymbol{V}_0^{-1} + \boldsymbol{\phi}_4,$$

with $\boldsymbol{V}_0^{-1} = \sigma_1^{-2}\boldsymbol{\phi}_1 + \sigma_2^{-2}\boldsymbol{\phi}_2 + \sigma_3^{-2}\boldsymbol{\phi}_3$,

resulting from the spectral decomposition $\boldsymbol{X}_1'\boldsymbol{V}_0^{-1}\boldsymbol{X}_1 = \boldsymbol{r}^\delta(\sum_{i=1}^{v-1}\varepsilon_i\boldsymbol{s}_i\boldsymbol{s}_i')\boldsymbol{r}^\delta$, where $\boldsymbol{r}^\delta = \mathrm{diag}[r_1, r_2, ..., r_v]$, applied similarly to (2.3.2) in Caliński and Kageyama (2000, p. 36); see also formula (3.8.15) there. (For the spectral decomposition forms see Rao and Mitra, 1971, pp. 5−7.)

It will also be interesting to note that, as $\boldsymbol{\tau} = \boldsymbol{r}^{-\delta}\boldsymbol{X}_1'\boldsymbol{X}_1\boldsymbol{\tau}$, the BLUE of $\boldsymbol{\tau}$ can be obtained, by (4) and (6), as

$$\hat{\boldsymbol{\tau}} = (\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{y}. \tag{8}$$

Its covariance (dispersion) matrix then takes the form

$$\begin{aligned}\mathrm{D}(\hat{\boldsymbol{\tau}}) &= (\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}^{-1}\mathrm{D}(\boldsymbol{y})\boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1} \\ &= (\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}.\end{aligned} \tag{9}$$

The results (7)−(9) can be checked by referring to Theorem 3.1 in Rao (1971). For this one has to show that the equality

$$\begin{bmatrix} \boldsymbol{V} & \boldsymbol{X}_1 \\ \boldsymbol{X}_1' & \mathbf{O} \end{bmatrix}^- = \begin{bmatrix} \boldsymbol{V}^{-1}(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})}) & \boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1} \\ (\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}^{-1} & -(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1} \end{bmatrix}$$

holds. In fact, this can easily be checked.

With these results the concept of testing the hypothesis

$$H_0 : (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')\boldsymbol{\tau} = \mathbf{0}, \tag{10}$$

can be considered. First one has to see whether the hypothesis (10) is consistent. For this, note that the BLUE of $\boldsymbol{\tau}_* = (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')\boldsymbol{\tau}$ is $\hat{\boldsymbol{\tau}}_* = (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')\hat{\boldsymbol{\tau}}$, with $\hat{\boldsymbol{\tau}}$ as given in (8). Its dispersion matrix, by (9), is of the form

$$\mathrm{D}(\hat{\boldsymbol{\tau}}_*) = (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}(\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}_v'), \tag{11}$$

with rank $v - 1$. It appears that as a g-inverse of $\mathrm{D}(\hat{\boldsymbol{\tau}}_*)$ one can take $[\mathrm{D}(\hat{\boldsymbol{\tau}}_*)]^- = \boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1$. Hence,

$$\mathrm{D}(\hat{\boldsymbol{\tau}}_*)[\mathrm{D}(\hat{\boldsymbol{\tau}}_*)]^-\hat{\boldsymbol{\tau}}_* = \hat{\boldsymbol{\tau}}_*, \tag{12}$$

as can be shown (see Appendix 1). The equality (12) indicates that H_0 in (10) is consistent; see formula (3.2.8) in Rao (1971).

Assuming now that $\boldsymbol{y} \sim N_n(\boldsymbol{X}_1\boldsymbol{\tau}, \boldsymbol{V})$ and, hence, that $\hat{\boldsymbol{\tau}}_* \sim N_v[\boldsymbol{\tau}_*, \mathrm{D}(\hat{\boldsymbol{\tau}}_*)]$, where $\boldsymbol{\tau}_*$ is as defined above, and $\mathrm{D}(\hat{\boldsymbol{\tau}}_*)$ is as in (11), one can test the hypothesis H_0 using the statistic

$$F = \frac{n-v}{v-1}\,\frac{\mathrm{SS}_V}{\mathrm{SS}_R} = \frac{n-v}{v-1}\,\frac{\hat{\boldsymbol{\tau}}_*'\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1\hat{\boldsymbol{\tau}}_*}{\boldsymbol{y}'\boldsymbol{V}^{-1}(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}}, \tag{13}$$

as follows from Theorem 3.2 in Rao (1971). Note, however, that the sums of squares in (13) can equivalently be written (see Appendix 2) as

$$\mathrm{SS}_V = \boldsymbol{y}'\boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{I}_v - \mathbf{1}_v\boldsymbol{r}'/n)(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}(\boldsymbol{I}_v - \boldsymbol{r}\mathbf{1}_v'/n)\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{y}, \tag{14}$$

$$\mathrm{SS}_R = \boldsymbol{y}'[\boldsymbol{V}^{-1} - \boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}^{-1}]\boldsymbol{y}. \tag{15}$$

Referring now to Theorems 9.2.1 and 9.4.1 in Rao and Mitra (1971), one can show that, independently,

$$\mathrm{SS}_V \sim \chi^2(v-1, \delta), \quad \text{with } \delta = \boldsymbol{\tau}_*'\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1\boldsymbol{\tau}_*, \tag{16}$$

$$\mathrm{SS}_R \sim \chi^2(n-v, 0). \tag{17}$$

Evidently, the distribution in (16) is central if the hypothesis H_0 is true, whereas that in (17) is central whether H_0 is true or not. These results imply that the statistic (13) has a noncentral F distribution with $v-1$ and $n-v$ degrees of freedom (d.f.), and with the noncentrality parameter δ as in (16). Thus, the distribution is central if H_0 is true.

It should be noted, however, that the above estimation and hypothesis testing procedures are applicable directly if the stratum variances σ_1^2, σ_2^2, σ_3^2 and σ_4^2 are known. In practice they are usually unknown and have to be estimated. To do this, it will be helpful to return to formula (7), writing it as

$$\begin{aligned}
&\|(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2_{V^{-1}} \\
&\quad = \boldsymbol{y}'(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})'\boldsymbol{V}^{-1}(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y} \\
&\quad = \sigma_1^{-2}\boldsymbol{y}'(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})'\boldsymbol{\phi}_1(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y} \\
&\qquad + \sigma_2^{-2}\boldsymbol{y}'(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})'\boldsymbol{\phi}_2(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y} \\
&\qquad + \sigma_3^{-2}\boldsymbol{y}'(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})'\boldsymbol{\phi}_3(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y},
\end{aligned} \tag{18}$$

which follows from the form of $\mathrm{D}(\boldsymbol{y}) \equiv \boldsymbol{V}$, given in (3). This form also implies, on account of the relation $\boldsymbol{\phi}_4 = n^{-1}\mathbf{1}_n\mathbf{1}_n' = n^{-1}\mathbf{1}_n\mathbf{1}_v'\boldsymbol{X}_1'$, that

$$\boldsymbol{\phi}_4(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})}) = \sigma_4^2\boldsymbol{\phi}_4\boldsymbol{V}^{-1}(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})}) = \mathbf{O}.$$

Now, from (18), one can write

$$\begin{aligned}\mathrm{E}\{\|(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2_{V^{-1}}\} &= \sigma_1^{-2}\mathrm{E}\{\|\boldsymbol{\phi}_1(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2\} \\ &\quad + \sigma_2^{-2}\mathrm{E}\{\|\boldsymbol{\phi}_2(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2\} \\ &\quad + \sigma_3^{-2}\mathrm{E}\{\|\boldsymbol{\phi}_3(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2\} \\ &= d'_1 + d'_2 + d'_3 = n - v, \end{aligned} \tag{19}$$

because, as can be shown,

$$\mathrm{E}\{\|\boldsymbol{\phi}_1(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2\} = \sigma_1^2 d'_1, \tag{20}$$

where $d'_1 = \mathrm{tr}[\boldsymbol{\phi}_1(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})]$,

$$\mathrm{E}\{\|\boldsymbol{\phi}_2(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2\} = \sigma_2^2 d'_2, \tag{21}$$

where $d'_2 = \mathrm{tr}[\boldsymbol{\phi}_2(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})]$,

$$\mathrm{E}\{\|\boldsymbol{\phi}_3(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2\} = \sigma_3^2 d'_3, \tag{22}$$

where $d'_3 = \mathrm{tr}[\boldsymbol{\phi}_3(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})]$.

With these results it is natural to use as estimators of σ_1^2, σ_2^2 and σ_3^2 the solutions of the equations

$$\|\boldsymbol{\phi}_1(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2 = \sigma_1^2 d'_1, \tag{23}$$

$$\|\boldsymbol{\phi}_2(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2 = \sigma_2^2 d'_2, \tag{24}$$

$$\|\boldsymbol{\phi}_3(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2 = \sigma_3^2 d'_3, \tag{25}$$

respectively (as suggested by Nelder, 1968, Section 3). This approach was also advocated by Houtman and Speed (1983, Section 4.5) and applied, for example, by Caliński and Łacka (2014, p. 959).

For completeness, it will be helpful to note that the equations (23), (24) and (25), with the formulae (20), (21) and (22), imply – on account of (19) – the equality

$$\begin{aligned}\hat{\sigma}_1^{-2}\|\boldsymbol{\phi}_1(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2 &+ \hat{\sigma}_2^{-2}\|\boldsymbol{\phi}_2(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2 \\ &+ \hat{\sigma}_3^{-2}\|\boldsymbol{\phi}_3(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2 \\ &\quad = d'_1 + d'_2 + d'_3 = n - v. \end{aligned} \tag{26}$$

Now, returning to (15), note that, after some algebraic transformations, it can be written equivalently as

$$\begin{aligned}\mathrm{SS}_R &= \boldsymbol{y}'[\boldsymbol{I}_n - \boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'](\sigma_1^{-2}\boldsymbol{\phi}_1 + \sigma_2^{-2}\boldsymbol{\phi}_2 + \sigma_3^{-2}\boldsymbol{\phi}_3)[\boldsymbol{I}_n \\ &\quad - \boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}^{-1}]\boldsymbol{y} \\ &= \sigma_1^{-2}\|\boldsymbol{\phi}_1(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2 + \sigma_2^{-2}\|\boldsymbol{\phi}_2(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2 \\ &\quad + \sigma_3^{-2}\|\boldsymbol{\phi}_3(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2. \end{aligned} \tag{27}$$

A comparison of formulae (26) and (27) shows that, if the stratum variances are estimated by solutions of the equations (23), (24) and (25), the result

$$\begin{aligned}\widehat{\mathrm{SS}}_R &= \hat\sigma_1^{-2}\|\boldsymbol{\phi}_1(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2 + \hat\sigma_2^{-2}\|\boldsymbol{\phi}_2(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2 \\ &\quad + \hat\sigma_3^{-2}\|\boldsymbol{\phi}_3(\boldsymbol{I}_n - \boldsymbol{P}_{X_1(V^{-1})})\boldsymbol{y}\|^2 = n - v \end{aligned} \tag{28}$$

then follows. By (28), the statistic F in (13) is reduced to the form

$$\widehat{F} = \frac{n-v}{v-1}\,\frac{\widehat{\mathrm{SS}}_V}{n-v} = \frac{\widehat{\mathrm{SS}}_V}{v-1}, \tag{29}$$

where $\widehat{\mathrm{SS}}_V$ is as in (14), but with σ_1^2, σ_2^2 and σ_3^2 there replaced by their estimates.

However, the χ^2 distribution of SS_V, indicated in (16), is valid exactly only if the true stratum variances are used in the applied matrix $\boldsymbol{V}^{-1} = \sigma_1^{-2}\boldsymbol{\phi}_1 + \sigma_2^{-2}\boldsymbol{\phi}_2 + \sigma_3^{-2}\boldsymbol{\phi}_3 + \sigma_4^{-2}\boldsymbol{\phi}_4$, resulting from (3). As for the component $\sigma_4^{-2}\boldsymbol{\phi}_4$, it does not in fact play any role in the application of formula (14) given for SS_V (as will be shown in the next section). Thus, when using in $\boldsymbol{V}^{-1}$ the estimates of σ_1^2, σ_2^2 and σ_3^2 obtained from (23), (24) and (25) respectively, the distribution (16) can be regarded as approximate only.

4. Some simplifying reformulations

According to certain remarks made in the previous section, the component $\sigma_4^{-2}\boldsymbol{\phi}_4$ in $\boldsymbol{V}^{-1} = \sigma_1^{-2}\boldsymbol{\phi}_1 + \sigma_2^{-2}\boldsymbol{\phi}_2 + \sigma_3^{-2}\boldsymbol{\phi}_3 + \sigma_4^{-2}\boldsymbol{\phi}_4$ seems to play no role in the formulae applicable in the considered analysis of experimental data. This suggests that some reformulation in the methodology presented in Section 3 would simplify the analysis without causing any changes in its results.

A desirable simplification can be obtained when the dispersion matrix $\boldsymbol{V}$ of the form given in (3) is replaced by the matrix

$$\boldsymbol{V}_* = \sigma_1^2\boldsymbol{\phi}_1 + \sigma_2^2\boldsymbol{\phi}_2 + \sigma_3^2(\boldsymbol{I}_n - \boldsymbol{\phi}_1 - \boldsymbol{\phi}_2),$$

i.e., when the inverted matrix $\boldsymbol{V}^{-1}$ is replaced by

$$\boldsymbol{V}_*^{-1} = \sigma_1^{-2}\boldsymbol{\phi}_1 + \sigma_2^{-2}\boldsymbol{\phi}_2 + \sigma_3^{-2}(\boldsymbol{I}_n - \boldsymbol{\phi}_1 - \boldsymbol{\phi}_2).$$

The relations between $\boldsymbol{V}$ and $\boldsymbol{V}_*$, and their inverses, are given by the equalities

$$\boldsymbol{V} = \boldsymbol{V}_* + (\sigma_4^2 - \sigma_3^2)n^{-1}\mathbf{1}_n\mathbf{1}_n' \quad \text{and}$$
$$\boldsymbol{V}^{-1} = \boldsymbol{V}_*^{-1} + (\sigma_4^{-2} - \sigma_3^{-2})n^{-1}\mathbf{1}_n\mathbf{1}_n'. \tag{30}$$

From (30) it follows (see Appendix 3) that

$$(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1} = (\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1} + (\sigma_4^2 - \sigma_3^2)n^{-1}\mathbf{1}_v\mathbf{1}_v'. \tag{31}$$

Applying the equality (31), it can be shown (see again Appendix 3) that the BLUE of $\boldsymbol{\tau}_* = (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')\boldsymbol{\tau}$ following from (8), i.e.,

$$\hat{\boldsymbol{\tau}}_* = (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')\hat{\boldsymbol{\tau}} = (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{y},$$

can equivalently be written as

$$\hat{\boldsymbol{\tau}}_* = (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{y}_*, \tag{32}$$

where $\boldsymbol{y}_* = (\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n')\boldsymbol{y}$, for which

$$\mathrm{E}(\boldsymbol{y}_*) = (\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n')\boldsymbol{X}_1\boldsymbol{\tau} = \boldsymbol{X}_1(\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')\boldsymbol{\tau} = \boldsymbol{X}_1\boldsymbol{\tau}_*$$

and

$$\mathrm{D}(\boldsymbol{y}_*) = (\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n')\boldsymbol{V}_*(\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n').$$

The dispersion matrix of $\hat{\boldsymbol{\tau}}_*$, given in(11), can on account of (31) be presented as

$$\mathrm{D}(\hat{\boldsymbol{\tau}}_*) = (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}(\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}_v'). \tag{33}$$

Furthermore, the formulae of SS_V and SS_R, given in (14) for treatments (varieties) and in (15) for residuals, can equivalently be written (see Appendices 3 and 4) as

$$\mathrm{SS}_V = \hat{\boldsymbol{\tau}}_*'\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1\hat{\boldsymbol{\tau}}_* = \boldsymbol{y}_*'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{y}_*, \tag{34}$$
$$\mathrm{SS}_R = \boldsymbol{y}_*'[\boldsymbol{V}_*^{-1} - \boldsymbol{V}_*^{-1}\boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}]\boldsymbol{y}_*, \tag{35}$$

with $\boldsymbol{y}_* = (\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n')\boldsymbol{y}$, as defined in relation to (32). The formulae (34) and (35) provide the sum

$$\mathrm{SS}_V + \mathrm{SS}_R = \boldsymbol{y}_*'\boldsymbol{V}_*^{-1}\boldsymbol{y}_* = \mathrm{SS}_T \quad \text{(say)}, \tag{36}$$

which can be called the total sum of squares. Referring again to Rao and Mitra (1971, Theorem 9.2.1), it can be shown that

$$\mathrm{SS}_T \ \sim \ \chi^2(n-1, \delta), \quad \text{with} \quad \delta = \boldsymbol{\tau}_*'\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1\boldsymbol{\tau}_*$$

equivalent to δ as given in (16). These results can be summarized in the form of an ANOVA table, as presented in Table 1.

Table 1. Analysis of variance for an experiment in a nested block design with orthogonal block structure

Source of variation	Degrees of freedom	Sum of squares	Expected mean square
Treatments	$v-1$	SS_V	$1+\delta/(v-1)$
Residuals	$n-v$	SS_R	1
Total	$n-1$	SS_T	—

The presentation of ANOVA results in Table 1 corresponds well with the formula (13) of the relevant F statistic.

Suppose now that after rejecting the hypothesis (10) one is interested in testing the hypothesis $H_{0,\mathrm{L}} : \boldsymbol{U}_\mathrm{L}'\boldsymbol{\tau} = \mathbf{0}$, where $\boldsymbol{U}_\mathrm{L}'\mathbf{1}_v = \mathbf{0}$. Note that this hypothesis, concerning a set of contrasts among treatment parameters, can also be written as

$$H_{0,\mathrm{L}} : \boldsymbol{U}_\mathrm{L}'\boldsymbol{\tau}_* = \mathbf{0}, \quad \text{where} \quad \boldsymbol{\tau}_* = (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')\boldsymbol{\tau}. \tag{37}$$

This shows that $H_{0,\mathrm{L}}$ is implied by H_0, given in (10). To find the relevant sum of squares, first note that the BLUE of $\boldsymbol{U}_\mathrm{L}'\boldsymbol{\tau}_*$ is, on account of (32), of the form

$$\boldsymbol{U}_\mathrm{L}'\hat{\boldsymbol{\tau}}_* = \boldsymbol{U}_\mathrm{L}'\hat{\boldsymbol{\tau}} = \boldsymbol{U}_\mathrm{L}'(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{y}_*. \tag{38}$$

Its dispersion matrix is, on account of (33), of the form

$$\mathrm{D}(\boldsymbol{U}_\mathrm{L}'\hat{\boldsymbol{\tau}}_*) = \boldsymbol{U}_\mathrm{L}'(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_\mathrm{L}. \tag{39}$$

Note that, applying Lemma 2.2.6(c) from Rao and Mitra (1971), one can write

$$\boldsymbol{U}'_{\mathrm{L}}(\boldsymbol{X}'_1\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\mathrm{L}}[\boldsymbol{U}'_{\mathrm{L}}(\boldsymbol{X}'_1\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\mathrm{L}}]^-\boldsymbol{U}'_{\mathrm{L}} = \boldsymbol{U}'_{\mathrm{L}},$$

which, with (39), gives the equality $\mathrm{D}(\boldsymbol{U}'_{\mathrm{L}}\hat{\boldsymbol{\tau}}_*)[\mathrm{D}(\boldsymbol{U}'_{\mathrm{L}}\hat{\boldsymbol{\tau}}_*)]^-\boldsymbol{U}'_{\mathrm{L}}\hat{\boldsymbol{\tau}}_* = \boldsymbol{U}'_{\mathrm{L}}\hat{\boldsymbol{\tau}}_*$. This shows that the hypothesis in (37) is consistent. The relevant sum of squares can then be obtained (following Theorem 3.2 of Rao, 1971) in the form

$$\begin{aligned}\mathrm{SS}(\boldsymbol{U}_{\mathrm{L}}) &= \hat{\boldsymbol{\tau}}'_*\boldsymbol{U}_{\mathrm{L}}[\mathrm{D}(\boldsymbol{U}'_{\mathrm{L}}\hat{\boldsymbol{\tau}}_*)]^-\boldsymbol{U}'_{\mathrm{L}}\hat{\boldsymbol{\tau}}_* \\ &= \hat{\boldsymbol{\tau}}'_*\boldsymbol{U}_{\mathrm{L}}[\boldsymbol{U}'_{\mathrm{L}}(\boldsymbol{X}'_1\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\mathrm{L}}]^-\boldsymbol{U}'_{\mathrm{L}}\hat{\boldsymbol{\tau}}_*, \qquad (40)\end{aligned}$$

with the d.f. equal to $\mathrm{rank}(\boldsymbol{U}_{\mathrm{L}})$, i.e., equal to $\mathrm{rank}[\mathrm{D}(\boldsymbol{U}'_{\mathrm{L}}\hat{\boldsymbol{\tau}}_*)]$. Note that $\boldsymbol{U}'_{\mathrm{L}}\hat{\boldsymbol{\tau}}_*$ is given in (38), and $[\boldsymbol{U}'_{\mathrm{L}}(\boldsymbol{X}'_1\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\mathrm{L}}]^-$ follows from (39). Also note, referring to Lemma 2.2.6(d) in Rao and Mitra (1971), that $\boldsymbol{U}_{\mathrm{L}}[\boldsymbol{U}'_{\mathrm{L}}(\boldsymbol{X}'_1\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\mathrm{L}}]^-\boldsymbol{U}'_{\mathrm{L}}$ is invariant for any choice of the appearing g-inverse, and is of rank equal to the rank of $\boldsymbol{U}_{\mathrm{L}}$. Of course, if the columns of $\boldsymbol{U}_{\mathrm{L}}$ are linearly independent, then

$$[\boldsymbol{U}'_{\mathrm{L}}(\boldsymbol{X}'_1\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\mathrm{L}}]^- \quad \text{becomes} \quad [\boldsymbol{U}'_{\mathrm{L}}(\boldsymbol{X}'_1\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\mathrm{L}}]^{-1}.$$

Now, following the assumption $\boldsymbol{y} \sim N_n(\boldsymbol{X}_1\boldsymbol{\tau}, \boldsymbol{V})$, adopted in Section 3, one may also assume that $\boldsymbol{U}'_{\mathrm{L}}\hat{\boldsymbol{\tau}}_* \sim N[\boldsymbol{U}'_{\mathrm{L}}\boldsymbol{\tau}_*, \mathrm{D}(\boldsymbol{U}'_{\mathrm{L}}\hat{\boldsymbol{\tau}}_*)]$. With this, applying Theorem 9.2.3 from Rao and Mitra (1971), it can be shown that

$$\mathrm{SS}(\boldsymbol{U}_{\mathrm{L}}) \sim \chi^2[\mathrm{rank}(\boldsymbol{U}_{\mathrm{L}}), \delta_L], \;\; \text{with} \;\; \delta_{\mathrm{L}} = \boldsymbol{\tau}'_*\boldsymbol{U}_{\mathrm{L}}[\mathrm{D}(\boldsymbol{U}'_{\mathrm{L}}\hat{\boldsymbol{\tau}}_*)]^-\boldsymbol{U}'_{\mathrm{L}}\boldsymbol{\tau}_*,$$

this distribution being central, i.e., with $\delta_{\mathrm{L}} = 0$, if $H_{0,\mathrm{L}}$ is true.

If there are several sets of contrasts for which individual hypothesis testing is of interest, then for each of them the sum of squares presented in (40) can be used accordingly. In some situations a relevant partition of the treatment sum of squares, given in (34), may be of interest in the application of ANOVA. The question then arises of what kind of conditions the chosen sets of contrasts have to satisfy. It can be shown (see Appendix 5) that for two such sets of contrasts, e.g. $\boldsymbol{U}'_{\mathrm{A}}\boldsymbol{\tau}_*$ and $\boldsymbol{U}'_{\mathrm{B}}\boldsymbol{\tau}_*$, the equality

$$\mathrm{SS}(\boldsymbol{U}_{\mathrm{A}}) + \mathrm{SS}(\boldsymbol{U}_{\mathrm{B}}) = \mathrm{SS}_V \qquad (41)$$

holds, for any vector $\hat{\boldsymbol{\tau}}_* = (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')\hat{\boldsymbol{\tau}}$, if and only if

$$\begin{aligned}&(\boldsymbol{X}'_1\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\mathrm{A}}[\boldsymbol{U}'_{\mathrm{A}}(\boldsymbol{X}'_1\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\mathrm{A}}]^-\boldsymbol{U}'_{\mathrm{A}} \\ &+ (\boldsymbol{X}'_1\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\mathrm{B}}[\boldsymbol{U}'_{\mathrm{B}}(\boldsymbol{X}'_1\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\mathrm{B}}]^-\boldsymbol{U}'_{\mathrm{B}} = \boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}'. \qquad (42)\end{aligned}$$

This, in turn, implies (on account of Lemma 2.2.6 in Rao and Mitra, 1971) that

$$\boldsymbol{U}_{\mathrm{B}}'(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\mathrm{A}} = \mathbf{O}. \tag{43}$$

These results can be extended for any number of considered sets of contrasts used in a partition of the type (41). The condition (43) can then be written as

$$\boldsymbol{U}_{\mathrm{L}}'(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\mathrm{L}^*} = \mathbf{O} \quad \text{for} \quad \mathrm{L} \neq \mathrm{L}^*. \tag{44}$$

It may be interesting to note that for some classes of designs the condition (44) is reduced to $\boldsymbol{U}_{\mathrm{L}}'\boldsymbol{U}_{\mathrm{L}^*} = \mathbf{O}$.

5. Application with estimated stratum variances

The hypothesis testing procedures presented in Section 4 are fully applicable if the stratum variances σ_1^2, σ_2^2 and σ_3^2 are known. As already mentioned at the end of Section 3, in practical applications these variances are usually unknown and have to be estimated. This can be done by solving the equations (23), (24) and (25). However, with these estimates the residual sum of squares SS_R, presented in (15) and equivalently in (35), is reduced to $n - v$, the corresponding d.f., as shown in formula (28). This leads to a corresponding reduction of the F statistic (13) to that presented in (29). The estimated treatment (variety) sum of squares appearing there, $\widehat{\mathrm{SS}}_V$, can, on account of formulae (34), (35) and (36), be written as

$$\widehat{\mathrm{SS}}_V = \boldsymbol{y}_*'\hat{\boldsymbol{V}}_*^{-1}\boldsymbol{y}_* - (n - v) \equiv \widehat{\mathrm{SS}}_T - n + v. \tag{45}$$

In the case of known (true) values of σ_1^2, σ_2^2 and σ_3^2 the quadratic form $\mathrm{SS}_T = \boldsymbol{y}_*'\boldsymbol{V}_*^{-1}\boldsymbol{y}_*$ is distributed as $\chi^2(n-1, \delta)$. If the hypothesis H_0 given in (10) is true, then $\delta = 0$ and the distribution is central. However, the indicated distribution of SS_T is fully applicable only if the true stratum variances σ_1^2, σ_2^2 and σ_3^2 appearing in $\boldsymbol{V}_*^{-1} = \sigma_1^{-2}\boldsymbol{\phi}_1 + \sigma_2^{-2}\boldsymbol{\phi}_2 + \sigma_3^{-2}(\boldsymbol{I}_n - \boldsymbol{\phi}_1 - \boldsymbol{\phi}_2)$ are used. Because now the matrix $\boldsymbol{V}_*^{-1}$ is replaced by

$$\hat{\boldsymbol{V}}_*^{-1} = \hat{\sigma}_1^{-2}\boldsymbol{\phi}_1 + \hat{\sigma}_2^{-2}\boldsymbol{\phi}_2 + \hat{\sigma}_3^{-2}(\boldsymbol{I}_n - \boldsymbol{\phi}_1 - \boldsymbol{\phi}_2),$$

the estimated total sum of squares $\widehat{\mathrm{SS}}_T$, appearing in (45), does not have an exact χ^2 distribution with $n - 1$ d.f. That distribution can, however, be considered as an approximation of the real distribution of $\widehat{\mathrm{SS}}_T$. This approximation will be the closer the larger is the number n, i.e., the size of the experiment.

With this approximation, the estimated mean square $\widehat{\mathrm{MS}}_V = \widehat{\mathrm{SS}}_V/(v-1)$, denoted by $\widehat{F}$ in (29), may be treated in a practical application as having (under H_0) approximately the distribution of $\chi^2(v-1,0)/(v-1)$, as follows from the relation in (45).

Thus, referring the test statistic (29) to the $\chi^2(v-1,0)/(v-1)$ distribution, one will obtain an approximate test of the hypothesis H_0 formulated in (10). This means that when calculating the relevant P values (i.e., the critical levels of significance) for testing H_0, or hypotheses implied by H_0, one has to consider them as approximate. The results obtained by Volaufova (2009) seem to suggest that the above ANOVA type F test approximation will in most cases provide reasonably accurate P values.

Finally, it may be interesting to recall the comments in Johnson, Kotz and Balakrishnan (1995, p. 338) according to which, if in the F statistic as in (13) the d.f. $n-v$ is large, then the natural approximation to be used is that this F statistic is distributed as $\chi^2(v-1,0)/(v-1)$. In fact, according to these comments the distribution of the statistic (29) corresponds to the F distribution with the second d.f. tending to infinity; see formula (27.27) there.

6. Examples

The methods considered in the previous sections will now be illustrated using data from three experiments conducted in different NB designs which induce the OBS property. The analysis concerning the first two of these experiments (Examples 1 and 2) illustrates the methods of obtaining the general ANOVA, as presented in Table 1, and also the partitioned ANOVA, usually of interest for factorial experiments. The analysis applied to the third experiment (Example 3) is confined to the general ANOVA. All required computations were performed using R (R Core Team, 2017).

Example 1. Brzeskwiniewicz (1994) analyzed data from a $v_{\mathrm{A}} \times v_{\mathrm{B}}$ factorial experiment with $v_{\mathrm{A}} = 3$ doses of nitrogen fertilizer (factor A) and $v_{\mathrm{B}} = 4$ varieties of potato (factor B). The experiment was conducted in an NB design with $\mathcal{D}^*$ based on the incidence matrix

$$\boldsymbol{N} = [\boldsymbol{N}_1 : \boldsymbol{N}_2 : \boldsymbol{N}_3 : \boldsymbol{N}_4 : \boldsymbol{N}_5 : \boldsymbol{N}_6 : \boldsymbol{N}_7 : \boldsymbol{N}_8 : \boldsymbol{N}_9 : \boldsymbol{N}_{10} : \boldsymbol{N}_{11} : \boldsymbol{N}_{12}]$$

of the following form, the rows of the matrix corresponding to the indicated treatment combinations (of the levels of factors A and B):

$$\begin{array}{cc} \mathrm{A} & \mathrm{B} \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 2 & 1 \\ 2 & 2 \\ 2 & 3 \\ 2 & 4 \\ 3 & 1 \\ 3 & 2 \\ 3 & 3 \\ 3 & 4 \end{array} \begin{bmatrix} 1\,0 & 1\,0 & 0\,0 & 0\,0 & 1\,0 & 1\,0 & 0\,0 & 0\,0 & 0\,0 & 0\,0 & 0\,0 & 0\,0 \\ 0\,0 & 0\,0 & 1\,0 & 1\,0 & 0\,0 & 0\,0 & 1\,0 & 1\,0 & 0\,0 & 0\,0 & 0\,0 & 0\,0 \\ 1\,0 & 0\,0 & 1\,0 & 0\,0 & 1\,0 & 0\,0 & 1\,0 & 0\,0 & 0\,0 & 0\,0 & 0\,0 & 0\,0 \\ 0\,0 & 1\,0 & 0\,0 & 1\,0 & 0\,0 & 1\,0 & 0\,0 & 1\,0 & 0\,0 & 0\,0 & 0\,0 & 0\,0 \\ 0\,0 & 0\,0 & 0\,0 & 0\,0 & 0\,1 & 0\,1 & 0\,0 & 0\,0 & 1\,0 & 1\,0 & 0\,0 & 0\,0 \\ 0\,0 & 0\,0 & 0\,0 & 0\,0 & 0\,0 & 0\,0 & 0\,1 & 0\,1 & 0\,0 & 0\,0 & 1\,0 & 1\,0 \\ 0\,0 & 0\,0 & 0\,0 & 0\,0 & 0\,1 & 0\,0 & 0\,1 & 0\,0 & 1\,0 & 0\,0 & 1\,0 & 0\,0 \\ 0\,0 & 0\,0 & 0\,0 & 0\,0 & 0\,0 & 0\,1 & 0\,0 & 0\,1 & 0\,0 & 1\,0 & 0\,0 & 1\,0 \\ 0\,1 & 0\,1 & 0\,0 & 0\,0 & 0\,0 & 0\,0 & 0\,0 & 0\,0 & 0\,1 & 0\,1 & 0\,0 & 0\,0 \\ 0\,0 & 0\,0 & 0\,1 & 0\,1 & 0\,0 & 0\,0 & 0\,0 & 0\,0 & 0\,0 & 0\,0 & 0\,1 & 0\,1 \\ 0\,1 & 0\,0 & 0\,1 & 0\,0 & 0\,0 & 0\,0 & 0\,0 & 0\,0 & 0\,1 & 0\,0 & 0\,1 & 0\,0 \\ 0\,0 & 0\,1 & 0\,0 & 0\,1 & 0\,0 & 0\,0 & 0\,0 & 0\,0 & 0\,0 & 0\,1 & 0\,0 & 0\,1 \end{bmatrix}.$$

The above partition of the matrix $\boldsymbol{N}$ into the matrices $\boldsymbol{N}_h$, $h = 1, 2, ..., 12$, each composed of two columns, provides the design $\mathcal{D}$ described by the incidence matrix

$$\boldsymbol{M} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

It is interesting to note that $\boldsymbol{M} = \boldsymbol{N}_{\mathrm{f(A)}} \otimes \boldsymbol{N}_{\mathrm{f(B)}}$, where

$$\boldsymbol{N}_{\mathrm{f(A)}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{N}_{\mathrm{f(B)}} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix},$$

the first defining the layout of the levels of factor A, the second defining that of the levels of factor B.

It is worth noting that here the matrix $\boldsymbol{N}_{\mathrm{f(A)}}$ is an incidence matrix of a balanced incomplete block (BIB) design and the matrix $\boldsymbol{N}_{\mathrm{f(B)}}$ is that of a partially balanced incomplete block (PBIB) design. The former, with parameters $v_{\mathrm{A}} = 3, r_{\mathrm{A}} = 2, k_{\mathrm{A}} = 2, b_{\mathrm{A}} = 3$ and $\lambda_{\mathrm{A}} = 1$, is exactly the design recorded at No. 1 in Table 8.2 of Caliński and Kageyama (2003), whereas the latter, with parameters $v_{\rm B} = 4, r_{\rm B} = 2, k_{\rm B} = 2, b_{\rm B} = 4, \lambda_{1,\rm B} = 0$ and $\lambda_{2,\rm B} = 1$, is a so-called semi-regular group divisible design (see, for example, Caliński and Kageyama, 2003, Section 6.0.2; Raghavarao and Padgett, 2005, Section 8.2).

It is assumed that the design used in the considered example was applied to available experimental units (field plots) grouped into blocks and those further into superblocks, all of them constructed in such a way as to allow the appropriate threefold randomization to be performed, as indicated in Section 2.

The individual plot observations (plot yields) obtained for the combinations of the levels of factors A and B in the experiment considered in this example are presented in Table 2. The order of blocks in this table corresponds to the order of the columns of the incidence matrix given above.

Table 2. Experimental observations of the field plot yield of the combinations of levels of the two factors analyzed in Example 1

Block	A	B	Observ.	Block	A	B	Observ.	Block	A	B	Observ.
1	1	1	35.8	9	1	1	38.4	17	2	1	35.3
1	1	3	30.8	9	1	3	30.7	17	2	3	43.8
2	3	1	40.2	10	2	1	28.5	18	3	1	38.0
2	3	3	47.6	10	2	3	37.0	18	3	3	45.4
3	1	1	33.0	11	1	1	36.6	19	2	1	33.6
3	1	4	46.1	11	1	4	44.9	19	2	4	46.0
4	3	1	41.5	12	2	1	30.2	20	3	1	45.0
4	3	4	46.6	12	2	4	50.5	20	3	4	52.0
5	1	2	49.1	13	1	2	48.3	21	2	2	36.0
5	1	3	36.3	13	1	3	35.2	21	2	3	45.5
6	3	2	57.1	14	2	2	41.2	22	3	2	50.6
6	3	3	43.4	14	2	3	47.5	22	3	3	49.0
7	1	2	46.6	15	1	2	49.5	23	2	2	38.5
7	1	4	43.5	15	1	4	44.5	23	2	4	42.3
8	3	2	57.5	16	2	2	46.3	24	3	2	53.3
8	3	4	51.4	16	2	4	42.6	24	3	4	47.0

When analyzing these data, the researcher (an agronomist) might be interested in estimating and testing certain sets of treatment parametric functions that can be defined as follows (assuming that the components of the vector $\boldsymbol{\tau}_*$ are ordered according to the order of the rows of the incidence matrix $\boldsymbol{N}$ given above):

$$\left[\left(\boldsymbol{I}_3 - \frac{1}{3}\mathbf{1}_3\mathbf{1}_3'\right)\otimes\frac{1}{4}\mathbf{1}_4'\right]\boldsymbol{\tau} = \boldsymbol{U}_{\mathrm{A}}'\boldsymbol{\tau} \equiv \boldsymbol{U}_{\mathrm{A}}'\boldsymbol{\tau}_*, \tag{46}$$

$$\left[\frac{1}{3}\mathbf{1}_3' \otimes \left(\boldsymbol{I}_4 - \frac{1}{4}\mathbf{1}_4\mathbf{1}_4'\right)\right]\boldsymbol{\tau} = \boldsymbol{U}_{\mathrm{B}}'\boldsymbol{\tau} \equiv \boldsymbol{U}_{\mathrm{B}}'\boldsymbol{\tau}_*, \tag{47}$$

$$\left[\left(\boldsymbol{I}_3 - \frac{1}{3}\mathbf{1}_3\mathbf{1}_3'\right)\otimes\left(\boldsymbol{I}_4 - \frac{1}{4}\mathbf{1}_4\mathbf{1}_4'\right)\right]\boldsymbol{\tau} = \boldsymbol{U}_{\mathrm{AB}}'\boldsymbol{\tau} \equiv \boldsymbol{U}_{\mathrm{AB}}'\boldsymbol{\tau}_*, \tag{48}$$

where (46) stands for the main effects of the levels of factor A, (47) stands for the main effects of the levels of factor B, and (48) represents the interaction effects of these two factors. All these linear functions can be seen as contrasts of treatment parameters. For each of these three sets of contrasts, say $\boldsymbol{U}_{\mathrm{L}}$, the BLUE is obtainable according to formula (38), and the relevant sum of squares, $\mathrm{SS}(\boldsymbol{U}_{\mathrm{L}})$, follows from (40).

To simplify the computations it may be useful to calculate first the matrices

$$\boldsymbol{U}_{\mathrm{L}}'(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1} \quad \text{for} \quad \mathrm{L} = \mathrm{A}, \ \mathrm{B} \text{ and } \mathrm{AB}.$$

For this, note that

$$\begin{aligned} \boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1 &= \sigma_1^{-2}\boldsymbol{X}_1'\boldsymbol{\phi}_1\boldsymbol{X}_1 + \sigma_2^{-2}\boldsymbol{X}_1'\boldsymbol{\phi}_2\boldsymbol{X}_1 + \sigma_3^{-2}\boldsymbol{X}_1'(\boldsymbol{I}_n - \boldsymbol{\phi}_1 - \boldsymbol{\phi}_2)\boldsymbol{X}_1 \\ &= \sigma_1^{-2}r\boldsymbol{I}_v - (\sigma_1^{-2} - \sigma_2^{-2})k^{-1}\boldsymbol{N}\boldsymbol{N}' - (\sigma_2^{-2} - \sigma_3^{-2})n_0^{-1}\boldsymbol{M}\boldsymbol{M}', \end{aligned}$$

where $r = 4, v = 12, k = 2, n_0 = 4$,

$$\boldsymbol{N}\boldsymbol{N}' = 2\boldsymbol{I}_3 \otimes \begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix}$$

and

$$\boldsymbol{M}\boldsymbol{M}' = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \otimes \begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix}.$$

With these results it can be checked that the condition (44) holds for any pair of the considered sets of contrasts, $\boldsymbol{U}_{\mathrm{A}}$, $\boldsymbol{U}_{\mathrm{B}}$ and $\boldsymbol{U}_{\mathrm{AB}}$, and then that the condition (42), extended to the three sets, also holds. With these conditions satisfied, one can proceed to the general ANOVA and to its partition into three components related to the three sets of contrasts. Note that the relevant extension of (42) here is of the form

$$
\begin{aligned}
&(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\mathrm{A}}[\boldsymbol{U}_{\mathrm{A}}'(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\mathrm{A}}]^{-}\boldsymbol{U}_{\mathrm{A}}' \\
&+ (\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\mathrm{B}}[\boldsymbol{U}_{\mathrm{B}}'(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\mathrm{B}}]^{-}\boldsymbol{U}_{\mathrm{B}}' \\
&+ (\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\mathrm{AB}}[\boldsymbol{U}_{\mathrm{AB}}'(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\mathrm{AB}}]^{-}\boldsymbol{U}_{\mathrm{AB}}' \\
&= \boldsymbol{I}_v - v^{-1}\mathbf{1}_v\mathbf{1}_v'. \qquad (49)
\end{aligned}
$$

If (49) holds, then

$$
\mathrm{SS}(\boldsymbol{U}_{\mathrm{A}}) + \mathrm{SS}(\boldsymbol{U}_{\mathrm{B}}) + \mathrm{SS}(\boldsymbol{U}_{\mathrm{AB}}) = \mathrm{SS}_V, \qquad (50)
$$

with each component on the left in (50) obtainable using formula (40). Of course, $\hat{\boldsymbol{\tau}}_*$ is obtainable by the use of formula (32).

Table 3. Analysis of variance for an experiment in a nested block design analyzed in Example 1

Source of variation	Degrees of freedom	Sum of squares	Mean square
Treatments	11	210.8489	19.1681
Residuals	36	36	1
Total	47	246.8489	—

Table 4. Analysis of variance for the sets of contrasts considered in Example 1

Source	Degrees of freedom	Sum of squares	Meane square	$\hat{F}$	P value
Treatments	11	210.8489	19.1681	19.1681	< 0.0001
A	2	71.3556	35.6778	35.6778	< 0.0001
B	3	97.1209	32.3736	32.3736	< 0.0001
AB	6	42.3724	7.0621	7.0621	< 0.0001
Residuals	36	36	1		
Total	47	246.8489			

The critical values, at the 1 percent level of significance, for the approximate distribution of the above test statistic $\hat{F}$, following from (29), are: 2.25 for 11 d.f., 4.61 for 2 d.f., 3.78 for 3 d.f. and 2.80 for 6 d.f.

The results presented in Tables 3 and 4 were obtained with the use of the empirical estimates (i.e., based on $\hat{\sigma}_1^2 = 9.77119$, $\hat{\sigma}_2^2 = 7.78197$ and $\hat{\sigma}_3^2 = 10.79420$)

$$\tilde{\boldsymbol{\tau}} = \begin{array}{l}[36.093,\ 48.159,\ 33.391,\ 44.536,\ 31.836,\ 40.546,\\ \ 43.494,\ 45.288,\ 41.139,\ 54.752,\ 46.247,\ 49.444]'\end{array}$$

and

$$\tilde{\boldsymbol{\tau}}_* = \begin{array}{l}[-6.818,\ 5.249,\ -9.520,\ 1.626,\ -11.074,\ -2.364,\\ \ 0.584,\ 2.378,\ -1.772,\ 11.842,\ 3.336,\ 6.534]',\end{array}$$

the former obtainable by the use of formula (8), the latter either from the relation $\tilde{\boldsymbol{\tau}}_* = (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')\tilde{\boldsymbol{\tau}}$, or directly by formula (32). Using $\tilde{\boldsymbol{\tau}}_*$ in the formula (34), i.e., replacing $\boldsymbol{V}_*^{-1}$ by $\hat{\boldsymbol{V}}_*^{-1}$ there, the estimated sum of squares $\widehat{\mathrm{SS}}_V$ is obtained. Similarly, using formula (40) in the same way, the relevant components of $\widehat{\mathrm{SS}}_V$ are obtained. Evidently, as follows from (28), the estimated residual sum of squares $\widehat{\mathrm{SS}}_R$ is reduced to $n - v$, its d.f. The term "empirical estimates" used above is taken from Rao and Kleffe (1988, p. 274).

Example 2. Caliński and Łacka (2014) analyzed data from a plant protection experiment. The experiment was carried out in laboratory conditions, in a growth chamber. Its aim was to evaluate the efficiency of 4 chemical substances (levels of factor B) applied in 3 concentrations − low, mid and high (levels of factor A) − to reduce plant damage caused by slugs *Arion lusitanicus*. Two of the chosen active substances, metaldehyde and methiocarb, are currently recommended for the slug control. Methiocarb in mid concentration is often considered as a standard. In the experiment, discs of Chinese cabbage leaves were treated with relevant solutions of the studied chemical compounds (henceforth called "chemicals"), and the amount of damage caused to them by slugs, given as percentages of their surface areas, was observed. Each box, as an experimental unit, contained three such discs and one *A. lusitanicus* slug placed inside. One camera covering $k = 2$ boxes was considered as forming a block of the design. During the experiment, $b_0 = 3$ cameras were working simultaneously. Each series of observations with the use of these 3 cameras was considered as one superblock of the design. In total, the experiment was composed of $a = 8$ such series, giving $b = 24$. Thus, in the experiment there were $n = 48$ experimental units, allowing each of the $v = 12$ treatments to be replicated $r = 4$ times.

The NB design so formulated has a $v \times b$ incidence matrix, defined as

$$\boldsymbol{N} = \boldsymbol{X}_1' \boldsymbol{X}_B = [\boldsymbol{N}_1 : \boldsymbol{N}_2 : \boldsymbol{N}_3 : \boldsymbol{N}_4 : \boldsymbol{N}_5 : \boldsymbol{N}_6 : \boldsymbol{N}_7 : \boldsymbol{N}_8],$$

of the form

$$\begin{array}{ccc} i & \mathrm{A} & \mathrm{B} \\ 1 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 1 & 3 \\ 4 & 1 & 4 \\ 5 & 2 & 1 \\ 6 & 2 & 2 \\ 7 & 2 & 3 \\ 8 & 2 & 4 \\ 9 & 3 & 1 \\ 10 & 3 & 2 \\ 11 & 3 & 3 \\ 12 & 3 & 4 \end{array} \begin{bmatrix} 1\,0\,0 & 1\,0\,0 & 0\,0\,0 & 0\,0\,0 & 1\,0\,0 & 1\,0\,0 & 0\,0\,0 & 0\,0\,0 \\ 0\,0\,0 & 0\,0\,0 & 1\,0\,0 & 1\,0\,0 & 0\,0\,0 & 0\,0\,0 & 1\,0\,0 & 1\,0\,0 \\ 1\,0\,0 & 0\,0\,0 & 1\,0\,0 & 0\,0\,0 & 1\,0\,0 & 0\,0\,0 & 1\,0\,0 & 0\,0\,0 \\ 0\,0\,0 & 1\,0\,0 & 0\,0\,0 & 1\,0\,0 & 0\,0\,0 & 1\,0\,0 & 0\,0\,0 & 1\,0\,0 \\ 0\,1\,0 & 0\,1\,0 & 0\,0\,0 & 0\,0\,0 & 0\,1\,0 & 0\,1\,0 & 0\,0\,0 & 0\,0\,0 \\ 0\,0\,0 & 0\,0\,0 & 0\,1\,0 & 0\,1\,0 & 0\,0\,0 & 0\,0\,0 & 0\,1\,0 & 0\,1\,0 \\ 0\,1\,0 & 0\,0\,0 & 0\,1\,0 & 0\,0\,0 & 0\,1\,0 & 0\,0\,0 & 0\,1\,0 & 0\,0\,0 \\ 0\,0\,0 & 0\,1\,0 & 0\,0\,0 & 0\,1\,0 & 0\,0\,0 & 0\,1\,0 & 0\,0\,0 & 0\,1\,0 \\ 0\,0\,1 & 0\,0\,1 & 0\,0\,0 & 0\,0\,0 & 0\,0\,1 & 0\,0\,1 & 0\,0\,0 & 0\,0\,0 \\ 0\,0\,0 & 0\,0\,0 & 0\,0\,1 & 0\,0\,1 & 0\,0\,0 & 0\,0\,0 & 0\,0\,1 & 0\,0\,1 \\ 0\,0\,1 & 0\,0\,0 & 0\,0\,1 & 0\,0\,0 & 0\,0\,1 & 0\,0\,0 & 0\,0\,1 & 0\,0\,0 \\ 0\,0\,0 & 0\,0\,1 & 0\,0\,0 & 0\,0\,1 & 0\,0\,0 & 0\,0\,1 & 0\,0\,0 & 0\,0\,1 \end{bmatrix},$$

where the rows correspond to the indicated treatment combinations (of the levels of factors A and B).

The above partition of the matrix $\boldsymbol{N}$ into the matrices $\boldsymbol{N}_h$, $h = 1, 2, ..., 8$, each of three columns, provides the design $\mathcal{D}$ described by the incidence matrix

$$\boldsymbol{M} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

The individual plot observations of the investigated characteristic, ordered according to the order of blocks of the design, are given in Table 5.

When analyzing these data, the researcher was particularly interested in estimating and testing a certain set of contrasts. These can be presented as certain basic contrasts (see Definition 3.4.1 in Caliński and Kageyama, 2000)

$$\{\boldsymbol{c}_i' \boldsymbol{\tau} \equiv \boldsymbol{c}_i' \boldsymbol{\tau}_*, i = 1, 2, ..., 11\}$$

Table 5. Observed damage to Chinese cabbage leaf discs (percentages of surface areas) at 3 concentrations (A) of 4 studied chemicals (B)

Block	A	B	Observ.	Block	A	B	Observ.	Block	A	B	Observ.
1	1	1	62.75	9	3	2	14.20	17	2	1	2.12
1	1	3	7.90	9	3	3	13.50	17	2	4	1.25
2	2	1	1.90	10	1	2	74.00	18	3	1	13.00
2	2	3	1.70	10	1	4	4.90	18	3	4	12.80
3	3	1	12.70	11	2	2	36.25	19	1	2	77.00
3	3	3	13.00	11	2	4	0.80	19	1	3	9.00
4	1	1	64.55	12	3	2	13.90	20	2	2	38.20
4	1	4	5.50	12	3	4	12.02	20	2	3	2.90
5	2	1	2.78	13	1	1	63.45	21	3	2	15.20
5	2	4	1.35	13	1	3	7.90	21	3	3	13.10
6	3	1	13.70	14	2	1	3.20	22	1	2	75.25
6	3	4	13.00	14	2	3	1.80	22	1	4	5.10
7	1	2	74.95	15	3	1	14.20	23	2	2	38.00
7	1	3	8.40	15	3	3	14.00	23	2	4	1.40
8	2	2	37.55	16	1	1	65.25	24	3	2	14.30
8	2	3	2.40	16	1	4	5.70	24	3	4	13.60

determined by the following vectors, with the corresponding stratum efficiency factors (from Caliński and Łacka, 2014, p. 968):

$$
\begin{array}{llll}
\boldsymbol{c}_1 = \mathbf{1}_3 \otimes [-1,-1,1,1]'/\sqrt{3}, & \varepsilon_{11} = 1, & \varepsilon_{21} = 0, & \varepsilon_{31} = 0, \\
\boldsymbol{c}_2 = [-2,1,1]' \otimes [-1,-1,1,1]'/\sqrt{6}, & \varepsilon_{12} = 1, & \varepsilon_{22} = 0, & \varepsilon_{32} = 0, \\
\boldsymbol{c}_3 = [0,-1,1]' \otimes [-1,-1,1,1]'/\sqrt{2}, & \varepsilon_{13} = 1, & \varepsilon_{23} = 0, & \varepsilon_{33} = 0, \\
\boldsymbol{c}_4 = \mathbf{1}_3 \otimes [-1,1,0,0]'\sqrt{2}/\sqrt{3}, & \varepsilon_{14} = 1/2, & \varepsilon_{24} = 0, & \varepsilon_{34} = 1/2, \\
\boldsymbol{c}_5 = \mathbf{1}_3 \otimes [0,0,-1,1]'\sqrt{2}/\sqrt{3}, & \varepsilon_{15} = 1/2, & \varepsilon_{25} = 0, & \varepsilon_{35} = 1/2, \\
\boldsymbol{c}_6 = [-2,1,1]' \otimes [-1,1,0,0]'/\sqrt{3}, & \varepsilon_{16} = 1/2, & \varepsilon_{26} = 1/2, & \varepsilon_{36} = 0, \\
\boldsymbol{c}_7 = [-2,1,1]' \otimes [0,0,-1,1]'/\sqrt{3}, & \varepsilon_{17} = 1/2, & \varepsilon_{27} = 1/2, & \varepsilon_{37} = 0, \\
\boldsymbol{c}_8 = [0,-1,1]' \otimes [-1,1,0,0]', & \varepsilon_{18} = 1/2, & \varepsilon_{28} = 1/2, & \varepsilon_{38} = 0, \\
\boldsymbol{c}_9 = [0,-1,1]' \otimes [0,0,-1,1]', & \varepsilon_{19} = 1/2, & \varepsilon_{29} = 1/2, & \varepsilon_{39} = 0, \\
\boldsymbol{c}_{10} = [-2,1,1]' \otimes \mathbf{1}_4/\sqrt{6}, & \varepsilon_{1,10} = 0, & \varepsilon_{2,10} = 1, & \varepsilon_{3,10} = 0, \\
\boldsymbol{c}_{11} = [0,-1,1]' \otimes \mathbf{1}_4/\sqrt{2}, & \varepsilon_{1,11} = 0, & \varepsilon_{2,11} = 1, & \varepsilon_{3,11} = 0.
\end{array}
$$

For each of these eleven basic contrasts the BLUE is obtainable by formula (38), with $\boldsymbol{U}_{\mathrm{L}}'$ replaced by $\boldsymbol{c}_i'$, and the relevant sum of squares, $\mathrm{SS}(\boldsymbol{c}_i)$, follows from (40) with the same replacement.

To simplify the computations it may be useful to calculate first the vectors

$$\boldsymbol{c}_i'(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1} \quad \text{for} \quad i = 1,2,...,11.$$

For this, note that

$$
\begin{aligned}
\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1 &= \sigma_1^{-2}\boldsymbol{X}_1'\boldsymbol{\phi}_1\boldsymbol{X}_1 + \sigma_2^{-2}\boldsymbol{X}_1'\boldsymbol{\phi}_2\boldsymbol{X}_1 + \sigma_3^{-2}\boldsymbol{X}_1'(\boldsymbol{I}_n - \boldsymbol{\phi}_1 - \boldsymbol{\phi}_2)\boldsymbol{X}_1 \\
&= (\sigma_1^{-2} - \sigma_3^{-2})\boldsymbol{C}_1 + (\sigma_2^{-2} - \sigma_3^{-2})\boldsymbol{C}_2 + \sigma_3^{-2}r\boldsymbol{I}_v,
\end{aligned}
$$

where $r = 4, v = 12$ and

$$\boldsymbol{C}_1 = \boldsymbol{X}_1'\boldsymbol{\phi}_1\boldsymbol{X}_1 = \boldsymbol{I}_3 \otimes \begin{bmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{bmatrix}, \text{ of rank 9,}$$

$$\boldsymbol{C}_2 = \boldsymbol{X}_1'\boldsymbol{\phi}_2\boldsymbol{X}_1 = 3^{-1} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \otimes \begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix}, \text{ of rank 6.}$$

With these results it can be checked whether the condition (44) holds for any pair of the considered contrasts, i.e., condition $\boldsymbol{c}_i'(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{c}_{i^*} = 0$ for any $i \neq i^*$, and then whether the condition (42), extended to all eleven basic contrasts, also holds. With these conditions satisfied, one can proceed to the general ANOVA and to its partition corresponding to the eleven basic contrasts. Note that the extension of (42) here is of the form

$$\begin{aligned}
&(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{c}_1[\boldsymbol{c}_1'(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{c}_1]^{-1}\boldsymbol{c}_1' \\
&+(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{c}_2[\boldsymbol{c}_2'(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{c}_2]^{-1}\boldsymbol{c}_2' \\
&+ ... + (\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{c}_{11}[\boldsymbol{c}_{11}'(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{c}_{11}]^{-1}\boldsymbol{c}_{11}' \\
&= \boldsymbol{I}_v - v^{-1}\mathbf{1}_v\mathbf{1}_v'.
\end{aligned}$$

If this holds, then

$$\mathrm{SS}(\boldsymbol{c}_1) + \mathrm{SS}(\boldsymbol{c}_2) + ... + \mathrm{SS}(\boldsymbol{c}_{11}) = \mathrm{SS}_V,$$

with each component on the left obtainable using formula (40), with $\boldsymbol{U}_{\mathrm{L}}$ replaced by $\boldsymbol{c}_i$. Of course $\hat{\boldsymbol{\tau}}_*$ is obtainable by formula (32).

Table 6. Analysis of variance for an experiment in a nested block design analyzed in Example 2

Source of variation	Degrees of freedom	Sum of squares	Mean square
Treatments	11	103246.2	9386.018
Residuals	36	36	1
Total	47	103282.2	—

Table 7. Analysis of variance for the contrasts considered in Example 2

Source	Degrees of freedom	Sum of squares	Mean square	$\hat{F}$	P value
Treatments	11	103246.20	9386.02	9386.02	< 0.0001
$\boldsymbol{c}_1'\boldsymbol{\tau}$	1	38874.34	38874.34	38874.34	< 0.0001
$\boldsymbol{c}_2'\boldsymbol{\tau}$	1	32927.98	32927.98	32927.98	< 0.0001
$\boldsymbol{c}_3'\boldsymbol{\tau}$	1	2669.24	2669.24	2669.24	< 0.0001
$\boldsymbol{c}_4'\boldsymbol{\tau}$	1	3690.55	3690.55	3690.55	< 0.0001
$\boldsymbol{c}_5'\boldsymbol{\tau}$	1	28.52	28.52	28.52	< 0.0001
$\boldsymbol{c}_6'\boldsymbol{\tau}$	1	214.89	214.89	214.89	< 0.0001
$\boldsymbol{c}_7'\boldsymbol{\tau}$	1	26.57	26.57	26.57	< 0.0001
$\boldsymbol{c}_8'\boldsymbol{\tau}$	1	4125.63	4125.63	4125.63	< 0.0001
$\boldsymbol{c}_9'\boldsymbol{\tau}$	1	0.69	0.69	0.69	$= 0.4107$
$\boldsymbol{c}_{10}'\boldsymbol{\tau}$	1	20526.73	20526.73	20526.73	< 0.0001
$\boldsymbol{c}_{11}'\boldsymbol{\tau}$	1	161.06	161.06	161.06	< 0.0001
Residuals	36	36	1		
Total	47	103282.20			

The critical values, at the 1 percent level of significance, for the approximate distribution of the above test statistic $\hat{F}$, following from (29), are: 2.25 for 11 d.f. and 6.63 for 1 d.f.

The results presented in Tables 6 and 7 were obtained with the use of the empirical estimates (i.e., based on $\hat{\sigma}_1^2 = 0.23019$, $\hat{\sigma}_2^2 = 0.35245$ and $\hat{\sigma}_3^2 = 1.53393$)

$$\tilde{\boldsymbol{\tau}} = \begin{array}{l} [64.037,\ 75.263,\ 8.275,\ 5.325,\ 2.564,\ 37.436, \\ \ 2.107,\ 1.293,\ 13.412,\ 14.388,\ 13.315,\ 12.940]' \end{array}$$

and

$$\tilde{\boldsymbol{\tau}}_* = \begin{array}{l} [43.174,\ 54.400,\ -12.588,\ -15.538,\ -18.299,\ 16.573, \\ -18.756,\ -19.570,\ -7.451,\ -6.475,\ -7.548,\ -7.923]', \end{array}$$

following the same approach as that applied in Example 1. These estimates also provide the empirical estimates of the considered basic contrasts: $\widetilde{\boldsymbol{c}_1'\boldsymbol{\tau}} = -94.596$, $\widetilde{\boldsymbol{c}_2'\boldsymbol{\tau}} = 87.061$, $\widetilde{\boldsymbol{c}_3'\boldsymbol{\tau}} = 24.788$, $\widetilde{\boldsymbol{c}_4'\boldsymbol{\tau}} = 38.436$, $\widetilde{\boldsymbol{c}_5'\boldsymbol{\tau}} = -3.379$, $\widetilde{\boldsymbol{c}_6'\boldsymbol{\tau}} = 7.736$, $\widetilde{\boldsymbol{c}_7'\boldsymbol{\tau}} = 2.721$, $\widetilde{\boldsymbol{c}_8'\boldsymbol{\tau}} = -33.896$, $\widetilde{\boldsymbol{c}_9'\boldsymbol{\tau}} = 0.439$, $\widetilde{\boldsymbol{c}_{10}'\boldsymbol{\tau}} = -85.056$ and $\widetilde{\boldsymbol{c}_{11}'\boldsymbol{\tau}} = 7.534$.

These results are very close to those presented in Caliński and Łacka (2014, pp. 970-971), obtained by the classical approach, namely by first performing the within-stratum analyses, based on the relevant submodels, and then combining their results. Here the results are obtainable from a direct analysis.

Example 3. Ceranka (1983) analyzed data from a plant-breeding field experiment with 25 breeding strains and 2 standard varieties of sunflower compared in an NB design based on an incidence matrix $\boldsymbol{N}^*$ of the type

$$\boldsymbol{N}^* = \begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{1}_s\boldsymbol{1}_b' \end{bmatrix},$$

with

$$\boldsymbol{N} = [\boldsymbol{N}_1 : \boldsymbol{N}_2 : \boldsymbol{N}_3 : \boldsymbol{N}_4 : \boldsymbol{N}_5 : \boldsymbol{N}_6],$$

where

$$\boldsymbol{N}_1' = \begin{bmatrix} 1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1 \end{bmatrix},$$

$$\boldsymbol{N}_2' = \begin{bmatrix} 1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0 \\ 0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0 \\ 0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0 \\ 0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0 \\ 0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1 \end{bmatrix},$$

$$\boldsymbol{N}_3' = \begin{bmatrix} 1&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&1 \\ 0&1&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&1&1&0&0&0&0 \\ 0&0&1&0&0&0&0&0&1&0&0&0&0&0&1&1&0&0&0&0&0&1&0&0&0 \\ 0&0&0&1&0&0&0&0&0&1&1&0&0&0&0&0&1&0&0&0&0&0&1&0&0 \\ 0&0&0&0&1&1&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&1&0 \end{bmatrix},$$

$$\boldsymbol{N}_4' = \begin{bmatrix} 1&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&1&0 \\ 0&1&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&1&0&0&0&0&0&0&1 \\ 0&0&1&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&1&0&1&0&0&0&0 \\ 0&0&0&1&0&1&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&1&0&0&0 \\ 0&0&0&0&1&0&1&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&1&0&0 \end{bmatrix},$$

$$
\boldsymbol{N}_5' = \begin{bmatrix}
1&0&0&0&0&0&0&0&1&0&0&1&0&0&0&0&0&0&0&1&0&0&1&0&0\\
0&1&0&0&0&0&0&0&0&1&0&0&1&0&0&1&0&0&0&0&0&0&0&1&0\\
0&0&1&0&0&1&0&0&0&0&0&0&0&1&0&0&1&0&0&0&0&0&0&0&1\\
0&0&0&1&0&0&1&0&0&0&0&0&0&0&1&0&0&1&0&0&1&0&0&0&0\\
0&0&0&0&1&0&0&1&0&0&1&0&0&0&0&0&0&0&1&0&0&1&0&0&0
\end{bmatrix},
$$

$$
\boldsymbol{N}_6' = \begin{bmatrix}
1&0&0&0&0&0&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0\\
0&1&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0&0&1&0&0&0&1&0&0\\
0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0&0&1&0\\
0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&1\\
0&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0
\end{bmatrix},
$$

and with

$$
\mathbf{1}_s\mathbf{1}_b' = [\mathbf{1}_2\mathbf{1}_5' : \mathbf{1}_2\mathbf{1}_5' : \mathbf{1}_2\mathbf{1}_5' : \mathbf{1}_2\mathbf{1}_5' : \mathbf{1}_2\mathbf{1}_5' : \mathbf{1}_2\mathbf{1}_5'].
$$

It has $b = 30$ blocks, each of size $k = 7$, grouped into $a = 6$ superblocks, each of size $n_0 = 35$. Note that the design by which the 27 treatments are arranged into 6 superblocks, denoted by $\mathcal{D}$, is here based on the 27×6 incidence matrix

$$
\boldsymbol{M} = \begin{bmatrix} \mathbf{1}_{25} & \mathbf{1}_{25} & \mathbf{1}_{25} & \mathbf{1}_{25} & \mathbf{1}_{25} & \mathbf{1}_{25} \\ 5\mathbf{1}_2 & 5\mathbf{1}_2 & 5\mathbf{1}_2 & 5\mathbf{1}_2 & 5\mathbf{1}_2 & 5\mathbf{1}_2 \end{bmatrix}.
$$

The plant trait observed on the experimental units (plots), and taken here for analysis, is the average diameter of the capitulum (head) in centimeteres. The individual plot observations are presented and analyzed in Caliński and Kageyama (2003, Example 7.3.22). These data have already been analyzed in Caliński and Siatkowski (2017, Example 2). That analysis, however, was performed without taking into account the grouping of blocks into superblocks. Here the analysis is conducted as for an NB design. The results are presented in Table 8.

Table 8. Analysis of variance for an experiment in a nested block design analyzed in Example 3

Source of variation	Degrees of freedom	Sum of squares	Meane square	$\hat{F}$	P value
Treatments	26	93.2401	3.5862	3.5862	< 0.0001
Residuals	183	183	1		
Total	209	276.2401			

The critical value, at the 1 percent level of significance, for the approximate distribution of the above test statistic $\hat{F}$, following from (29), is here 1.76 for 26 d.f.

The results presented in Table 8 were obtained with the use of the empirical estimates (i.e., based on $\hat{\sigma}_1^2 = 0.91939$, $\hat{\sigma}_2^2 = 2.03387$ and $\hat{\sigma}_3^2 = 201.50110$)

$$\tilde{\boldsymbol{\tau}} = \begin{array}{l}[15.552,\ 15.808,\ 15.099,\ 15.627,\ 16.347,\ 15.725,\ 15.426,\\ \ 15.747,\ 16.025,\ 15.290,\ 16.418,\ 15.454,\ 15.430,\ 15.664,\\ \ 15.591,\ 15.811,\ 15.300,\ 16.564,\ 15.473,\ 17.628,\ 14.937,\\ \ 15.211,\ 14.618,\ 14.678,\ 14.326,\ 14.690,\ 15.780]'\end{array}$$

and

$$\tilde{\boldsymbol{\tau}}_* = \begin{array}{rrrrrrr}[0.063, & 0.319, & -0.390, & 0.138, & 0.858, & 0.237, & -0.062,\\ 0.259, & 0.537, & -0.198, & 0.929, & -0.034, & -0.058, & 0.176,\\ 0.102, & 0.322, & -0.189, & 1.076, & -0.015, & 2.140, & -0.552,\\ -0.277, & -0.871, & -0.811, & -1.163, & -0.799, & 0.291]', & \end{array}$$

following the same approach as that applied in Example 1.

It may be interesting to note that the test statistic $\hat{F}$ is here slightly larger than in Example 2 of the previous paper (Caliński and Siatkowski, 2017), where the grouping of blocks into superblocks is ignored.

7. Concluding remarks

The present paper is the second in a series concerning a new approach to the analysis of experiments with the OBS property. The first paper in this series, by the same authors (Caliński and Siatkowski, 2017), concerns experiments conducted in proper block designs. Here the new approach is applied to experiments in nested block designs that induce the OBS property.

Exactly as in the first work, it appears that when the unknown stratum variances within the covariance (dispersion) matrix $\boldsymbol{V}$, given in (3), are replaced by their estimates, obtained from the estimation procedure suggested by Nelder (1968), the residual sum of squares SS_R is reduced to its d.f., that is, its expectation. This result is obtainable due to the proposed new approach to the analysis of experimental data.

The indicated result, presented in Section 3, follows from the use of a covariance matrix $\boldsymbol{V}$ not in the form

$$\boldsymbol{V} = \sigma_1^2[\boldsymbol{\phi}_1 + (\sigma_2^2/\sigma_1^2)\boldsymbol{\phi}_2 + (\sigma_3^2/\sigma_1^2)\boldsymbol{\phi}_3 + (\sigma_4^2/\sigma_1^2)\boldsymbol{\phi}_4] = \sigma_1^2\boldsymbol{F} \quad \text{(say)}$$

(appearing in the general Gauss−Markov model), as usually applied in the literature (as recalled by Kala, 2017), but in its original form $\boldsymbol{V} = \sigma_1^2\boldsymbol{\phi}_1 +$

$\sigma_2^2\boldsymbol{\phi}_2 + \sigma_3^2\boldsymbol{\phi}_3 + \sigma_4^2\boldsymbol{\phi}_4$. This ensures that $\mathrm{E}(\mathrm{SS}_R) = n - v$, as follows from (19). As a consequence of this application, the test statistic (13) is reduced to the form (29), i.e., to the estimated treatment mean square, $\widehat{\mathrm{MS}}_V = \widehat{\mathrm{SS}}_V/(v-1)$. This can be seen as an advantage, particularly with regard to the approximation of the relevant distribution, indicated at the end of Section 5.

Another feature of the proposed approach relates to simplification of the main analytical procedures, as presented in Section 4. One of the resulting advantages is the reduction of the number of stratum variances involved, from four to three, that is, to σ_1^2, σ_2^2 and σ_3^2 only. This substantially simplifies the computations.

However, as can be seen from the examples analyzed in Section 6, the main advantage of the proposed approach is the fact that the ANOVA results are obtainable directly, not by first performing some partial analyses, under relevant stratum submodels, and then combining their results (as is done in most of the relevant literature).

The indicated advantages are similar to those presented in the first paper in the series, planned for different classes of designs inducing the OBS property.

Finally, returning to the examples presented in Section 6, it may be noted that the design considered in Example 1 can be regarded as a split-plot type design. On the other hand, the design applied in Example 3, when restricted to the part represented by the incidence matrix $\boldsymbol{N}$, can be seen as a resolvable block type design. Because these two types of design are often used in practice, it may be interesting to devote to them separate papers in the present series.

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Appendix

Appendix 1

For formula (12) one has first to show that $\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1$ can be taken as a g-inverse of $\mathrm{D}(\hat{\boldsymbol{\tau}}_*)$ as given in (11), i.e., that the equality

$$(\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}(\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}_v')\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}(\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}_v') = (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}(\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}_v')$$

holds. For this, it is sufficient to consider the equalities

$$\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}') = (\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}_v')\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1 \quad \text{and}$$
$$(\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}_v')(\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}_v') = (\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}_v').$$

The second equality is obvious. To prove the first, one may use the equalities

$$\boldsymbol{X}_1(\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}') = (\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n')\boldsymbol{X}_1 \quad \text{and}$$
$$\boldsymbol{V}^{-1}(\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n') = (\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n')\boldsymbol{V}^{-1},$$

which can easily be checked remembering that $\boldsymbol{X}_1\mathbf{1}_v = \mathbf{1}_n$ and $\mathbf{1}_n'\boldsymbol{X}_1 = \boldsymbol{r}'$, and also recalling the properties of the matrices $\boldsymbol{\phi}_1$, $\boldsymbol{\phi}_2$, $\boldsymbol{\phi}_3$ and $\boldsymbol{\phi}_4$ in formula (3).

Now, with $\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1$ as a g-inverse of $\mathrm{D}(\hat{\boldsymbol{\tau}}_*)$, the equality (12) follows, which can easily be checked noting that $(\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')\hat{\boldsymbol{\tau}}_* = \hat{\boldsymbol{\tau}}_*$.

Appendix 2

For formula (13) note that the sum of squares SS_V can, on account of (8) and the relation $\hat{\boldsymbol{\tau}}_* = (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')\hat{\boldsymbol{\tau}}$, be written as

$$\mathrm{SS}_V = \boldsymbol{y}'\boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}(\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}_v')\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{y},$$

which, by the equalities

$$\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1} = \boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}_v' \quad \text{and}$$
$$(\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1} = (\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}(\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}_v')$$

(see Appendix 1), can be reduced to the form in (14). As to the sum of squares SS_R, its formula (15) follows directly from (13) on account of (4).

Appendix 3

For the formulae in (30) note that, using the well-known formula

$$(\boldsymbol{A} + \boldsymbol{BCD})^{-1} = \boldsymbol{A}^{-1} - \boldsymbol{A}^{-1}\boldsymbol{B}(\boldsymbol{C}^{-1} + \boldsymbol{DA}^{-1}\boldsymbol{B})^{-1}\boldsymbol{DA}^{-1},$$

one can write

$$\boldsymbol{V}^{-1} = \boldsymbol{V}_*^{-1} - \boldsymbol{V}_*^{-1} n^{-1}\mathbf{1}_n\mathbf{1}_n'[(\sigma_4^2-\sigma_3^2)^{-1}\boldsymbol{I}_n + n^{-1}\mathbf{1}_n\mathbf{1}_n'\boldsymbol{V}_*^{-1}n^{-1}\mathbf{1}_n\mathbf{1}_n']^{-1}n^{-1}\mathbf{1}_n\mathbf{1}_n'\boldsymbol{V}_*^{-1},$$

from which

$$\begin{aligned}\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1 &= \boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1 - \boldsymbol{X}_1'\boldsymbol{V}_*^{-1}n^{-1}\mathbf{1}_n\mathbf{1}_n'[(\sigma_4^2-\sigma_3^2)^{-1}\boldsymbol{I}_n \\ &\quad + n^{-1}\mathbf{1}_n\mathbf{1}_n'\boldsymbol{V}_*^{-1}n^{-1}\mathbf{1}_n\mathbf{1}_n']^{-1}n^{-1}\mathbf{1}_n\mathbf{1}_n'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1 \\ &= \boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1 - \boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1\mathbf{1}_v n^{-1}\mathbf{1}_n'[(\sigma_4^2-\sigma_3^2)^{-1}\boldsymbol{I}_n \\ &\quad + n^{-1}\mathbf{1}_n\mathbf{1}_v'\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1\mathbf{1}_v n^{-1}\mathbf{1}_n']^{-1}n^{-1}\mathbf{1}_n\mathbf{1}_v'\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1 \\ &= [(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1} + n^{-1}\mathbf{1}_v\mathbf{1}_n'(\sigma_4^2-\sigma_3^2)n^{-1}\mathbf{1}_n\mathbf{1}_v']^{-1} \\ &= [(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1} + (\sigma_4^2-\sigma_3^2)n^{-1}\mathbf{1}_v\mathbf{1}_v']^{-1}.\end{aligned}$$

Taking the inverse of this, one obtains the formula (31). From (30) it also follows that

$$\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1 = \boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1 + (\sigma_4^{-2}-\sigma_3^{-2})n^{-1}\boldsymbol{r}\boldsymbol{r}',$$

due to the relation $\mathbf{1}_n'\boldsymbol{X}_1 = \boldsymbol{r}'$. Furthermore, with these results the equality (32) can be proved, proceeding as follows:

$$\begin{aligned}\hat{\boldsymbol{\tau}}_* &= (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{y} \\ &= (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{y} \\ &= (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')(\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{y} \\ &= (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}(\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}_v')\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{y} \\ &= (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'(\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n')\boldsymbol{V}^{-1}\boldsymbol{y} \\ &= (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'(\boldsymbol{I}_v - n^{-1}\mathbf{1}_n\mathbf{1}_n')[\boldsymbol{V}_*^{-1} \\ &\quad + (\sigma_4^{-2}-\sigma_3^{-2})n^{-1}\mathbf{1}_n\mathbf{1}_n']\boldsymbol{y} \\ &= (\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}(\boldsymbol{I}_v - n^{-1}\mathbf{1}_n\mathbf{1}_n')\boldsymbol{y},\end{aligned}$$

because $(\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}_v')\boldsymbol{X}_1' = \boldsymbol{X}_1'(\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n')$ and $(\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n')\boldsymbol{V}_*^{-1} = \boldsymbol{V}_*^{-1}(\boldsymbol{I}_v - n^{-1}\mathbf{1}_n\mathbf{1}_n')$, as can easily be checked (see also Appendix 1).

Appendix 4

Formulae (34) and (35) are to be shown to be equivalent to formulae (14) and (15) respectively. To prove this, it may be helpful first to note the following equalities, which can easily be checked (see also Appendices 1 and 3):

$$\begin{aligned}(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}(\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}_v') &= (\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}(\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}_v'), \\ \boldsymbol{X}_1(\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}') &= (\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n')\boldsymbol{X}_1, \\ (\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}_v')\boldsymbol{X}_1' &= \boldsymbol{X}_1'(\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n'), \\ \boldsymbol{V}^{-1}(\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n') &= \boldsymbol{V}_*^{-1}(\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n'), \\ (\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n')\boldsymbol{V}^{-1} &= (\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n')\boldsymbol{V}_*^{-1}, \\ \boldsymbol{V}_*^{-1}(\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n') &= (\boldsymbol{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n')\boldsymbol{V}_*^{-1}.\end{aligned}$$

With these observations, it is easy to proceed as follows:

$$\begin{aligned}\mathrm{SS}_V &= \boldsymbol{y}'\boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{I}_v - n^{-1}\boldsymbol{1}_v\boldsymbol{r}')(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}(\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\boldsymbol{1}_v')\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{y}\\
&= \boldsymbol{y}'\boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{I}_v - n^{-1}\boldsymbol{1}_v\boldsymbol{r}')(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}(\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\boldsymbol{1}_v')\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{y}\\
&= \boldsymbol{y}'\boldsymbol{V}^{-1}(\boldsymbol{I}_n - n^{-1}\boldsymbol{1}_n\boldsymbol{1}_n')\boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'(\boldsymbol{I}_n - n^{-1}\boldsymbol{1}_n\boldsymbol{1}_n')\boldsymbol{V}^{-1}\boldsymbol{y}\\
&= \boldsymbol{y}'(\boldsymbol{I}_n - n^{-1}\boldsymbol{1}_n\boldsymbol{1}_n')\boldsymbol{V}_*^{-1}\boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}(\boldsymbol{I}_n - n^{-1}\boldsymbol{1}_n\boldsymbol{1}_n')\boldsymbol{y},\end{aligned}$$

which, with $\boldsymbol{y}_* = (\boldsymbol{I}_n - n^{-1}\boldsymbol{1}_n\boldsymbol{1}_n')\boldsymbol{y}$, is equivalent to the formula (34).

Now, considering formula (15), first note (recalling Appendix 1) that

$$\begin{aligned}&[\boldsymbol{V}^{-1} - \boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}^{-1}]n^{-1}\boldsymbol{1}_n\boldsymbol{1}_n'\\
&= \boldsymbol{V}^{-1}n^{-1}\boldsymbol{1}_n\boldsymbol{1}_n' - \boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}^{-1}n^{-1}\boldsymbol{1}_n\boldsymbol{1}_n'\\
&= \boldsymbol{V}^{-1}n^{-1}\boldsymbol{1}_n\boldsymbol{1}_n' - \boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1\boldsymbol{1}_v n^{-1}\boldsymbol{1}_n'\\
&= \boldsymbol{V}^{-1}n^{-1}\boldsymbol{1}_n\boldsymbol{1}_n' - \boldsymbol{V}^{-1}\boldsymbol{X}_1\boldsymbol{1}_v n^{-1}\boldsymbol{1}_n' = \boldsymbol{V}^{-1}n^{-1}\boldsymbol{1}_n\boldsymbol{1}_n' - \boldsymbol{V}^{-1}n^{-1}\boldsymbol{1}_n\boldsymbol{1}_n' = \mathbf{O}.\end{aligned}$$

With this result, formula (15) can be written as

$$\begin{aligned}\mathrm{SS}_R &= \boldsymbol{y}'[\boldsymbol{V}^{-1} - \boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}^{-1}]\boldsymbol{y}\\
&= \boldsymbol{y}'[\boldsymbol{V}^{-1} - \boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}^{-1}](\boldsymbol{I}_n - n^{-1}\boldsymbol{1}_n\boldsymbol{1}_n')\boldsymbol{y}\\
&= \boldsymbol{y}'[\boldsymbol{V}^{-1} - \boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}^{-1}](\boldsymbol{I}_n - n^{-1}\boldsymbol{1}_n\boldsymbol{1}_n')(\boldsymbol{I}_n - n^{-1}\boldsymbol{1}_n\boldsymbol{1}_n')\boldsymbol{y}\\
&= \boldsymbol{y}'[\boldsymbol{V}^{-1}(\boldsymbol{I}_n - n^{-1}\boldsymbol{1}_n\boldsymbol{1}_n') - \boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}^{-1}(\boldsymbol{I}_n - n^{-1}\boldsymbol{1}_n\boldsymbol{1}_n')]\boldsymbol{y}_*\\
&= \boldsymbol{y}'[(\boldsymbol{I}_n - n^{-1}\boldsymbol{1}_n\boldsymbol{1}_n')\boldsymbol{V}^{-1} - \boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'(\boldsymbol{I}_n - n^{-1}\boldsymbol{1}_n\boldsymbol{1}_n')\boldsymbol{V}^{-1}]\boldsymbol{y}_*\\
&= \boldsymbol{y}'[(\boldsymbol{I}_n - n^{-1}\boldsymbol{1}_n\boldsymbol{1}_n')\boldsymbol{V}_*^{-1} - \boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'(\boldsymbol{I}_n - n^{-1}\boldsymbol{1}_n\boldsymbol{1}_n')\boldsymbol{V}_*^{-1}]\boldsymbol{y}_*\\
&= \boldsymbol{y}'[(\boldsymbol{I}_n - n^{-1}\boldsymbol{1}_n\boldsymbol{1}_n')\boldsymbol{V}_*^{-1} - \boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1)^{-1}(\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\boldsymbol{1}_v')\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}]\boldsymbol{y}_*\\
&= \boldsymbol{y}'[(\boldsymbol{I}_n - n^{-1}\boldsymbol{1}_n\boldsymbol{1}_n')\boldsymbol{V}_*^{-1} - \boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}(\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\boldsymbol{1}_v')\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}]\boldsymbol{y}_*\\
&= \boldsymbol{y}'[(\boldsymbol{I}_n - n^{-1}\boldsymbol{1}_n\boldsymbol{1}_n')\boldsymbol{V}_*^{-1} - \boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{I}_v - n^{-1}\boldsymbol{1}_v\boldsymbol{r}')(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}]\boldsymbol{y}_*\\
&= \boldsymbol{y}'[(\boldsymbol{I}_n - n^{-1}\boldsymbol{1}_n\boldsymbol{1}_n')\boldsymbol{V}_*^{-1} - \boldsymbol{V}^{-1}(\boldsymbol{I}_n - n^{-1}\boldsymbol{1}_n\boldsymbol{1}_n')\boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}]\boldsymbol{y}_*\\
&= \boldsymbol{y}'[(\boldsymbol{I}_n - n^{-1}\boldsymbol{1}_n\boldsymbol{1}_n')\boldsymbol{V}_*^{-1} - (\boldsymbol{I}_n - n^{-1}\boldsymbol{1}_n\boldsymbol{1}_n')\boldsymbol{V}_*^{-1}\boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}]\boldsymbol{y}_*\\
&= \boldsymbol{y}_*'[\boldsymbol{V}_*^{-1} - \boldsymbol{V}_*^{-1}\boldsymbol{X}_1(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}]\boldsymbol{y}_*,\end{aligned}$$

which is equivalent to (35).

Appendix 5

For formula (41) note that, from (40),

$$\begin{aligned}\mathrm{SS}(\boldsymbol{U}_\mathrm{A}) + \mathrm{SS}(\boldsymbol{U}_\mathrm{B}) &= \hat{\boldsymbol{\tau}}_*'\{\boldsymbol{U}_\mathrm{A}[\boldsymbol{U}_\mathrm{A}'(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_\mathrm{A}]^-\boldsymbol{U}_\mathrm{A}'\\
&\quad + \boldsymbol{U}_\mathrm{B}[\boldsymbol{U}_\mathrm{B}'(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_\mathrm{B}]^-\boldsymbol{U}_\mathrm{B}'\}\hat{\boldsymbol{\tau}}_*\\
&= \hat{\boldsymbol{\tau}}'(\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\boldsymbol{1}_v')\{\boldsymbol{U}_\mathrm{A}[\boldsymbol{U}_\mathrm{A}'(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_\mathrm{A}]^-\boldsymbol{U}_\mathrm{A}'\\
&\quad + \boldsymbol{U}_\mathrm{B}[\boldsymbol{U}_\mathrm{B}'(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_\mathrm{B}]^-\boldsymbol{U}_\mathrm{B}'\}(\boldsymbol{I}_v - n^{-1}\boldsymbol{1}_v\boldsymbol{r}')\hat{\boldsymbol{\tau}},\end{aligned}$$

which, because $\boldsymbol{U}_{\rm A}'\mathbf{1}_v = \mathbf{0} = \boldsymbol{U}_{\rm B}'\mathbf{1}_v$, reduces to

$$\begin{aligned}\mathrm{SS}(\boldsymbol{U}_{\rm A}) + \mathrm{SS}(\boldsymbol{U}_{\rm B}) = \hat{\boldsymbol{\tau}}'\{&\boldsymbol{U}_{\rm A}[\boldsymbol{U}_{\rm A}'(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\rm A}]^-\boldsymbol{U}_{\rm A}' \\ &+ \boldsymbol{U}_{\rm B}[\boldsymbol{U}_{\rm B}'(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\rm B}]^-\boldsymbol{U}_{\rm B}'\}\hat{\boldsymbol{\tau}}.\end{aligned}$$

On the other hand, from the equivalence of SS_V in (13) and (34), one can write (also from Appendix 3) that

$$\begin{aligned}\mathrm{SS}_V = \hat{\boldsymbol{\tau}}_*'\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1\hat{\boldsymbol{\tau}}_* &= \hat{\boldsymbol{\tau}}'(\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}_v')\boldsymbol{X}_1'\boldsymbol{V}^{-1}\boldsymbol{X}_1(\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')\hat{\boldsymbol{\tau}} \\ &= \hat{\boldsymbol{\tau}}'(\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}_v')\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1(\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')\hat{\boldsymbol{\tau}}.\end{aligned}$$

Hence, for any $\hat{\boldsymbol{\tau}}$, the equality (41) holds if and only if

$$\begin{aligned}\boldsymbol{U}_{\rm A}[\boldsymbol{U}_{\rm A}'(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\rm A}]^-\boldsymbol{U}_{\rm A}' &+ \boldsymbol{U}_{\rm B}[\boldsymbol{U}_{\rm B}'(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}\boldsymbol{U}_{\rm B}]^-\boldsymbol{U}_{\rm B}' \\ &= (\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}_v')\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1(\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}') \\ &= \boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1(\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}'),\end{aligned}$$

because $(\boldsymbol{I}_v - n^{-1}\boldsymbol{r}\mathbf{1}_v')\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1 = \boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1(\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}')$. Now, premultiplying by $(\boldsymbol{X}_1'\boldsymbol{V}_*^{-1}\boldsymbol{X}_1)^{-1}$, one obtains the condition (42).