Use of α-resolvable designs in the construction of two-factor experiments of split-plot type

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Summary

We consider an incomplete split-plot design (ISPD) with two factors generated by the semi-Kronecker product of two α-resolvable designs. We use an α-resolvable design for the whole plot treatments and an affine α-resolvable design for the subplot treatments. We characterize the ISPDs with respect to the general balance property, and we give the stratum efficiency factors for the ISPDs.

Key words: α-resolvable designs; Affine α-resolvable designs; General balance property; Incomplete split-plot designs; Stratum efficiency factors.

1.   Introduction

We consider a two-factor experiment of split-plot type. The first factor A has v_1 levels $A_1, A_2, \ldots, A_{v_1}$, and the second factor B has v_2 levels $B_1, B_2, \ldots, B_{v_2}$. In a split-plot design, each of b blocks is divided into k_1 whole plots, and each whole plot is divided into k_2 subplots. The levels of factors A and B are applied to the whole plots (called whole plot treatments) and the subplots (called subplot treatments), respectively. We consider an incomplete split-plot design (ISPD) such that $k_1 < v_1$ or $k_2 < v_2$.

ISPDs are used in biological, agricultural and environmental sciences, and the analysis of data and the constructions of the ISPDs have been studied by Bhargava and Shah (1975), Rees (1969), Robinson (1967a, 1967b), Mejza and Mejza (1984, 1996), Mejza (1987), and others. In Mejza (1987)

and Mejza and Mejza (1996), a mixed linear model with fixed treatment effects and random block, whole plot and subplot effects was given. The hth factorial treatment combination effect τ_h is defined by

$$\tau_h = \mu + \alpha_i + \gamma_j + (\alpha\gamma)_{ij}$$

for $h = (i-1)v_2 + j$, $i = 1, 2, \ldots, v_1$ and $j = 1, 2, \ldots, v_2$, where μ is the overall mean, α_i denotes the main effect of the level A_i of the factor A, γ_j denotes the main effect of the level B_j of the factor B and $(\alpha\gamma)_{ij}$ denotes the interaction effect of A_i and B_j. Here $\Sigma_{i=1}^{v_1}\alpha_i = 0$, $\Sigma_{j=1}^{v_2}\gamma_j = 0$, $\Sigma_{i=1}^{v_1}(\alpha\gamma)_{ij} = 0$ for $j = 1, 2, \ldots, v_2$ and $\Sigma_{j=1}^{v_2}(\alpha\gamma)_{ij} = 0$ for $i = 1, 2, \ldots, v_1$. The mixed linear model results from a three-step randomization, i.e., the randomization of blocks, the randomization of whole plots within each block and the randomization of subplots within each whole plot. This kind of randomization leads us to the experiments with orthogonal block structure as defined by Nelder (1965a, 1965b), and the multistratum analysis proposed by Nelder (1965a, 1965b) and Houtman and Speed (1983) can be applied to the analysis of the experiments. In the case of the ISPD, we have three strata, apart from the zero stratum connected with the overall mean only: (I) the inter-block stratum, (II) the inter-whole plot stratum and (III) the intra-unit stratum. The statistical properties of the ISPD are strongly related to the eigenvalues and the eigenvectors of the stratum information matrices for the treatment combinations. Here, we assume that every treatment combination A_iB_j $(i = 1, 2, \ldots, v_1, j = 1, 2, \ldots, v_2)$ occurs in precisely r blocks and the treatment combinations are arranged in the lexicographic order. The stratum information matrices are

$$\mathbf{A}_1 = \frac{1}{k_1 k_2}\mathbf{N}_1\mathbf{N}_1' - \frac{r}{v}\mathbf{J}_v, \quad \mathbf{A}_2 = \frac{1}{k_2}\mathbf{N}_2\mathbf{N}_2' - \frac{1}{k_1 k_2}\mathbf{N}_1\mathbf{N}_1'$$

$$\text{and} \quad \mathbf{A}_3 = r\mathbf{I}_v - \frac{1}{k_2}\mathbf{N}_2\mathbf{N}_2', \tag{1}$$

where $v = v_1 v_2$, $\mathbf{N}_1$ and $\mathbf{N}_2$ are the incidence matrices for treatment combinations vs. blocks and treatment combinations vs. whole plots, respectively, $\mathbf{I}_v$ is the identity matrix of order v and $\mathbf{J}_v$ is a $v \times v$ matrix whose elements are all unity. The eigenvalues of a matrix $\mathbf{A}_f^* = r^{-1}\mathbf{A}_f$ can be identified as the stratum efficiency factors of the design for $f = 1, 2, 3$ (see Houtman and Speed, 1983). Moreover, the eigenvectors define contrasts of treatment effects, which are called the basic contrasts (see Pearce, Caliński and Marshall, 1974).

Mejza (1987) and Mejza and Mejza (1996) considered the constructions of ISPDs by the Kronecker product of the incidence matrices of two designs. Ozawa et al. (2004), Ozawa and Kuriki (2006) and Kuriki and Nakajima (2007) considered the constructions of ISPDs by a modified Kronecker product (called the semi-Kronecker product) of the incidence matrices of two resolvable designs. In this paper, we construct a new ISPD by the semi-Kronecker product of the incidence matrices of an α-resolvable design for the whole plot treatments and an affine α-resolvable design for the subplot treatments, and these new ISPDs also include those obtained by Ozawa et al. (2004), Ozawa and Kuriki (2006) and Kuriki and Nakajima (2007). We characterize the ISPDs with respect to the general balance property and we give the stratum efficiency factors for the ISPDs. These ISPDs have smaller numbers of blocks than those designs constructed by the Kronecker product of two designs, and they would be useful in many practical situations.

2.   Method of construction of ISPDs

Firstly, we need the semi-Kronecker product of two matrices in order to construct ISPDs (see Mejza, Kuriki and Mejza, 2001). Suppose that two matrices $\mathbf{E}$ and $\mathbf{F}$ are divided into the same number t of submatrices as follows:

$$\mathbf{E} = (\mathbf{E}_1 : \mathbf{E}_2 : \cdots : \mathbf{E}_t) \quad \text{and} \quad \mathbf{F} = (\mathbf{F}_1 : \mathbf{F}_2 : \cdots : \mathbf{F}_t).$$

Then, the semi-Kronecker product $\mathbf{E} \,\tilde{\otimes}\, \mathbf{F}$ is given by

$$\mathbf{E} \,\tilde{\otimes}\, \mathbf{F} = (\mathbf{E}_1 \otimes \mathbf{F}_1 : \mathbf{E}_2 \otimes \mathbf{F}_2 : \cdots : \mathbf{E}_t \otimes \mathbf{F}_t),$$

where $\otimes$ denotes the usual Kronecker product. This type of Kronecker product was first considered by Khatri and Rao (1968).

Next, we need an α-resolvable design and an affine α-resolvable design. A design with v treatments, r replications of each treatment and k plots within each block is denoted by $\mathrm{D}(v, r, k)$. If the collection of blocks of a $\mathrm{D}(v, r, k)$ can be partitioned into classes (called resolution classes) such that every treatment occurs precisely α times in every class, the design is said to be α-resolvable (resolvable for brevity, if $\alpha = 1$) and it is denoted by α-$\mathrm{RD}(v, r, k)$. The parameters of an α-$\mathrm{RD}(v, r, k)$ satisfy the following conditions:

$$\alpha v = k\beta, \quad b = t\beta \quad \text{and} \quad r = \alpha t,$$

where β is the number of blocks within each resolution class, b is the total number of blocks and t is the number of resolution classes. Moreover, an α-resolvable design is called an affine α-resolvable design if the number of common treatments appearing in two distinct blocks in the same resolution class is q_1, and if the number of common treatments appearing in two distinct blocks belonging to different resolution classes is q_2. The parameters of an affine α-RD(v, r, k) satisfy the additional following conditions:

$$q_1 = \frac{\alpha - 1}{\beta - 1}k, \quad q_2 = \frac{k^2}{v}, \quad \beta > 1 \quad \text{and} \quad k > q_1.$$

Many authors have given constructions of α-resolvable and affine α-resolvable balanced and partially balanced incomplete block designs (see, for example, Kageyama, 1973, Bailey, Monod and Morgan, 1995, Caliński and Kageyama, 2003, Kadowaki and Kageyama, 2009). By taking some resolution classes of these designs, we can obtain α-resolvable and affine α-resolvable designs considered above. In particular, an affine α-resolvable design with $\alpha = 1$, $q_1 = 0$ and $q_2 = 1$ is called a square lattice design.

We construct an ISPD from an α-resolvable design and an affine α-resolvable design. Let

$$\mathbf{N}_A = (\mathbf{N}_{A1} : \mathbf{N}_{A2} : \cdots : \mathbf{N}_{At}) \quad \text{and} \quad \mathbf{N}_B = (\mathbf{N}_{B1} : \mathbf{N}_{B2} : \cdots : \mathbf{N}_{Bt})$$

be the incidence matrices of an α_1-RD(v_1, r_1, k_1) with $\beta_1 = \alpha_1 v_1 / k_1$ and $b_1 = t\beta_1$, and an affine α_2-RD(v_2, r_2, k_2) with $\beta_2 = \alpha_2 v_2 / k_2$ and $b_2 = t\beta_2$, respectively, where the designs have the same number t of resolution classes, and $\mathbf{N}_{Ai}$ and $\mathbf{N}_{Bi}$ correspond to the ith resolution class. By the definition of the affine α-resolvable design,

$$\mathbf{N}'_{Bi}\mathbf{N}_{Bi} = (k_2 - q_{21})\mathbf{I}_{\beta_2} + q_{21}\mathbf{J}_{\beta_2} \quad \text{and} \quad \mathbf{N}'_{Bi}\mathbf{N}_{Bj} = q_{22}\mathbf{J}_{\beta_2} \qquad (2)$$

hold for $i, j = 1, 2, \ldots, t$, $i \neq j$, where $q_{21} = k_2(\alpha_2 - 1)/(\beta_2 - 1)$ and $q_{22} = k_2^2/v_2$. Now we construct an ISPD, say $\tilde{\mathcal{D}}$, such that its incidence matrix $\mathbf{N}_1$ is obtained by the semi-Kronecker product of $\mathbf{N}_A$ and $\mathbf{N}_B$, i.e.,

$$\mathbf{N}_1 = \mathbf{N}_A \,\tilde{\otimes}\, \mathbf{N}_B = (\mathbf{N}_{A1} \otimes \mathbf{N}_{B1} : \mathbf{N}_{A2} \otimes \mathbf{N}_{B2} : \cdots : \mathbf{N}_{At} \otimes \mathbf{N}_{Bt}). \quad (3)$$

Here, we use the α-resolvable design for whole plot treatments and the affine α-resolvable design for subplot treatments, respectively, and we arrange the same treatments of the affine α-resolvable design on the subplots in each whole plot within each block. The ISPD $\tilde{\mathcal{D}}$ has v_1 whole plot treatments, v_2 subplot treatments and $t\beta_1\beta_2$ blocks with k_1 whole plots and k_2 subplots in

each whole plot. For $\tilde{\mathcal{D}}$, we can express $\mathbf{N}_2$ as

$$\mathbf{N}_2 = (\mathbf{1}'_{\alpha_1} \otimes \mathbf{I}_{v_1} \otimes \mathbf{N}_{B1} : \mathbf{1}'_{\alpha_1} \otimes \mathbf{I}_{v_1} \otimes \mathbf{N}_{B2} : \cdots : \mathbf{1}'_{\alpha_1} \otimes \mathbf{I}_{v_1} \otimes \mathbf{N}_{Bt}), \quad (4)$$

arranging the whole plots in a suitable order, where $\mathbf{1}_v$ is a $v \times 1$ vector whose elements are all unity. Therefore, from (3) and (4), we have

$$\begin{aligned}
\mathbf{N}_1\mathbf{N}'_1 &= \textstyle\sum_{i=1}^{t}(\mathbf{N}_{Ai}\mathbf{N}'_{Ai} \otimes \mathbf{N}_{Bi}\mathbf{N}'_{Bi}) \quad \text{and} \\
\mathbf{N}_2\mathbf{N}'_2 &= \textstyle\sum_{i=1}^{t}(\alpha_1\mathbf{I}_{v_1} \otimes \mathbf{N}_{Bi}\mathbf{N}'_{Bi}) = \alpha_1\mathbf{I}_{v_1} \otimes \mathbf{N}_B\mathbf{N}'_B.
\end{aligned} \quad (5)$$

Note that $\tilde{\mathcal{D}}$ is also an α-resolvable design with $\alpha = \alpha_1\alpha_2$ when the treatment combinations are regarded as the usual treatments.

Example 1. We consider a 2-RD(6,4,4) and an affine 2-RD(9,4,6) with $\beta_1 = \beta_2 = 3$, $b_1 = b_2 = 6$, $t = 2$, $q_{21} = 3$, $q_{22} = 4$ and the incidence matrices $\mathbf{N}_A = (\mathbf{N}_{A1} : \mathbf{N}_{A2})$ and $\mathbf{N}_B = (\mathbf{N}_{B1} : \mathbf{N}_{B2})$, where

$$\mathbf{N}_{A1} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad \mathbf{N}_{A2} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

$$\mathbf{N}_{B1} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad \mathbf{N}_{B2} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

The ISPD $\tilde{\mathcal{D}}$ constructed by the semi-Kronecker product has 18 blocks with 4 whole plots and 6 subplots in each whole plot. The ISPD can be expressed by the following way:

$\{A_1, A_2, A_3, A_4 \,|\, B_1, B_2, B_3, B_4, B_5, B_6\}$, $\{A_1, A_2, A_3, A_4 \,|\, B_4, B_5, B_6, B_7, B_8, B_9\}$,

$\{A_1, A_2, A_3, A_4 \,|\, B_1, B_2, B_3, B_7, B_8, B_9\}$, $\{A_1, A_2, A_5, A_6 \,|\, B_1, B_2, B_3, B_4, B_5, B_6\}$,

$\{A_1, A_2, A_5, A_6 \,|\, B_4, B_5, B_6, B_7, B_8, B_9\}$, $\{A_1, A_2, A_5, A_6 \,|\, B_1, B_2, B_3, B_7, B_8, B_9\}$,

$\{A_3, A_4, A_5, A_6 \mid B_1, B_2, B_3, B_4, B_5, B_6\}$, $\{A_3, A_4, A_5, A_6 \mid B_4, B_5, B_6, B_7, B_8, B_9\}$,

$\{A_3, A_4, A_5, A_6 \mid B_1, B_2, B_3, B_7, B_8, B_9\}$, $\{A_1, A_2, A_5, A_6 \mid B_1, B_2, B_4, B_5, B_7, B_8\}$,

$\{A_1, A_2, A_5, A_6 \mid B_2, B_3, B_5, B_6, B_8, B_9\}$, $\{A_1, A_2, A_5, A_6 \mid B_1, B_3, B_4, B_6, B_7, B_9\}$,

$\{A_2, A_3, A_4, A_6 \mid B_1, B_2, B_4, B_5, B_7, B_8\}$, $\{A_2, A_3, A_4, A_6 \mid B_2, B_3, B_5, B_6, B_8, B_9\}$,

$\{A_2, A_3, A_4, A_6 \mid B_1, B_3, B_4, B_6, B_7, B_9\}$, $\{A_1, A_3, A_4, A_5 \mid B_1, B_2, B_4, B_5, B_7, B_8\}$,

$\{A_1, A_3, A_4, A_5 \mid B_2, B_3, B_5, B_6, B_8, B_9\}$, $\{A_1, A_3, A_4, A_5 \mid B_1, B_3, B_4, B_6, B_7, B_9\}$,

where $\{A_{i_1}, A_{i_2}, A_{i_3}, A_{i_4} \mid B_{j_1}, B_{j_2}, B_{j_3}, B_{j_4}, B_{j_5}, B_{j_6}\}$ denotes a block such that $A_{i_1}, A_{i_2}, A_{i_3}, A_{i_4}$ are whole plot treatments and $B_{j_1}, B_{j_2}, B_{j_3}, B_{j_4}, B_{j_5},$ B_{j_6} are subplot treatments in each whole plot. We note that if the ISPD is constructed by the usual Kronecker product of the incidence matrices, then the number of blocks becomes 36. Generally, the number of blocks of an ISPD constructed by the Kronecker product is t times larger than those of an ISPD constructed by the semi-Kronecker product.

3. Stratum efficiency factors for the ISPD $\tilde{\mathcal{D}}$

In this section, we consider the stratum efficiency factors for the ISPD $\tilde{\mathcal{D}}$ constructed by the semi-Kronecker product of two α-resolvable designs. To find the stratum efficiency factors, we need the eigenvalues of the stratum information matrices $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ of $\tilde{\mathcal{D}}$. If the stratum information matrices $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ have common eigenvectors, i.e., if $\tilde{\mathcal{D}}$ is generally balanced, then we can easily find the eigenvalues of the matrices. From (2),

$$\mathbf{N}_{Bi}\mathbf{N}'_{Bi}\mathbf{N}_{Bj}\mathbf{N}'_{Bj} = \alpha_2^2 q_{22}\mathbf{J}_{v_2} \tag{6}$$

holds for $i, j = 1, 2, \ldots, t$, $i \neq j$. From (5) and (6), we see that the concurrence matrices $\mathbf{N}_1\mathbf{N}'_1$ and $\mathbf{N}_2\mathbf{N}'_2$ are commutative, which implies, by (1), that the stratum information matrices $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ are mutually commutative. Therefore $\tilde{\mathcal{D}}$ is generally balanced.

We consider the eigenvectors of the concurrence matrices in the α_1-RD(v_1, r_1, k_1) and the affine α_2-RD(v_2, r_2, k_2) in order to find the common eigenvectors of $\mathbf{N}_1\mathbf{N}'_1$ and $\mathbf{N}_2\mathbf{N}'_2$.

Let $\theta_j^{(i)}$ and $\boldsymbol{x}_{ij}$ $(j = 0, 1, \ldots, v_1 - 1)$ be the eigenvalues and the corresponding mutually orthonormal eigenvectors of $\mathbf{N}_{Ai}\mathbf{N}'_{Ai}$ for each $i = 1, 2, \ldots,$ t. In particular, the eigenvalue of $\mathbf{N}_{Ai}\mathbf{N}'_{Ai}$ corresponding to the eigenvector

$\boldsymbol{x}_{i0} = (\sqrt{v_1})^{-1}\mathbf{1}_{v_1}$ is given by $\theta_0^{(i)} = \alpha_1 k_1$ for each $i = 1, 2, \ldots, t$. In Table 1, we see the eigenvalues and the eigenvectors of the concurrence matrix $\mathbf{N}_{Ai}\mathbf{N}'_{Ai}$ for each $i = 1, 2, \ldots, t$ in the α_1-RD(v_1, r_1, k_1). Furthermore, the

Table 1. Eigenvalues and eigenvectors of $\mathbf{N}_{Ai}\mathbf{N}'_{Ai}$ in α_1-RD(v_1, r_1, k_1)

Eigenvalues	Eigenvectors
$\alpha_1 k_1$	$\boldsymbol{x}_{i0} = (\sqrt{v_1})^{-1}\mathbf{1}_{v_1}$
$\theta_j^{(i)}$	$\boldsymbol{x}_{ij}$ $(j = 1, 2, \ldots, v_1 - 1)$

eigenvalues and the corresponding mutually orthonormal eigenvectors of $\mathbf{N}_A\mathbf{N}'_A$ are denoted by θ_j and $\boldsymbol{d}_j$ for $j = 0, 1, \ldots, v_1 - 1$. In particular, $\theta_0 = t\alpha_1 k_1$ and $\boldsymbol{d}_0 = (\sqrt{v_1})^{-1}\mathbf{1}_{v_1}$.

For the incidence matrix of the affine α_2-RD(v_2, r_2, k_2), by (2), $\mathbf{N}_{Bi}\mathbf{N}'_{Bi}$ has the eigenvalues $\alpha_2 k_2$, $k_2 - q_{21}$ and 0 with multiplicities 1, $\beta_2 - 1$ and $v_2 - \beta_2$, respectively, for each $i = 1, 2, \ldots, t$. From (6), $\mathbf{N}_{B1}\mathbf{N}'_{B1}$, $\mathbf{N}_{B2}\mathbf{N}'_{B2}, \ldots,$ $\mathbf{N}_{Bt}\mathbf{N}'_{Bt}$ are mutually commutative, so these concurrence matrices have common eigenvectors. Let $\mathbf{Q} = (\boldsymbol{q}_0, \boldsymbol{q}_1, \ldots, \boldsymbol{q}_{\beta_2-1})$ be an orthogonal matrix of order β_2 with $\boldsymbol{q}_0 = (\sqrt{\beta_2})^{-1}\mathbf{1}_{\beta_2}$. For the ith resolution class, by (2), the mutually orthonormal eigenvectors of $\mathbf{N}_{Bi}\mathbf{N}'_{Bi}$ corresponding to the eigenvalues $\alpha_2 k_2$ and $k_2 - q_{21}$ are given by

$$\boldsymbol{z}_{i0} = \frac{1}{\sqrt{\alpha_2 k_2}}\mathbf{N}_{Bi}\boldsymbol{q}_0 \quad \text{and} \quad \boldsymbol{z}_{ip} = \frac{1}{\sqrt{k_2 - q_{21}}}\mathbf{N}_{Bi}\boldsymbol{q}_p$$

for $p = 1, 2, \ldots, \beta_2 - 1$, respectively. In particular, $\boldsymbol{z}_{i0} = (\sqrt{v_2})^{-1}\mathbf{1}_{v_2}$. By (2), the eigenvectors $\boldsymbol{z}_{ip}$ $(p = 0, 1, \ldots, \beta_2 - 1)$ are also the eigenvectors of the concurrence matrix $\mathbf{N}_{Bh}\mathbf{N}'_{Bh}$ $(h \neq i)$ for the other resolution class, and the eigenvalues of $\mathbf{N}_{Bh}\mathbf{N}'_{Bh}$ corresponding to $\boldsymbol{z}_{i0}$ and $\boldsymbol{z}_{ip}$ $(p \neq 0)$ are $\alpha_2 k_2$ and 0, respectively. The mutually orthonormal common eigenvectors of $\mathbf{N}_{B1}\mathbf{N}'_{B1}, \mathbf{N}_{B2}\mathbf{N}'_{B2}, \ldots, \mathbf{N}_{Bt}\mathbf{N}'_{Bt}$ corresponding to the eigenvalue 0 are denoted by $\boldsymbol{z}_q^*$ for $q = 1, 2, \ldots, v_2 - t(\beta_2 - 1) - 1$. These eigenvalues and eigenvectors are summarized in the following table:

The common eigenvectors in Table 2 are mutually orthogonal, and these eigenvectors play an important role in the succeeding argument.

Combining the eigenvectors of Table 1 and Table 2, we consider the following six sets of vectors:

(i) $\dfrac{1}{\sqrt{v_1}}\mathbf{1}_{v_1} \otimes \dfrac{1}{\sqrt{v_2}}\mathbf{1}_{v_2}$,   (ii) $\boldsymbol{d}_j \otimes \dfrac{1}{\sqrt{v_2}}\mathbf{1}_{v_2}$,   (iii) $\dfrac{1}{\sqrt{v_1}}\mathbf{1}_{v_1} \otimes \boldsymbol{z}_{ip}$,

(iv) $\dfrac{1}{\sqrt{v_1}}\mathbf{1}_{v_1} \otimes \boldsymbol{z}_q^*$,   (v) $\boldsymbol{x}_{ij} \otimes \boldsymbol{z}_{ip}$,   (vi) $\boldsymbol{d}_j \otimes \boldsymbol{z}_q^*$,

Table 2. Eigenvalues and common eigenvectors of $\mathbf{N}_{Bi}\mathbf{N}'_{Bi}$ in affine $\alpha_2\text{-RD}(v_2, r_2, k_2)$

Eigenvalues				Common eigenvectors
$\mathbf{N}_{B1}\mathbf{N}'_{B1}$	$\mathbf{N}_{B2}\mathbf{N}'_{B2}$	$\cdots$	$\mathbf{N}_{Bt}\mathbf{N}'_{Bt}$	
$\alpha_2 k_2$	$\alpha_2 k_2$	$\cdots$	$\alpha_2 k_2$	$\boldsymbol{z}_{i0} = (\sqrt{v_2})^{-1}\mathbf{1}_{v_2}$
$k_2 - q_{21}$	0	$\cdots$	0	$\boldsymbol{z}_{1p}$ $(p = 1, 2, \ldots, \beta_2 - 1)$
0	$k_2 - q_{21}$	$\cdots$	0	$\boldsymbol{z}_{2p}$ $(p = 1, 2, \ldots, \beta_2 - 1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
0	0	$\cdots$	$k_2 - q_{21}$	$\boldsymbol{z}_{tp}$ $(p = 1, 2, \ldots, \beta_2 - 1)$
0	0	$\cdots$	0	$\boldsymbol{z}_q^*$ $(q = 1, 2, \ldots, v_2 - t(\beta_2 - 1) - 1)$

for $i = 1, 2, \ldots, t$, $j = 1, 2, \ldots, v_1 - 1$, $p = 1, 2, \ldots, \beta_2 - 1$ and $q = 1, 2, \ldots, v_2 - t(\beta_2 - 1) - 1$. The vectors of (i)–(vi) are mutually orthonormal and the total number of vectors is $v_1 v_2$. We show that the vectors of (i)–(vi) are the common eigenvectors of $\mathbf{N}_1\mathbf{N}'_1$ and $\mathbf{N}_2\mathbf{N}'_2$ and we find the corresponding eigenvalues of $\mathbf{N}_1\mathbf{N}'_1$ and $\mathbf{N}_2\mathbf{N}'_2$.

Firstly, we consider the matrix $\mathbf{N}_1\mathbf{N}'_1$. For (i), we have, from (5), Table 1 and Table 2,

$$\mathbf{N}_1\mathbf{N}'_1 \left(\frac{1}{\sqrt{v_1}}\mathbf{1}_{v_1} \otimes \frac{1}{\sqrt{v_2}}\mathbf{1}_{v_2} \right) =$$
$$= \sum_{i=1}^{t}(\mathbf{N}_{Ai}\mathbf{N}'_{Ai} \otimes \mathbf{N}_{Bi}\mathbf{N}'_{Bi}) \left(\frac{1}{\sqrt{v_1}}\mathbf{1}_{v_1} \otimes \frac{1}{\sqrt{v_2}}\mathbf{1}_{v_2} \right) =$$
$$= \sum_{i=1}^{t} \left(\mathbf{N}_{Ai}\mathbf{N}'_{Ai}\frac{1}{\sqrt{v_1}}\mathbf{1}_{v_1} \otimes \mathbf{N}_{Bi}\mathbf{N}'_{Bi}\frac{1}{\sqrt{v_2}}\mathbf{1}_{v_2} \right) =$$
$$= \sum_{i=1}^{t} \left(\alpha_1 k_1 \frac{1}{\sqrt{v_1}}\mathbf{1}_{v_1} \otimes \alpha_2 k_2 \frac{1}{\sqrt{v_2}}\mathbf{1}_{v_2} \right) = t\alpha_1 k_1 \alpha_2 k_2 \left(\frac{1}{\sqrt{v_1}}\mathbf{1}_{v_1} \otimes \frac{1}{\sqrt{v_2}}\mathbf{1}_{v_2} \right).$$

The corresponding eigenvalue is $t\alpha_1 k_1 \alpha_2 k_2$.

For (ii), we have

$$\mathbf{N}_1\mathbf{N}'_1 \left(\boldsymbol{d}_j \otimes \frac{1}{\sqrt{v_2}}\mathbf{1}_{v_2} \right) = \sum_{i=1}^{t} \left(\mathbf{N}_{Ai}\mathbf{N}'_{Ai}\boldsymbol{d}_j \otimes \mathbf{N}_{Bi}\mathbf{N}'_{Bi}\frac{1}{\sqrt{v_2}}\mathbf{1}_{v_2} \right) =$$
$$= \sum_{i=1}^{t} \left(\mathbf{N}_{Ai}\mathbf{N}'_{Ai}\boldsymbol{d}_j \otimes \alpha_2 k_2 \frac{1}{\sqrt{v_2}}\mathbf{1}_{v_2} \right) = \mathbf{N}_A\mathbf{N}'_A\boldsymbol{d}_j \otimes \alpha_2 k_2 \frac{1}{\sqrt{v_2}}\mathbf{1}_{v_2} =$$
$$= \theta_j \alpha_2 k_2 \left(\boldsymbol{d}_j \otimes \frac{1}{\sqrt{v_2}}\mathbf{1}_{v_2} \right).$$

The corresponding eigenvalue is $\theta_j \alpha_2 k_2$ for $j = 1, 2, \ldots, v_1 - 1$.

For (iii), we have

$$\mathbf{N}_1\mathbf{N}'_1 \left(\frac{1}{\sqrt{v_1}}\mathbf{1}_{v_1} \otimes \boldsymbol{z}_{ip} \right) = \sum_{i'=1}^{t} \left(\mathbf{N}_{Ai'}\mathbf{N}'_{Ai'}\frac{1}{\sqrt{v_1}}\mathbf{1}_{v_1} \otimes \mathbf{N}_{Bi'}\mathbf{N}'_{Bi'}\boldsymbol{z}_{ip} \right) =$$
$$= \alpha_1 k_1(k_2 - q_{21}) \left(\frac{1}{\sqrt{v_1}}\mathbf{1}_{v_1} \otimes \boldsymbol{z}_{ip} \right).$$

The corresponding eigenvalue is $\alpha_1 k_1(k_2 - q_{21})$ for $i = 1, 2, \ldots, t$ and $p = 1, 2, \ldots, \beta_2 - 1$.

For (iv), we have

$$\mathbf{N}_1\mathbf{N}_1' \left(\frac{1}{\sqrt{v_1}}\mathbf{1}_{v_1} \otimes \boldsymbol{z}_q^* \right) = \sum_{i=1}^{t} \left(\mathbf{N}_{Ai}\mathbf{N}_{Ai}'\frac{1}{\sqrt{v_1}}\mathbf{1}_{v_1} \otimes \mathbf{N}_{Bi}\mathbf{N}_{Bi}'\boldsymbol{z}_q^* \right) = \boldsymbol{0}.$$

The corresponding eigenvalue is zero for $q = 1, 2, \ldots, v_2 - t(\beta_2 - 1) - 1$. Similarly, for (vi), the eigenvalue is also zero.

For (v), we have

$$\mathbf{N}_1\mathbf{N}_1' \left(\boldsymbol{x}_{ij} \otimes \boldsymbol{z}_{ip} \right) = \sum_{i'=1}^{t} (\mathbf{N}_{Ai'}\mathbf{N}_{Ai'}'\boldsymbol{x}_{ij} \otimes \mathbf{N}_{Bi'}\mathbf{N}_{Bi'}'\boldsymbol{z}_{ip}) =$$
$$= \mathbf{N}_{Ai}\mathbf{N}_{Ai}'\boldsymbol{x}_{ij} \otimes \mathbf{N}_{Bi}\mathbf{N}_{Bi}'\boldsymbol{z}_{ip} = \theta_j^{(i)}(k_2 - q_{21}) \left(\boldsymbol{x}_{ij} \otimes \boldsymbol{z}_{ip} \right).$$

The corresponding eigenvalue is $\theta_j^{(i)}(k_2 - q_{21})$ for $i = 1, 2, \ldots, t$, $j = 1, 2, \ldots, v_1 - 1$ and $p = 1, 2, \ldots, \beta_2 - 1$.

Next, we consider the matrix $\mathbf{N}_2\mathbf{N}_2'$. For (i), we have, from (5), Table 1 and Table 2,

$$\mathbf{N}_2\mathbf{N}_2' \left(\frac{1}{\sqrt{v_1}}\mathbf{1}_{v_1} \otimes \frac{1}{\sqrt{v_2}}\mathbf{1}_{v_2} \right) = (\alpha_1\mathbf{I}_{v_1} \otimes \mathbf{N}_B\mathbf{N}_B') \left(\frac{1}{\sqrt{v_1}}\mathbf{1}_{v_1} \otimes \frac{1}{\sqrt{v_2}}\mathbf{1}_{v_2} \right) =$$
$$= \left(\alpha_1\frac{1}{\sqrt{v_1}}\mathbf{1}_{v_1} \right) \otimes \left(\mathbf{N}_B\mathbf{N}_B'\frac{1}{\sqrt{v_2}}\mathbf{1}_{v_2} \right) = t\alpha_1\alpha_2 k_2 \left(\frac{1}{\sqrt{v_1}}\mathbf{1}_{v_1} \otimes \frac{1}{\sqrt{v_2}}\mathbf{1}_{v_2} \right).$$

The corresponding eigenvalue is $t\alpha_1\alpha_2 k_2$.

For (ii), we have

$$\mathbf{N}_2\mathbf{N}_2' \left(\boldsymbol{d}_j \otimes \frac{1}{\sqrt{v_2}}\mathbf{1}_{v_2} \right) = (\alpha_1\boldsymbol{d}_j) \otimes \left(\mathbf{N}_B\mathbf{N}_B'\frac{1}{\sqrt{v_2}}\mathbf{1}_{v_2} \right) =$$
$$= t\alpha_1\alpha_2 k_2 \left(\boldsymbol{d}_j \otimes \frac{1}{\sqrt{v_2}}\mathbf{1}_{v_2} \right).$$

The corresponding eigenvalue is $t\alpha_1\alpha_2 k_2$ for $j = 1, 2, \ldots, v_1 - 1$.

For (iii), we have

$$\mathbf{N}_2\mathbf{N}_2' \left(\frac{1}{\sqrt{v_1}}\mathbf{1}_{v_1} \otimes \boldsymbol{z}_{ip} \right) = \left(\alpha_1\frac{1}{\sqrt{v_1}}\mathbf{1}_{v_1} \right) \otimes (\mathbf{N}_B\mathbf{N}_B'\boldsymbol{z}_{ip}) =$$
$$= \alpha_1(k_2 - q_{21}) \left(\frac{1}{\sqrt{v_1}}\mathbf{1}_{v_1} \otimes \boldsymbol{z}_{ip} \right).$$

The corresponding eigenvalue is $\alpha_1(k_2 - q_{21})$ for $i = 1, 2, \ldots, t$ and $p = 1, 2, \ldots, \beta_2 - 1$.

For (iv), we have

$$\mathbf{N}_2\mathbf{N}_2' \left(\frac{1}{\sqrt{v_1}}\mathbf{1}_{v_1} \otimes \boldsymbol{z}_q^* \right) = \left(\alpha_1\frac{1}{\sqrt{v_1}}\mathbf{1}_{v_1} \right) \otimes \left(\mathbf{N}_B\mathbf{N}_B'\boldsymbol{z}_q^* \right) = \boldsymbol{0}.$$

The corresponding eigenvalue is zero for $q = 1, 2, \ldots, v_2 - t(\beta_2 - 1) - 1$. Similarly, for (vi), the eigenvalue is also zero.

For (v), we have

$$\mathbf{N}_2\mathbf{N}_2' \left(\boldsymbol{x}_{ij} \otimes \boldsymbol{z}_{ip} \right) = (\alpha_1 \boldsymbol{x}_{ij}) \otimes \left(\mathbf{N}_B\mathbf{N}_B' \boldsymbol{z}_{ip} \right) = \alpha_1 (k_2 - q_{21}) \left(\boldsymbol{x}_{ij} \otimes \boldsymbol{z}_{ip} \right).$$

The corresponding eigenvalue is $\alpha_1(k_2 - q_{21})$ for $i = 1, 2, \ldots, t$, $j = 1, 2, \ldots, v_1 - 1$ and $p = 1, 2, \ldots, \beta_2 - 1$. The corresponding eigenvalues of $\mathbf{N}_1\mathbf{N}_1'$ and $\mathbf{N}_2\mathbf{N}_2'$ are summarized in the table below:

Table 3. Eigenvalues and common eigenvectors of $\mathbf{N}_1\mathbf{N}_1'$ and $\mathbf{N}_2\mathbf{N}_2'$

Eigenvalues		Common eigenvectors
$\mathbf{N}_1\mathbf{N}_1'$	$\mathbf{N}_2\mathbf{N}_2'$	
$t\alpha_1 k_1 \alpha_2 k_2$	$t\alpha_1 \alpha_2 k_2$	(i)
$\theta_j \alpha_2 k_2$	$t\alpha_1 \alpha_2 k_2$	(ii)
$\alpha_1 k_1 (k_2 - q_{21})$	$\alpha_1 (k_2 - q_{21})$	(iii)
$\theta_j^{(i)} (k_2 - q_{21})$	$\alpha_1 (k_2 - q_{21})$	(v)
0	0	(iv), (vi)

Here $i = 1, 2, \ldots, t$ and $j = 1, 2, \ldots, v_1 - 1$.

The vectors of (i)–(vi) are also the common eigenvectors of the stratum information matrices $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$. Using (1) and Table 3, we give the stratum efficiency factors for $\tilde{\mathcal{D}}$ in the following table:

Table 4. Stratum efficiency factors for the ISPD $\tilde{\mathcal{D}}$

Type of contrasts	No.	I	II	III
A	$v_1 - 1$	ξ_j	$1 - \xi_j$	0
B	$t(\beta_2 - 1)$	ω	0	$1 - \omega$
	$v_2 - t(\beta_2 - 1) - 1$	0	0	1
$A \times B$	$t(v_1 - 1)(\beta_2 - 1)$	$t\omega \xi_j^{(i)}$	$\omega(1 - t\xi_j^{(i)})$	$1 - \omega$
	$(v_1 - 1)\{v_2 - t(\beta_2 - 1) - 1\}$	0	0	1

Here $\xi_j = \theta_j/(t\alpha_1 k_1)$, $\xi_j^{(i)} = \theta_j^{(i)}/(t\alpha_1 k_1)$ for $i = 1, 2, \ldots, t$ and $j = 1, 2, \ldots, v_1 - 1$, $\omega = (k_2 - q_{21})/(t\alpha_2 k_2)$, A and B denote the basic contrasts among the main effects of whole plot and subplot treatments and $A \times B$ denotes the basic contrasts among the interaction effects. The eigenvectors of (ii), (iii)–(iv) and (v)–(vi) define the basic contrasts A, B and $A \times B$, respectively. We use Table 4 to improve the estimators for the basic contrasts of the treatment effects combining the estimators obtained from

the strata I, II and III. This procedure was proposed by Nelder (1965a, 1965b) and Houtman and Speed (1983). In particular, we see that some basic contrasts of B and $A \times B$ are estimable with full efficiency in stratum III. For a special case where the α-resolvable design is a resolvable design for whole plot treatments and the affine α-resolvable design is a square lattice design for subplot treatments, the stratum efficiency factors are given in Kuriki and Nakajima (2007).

Example 2. For the ISPD $\tilde{\mathcal{D}}$ given in Example 1, $\omega = 1/8$, $\theta_1 = 4$, $\theta_2 = \theta_3 = 2$, $\theta_4 = \theta_5 = 0$, $\theta_1^{(i)} = \theta_2^{(i)} = 2$ and $\theta_3^{(i)} = \theta_4^{(i)} = \theta_5^{(i)} = 0$ for $i = 1, 2$. Thus, using Table 4, the stratum efficiency factors can be calculated as in the following table:

Type of contrasts	No.	I	II	III
A	1	1/4	3/4	0
	2	1/8	7/8	0
	2	0	1	0
B	4	1/8	0	7/8
	4	0	0	1
$A \times B$	8	1/32	3/32	7/8
	12	0	1/8	7/8
	20	0	0	1

4.   Applications

The paper deals with a few aspects of optimal (statistical and practical) properties of experimental designs. One of the aspects relates to the proper use of the available structure of experimental units. This is very important in agricultural field experiments. The units (field plots) are natural and there is no way of changing them to a structure appriopriate to a theoretical (master) optimal design. The split-plot design, considered here, is one of the most utilized designs in agricultural field experiments (see, for example, Gomez and Gomez, 1984, Little and Hills, 1978). This results from statistical as well as practical considerations (ease of mechanically carrying out the experiment). Incomplete non-orthogonal designs (as considered here) have a much wider scope for optimal use in practice.

The second aspect concerns statistical properties of designs. Statistical properties depend on a linear model of observations. In this work we examine statistical properties of designs under a randomization-derived linear model of observations (random block effects describing the structure of units) with treatment (combination) effects being fixed (cf. Caliński and Kageyama, 2000, 2003, Hinkelmann and Kempthorne, 2005, 2007, Hinkelmann, 2012). The experimenter usually makes a

ranking of factors as well as linear functions of level effects (contrasts) with respect to the scientific interest and aim of the experiment. Then the property called general balance plays a crucial role (see, for example, Mejza, 1992, Bailey, 1994) in satisfying the experimenter's wishes as regards the estimability and efficiency of the treatment contrasts. In the generally balanced design we know the efficiencies of all estimable treatment effect functions. Hence this property is examined here in the detail. General balance aids interpretation; the design which is generally balanced with respect to meaningful contrasts may be superior to a technically optimal design. Another problem concerns choosing an incomplete design that fits the structure of experimental units, is optimal for the most interesting treatment effect functions, and is not too expensive (it utilizes as small as possible a number of units of proper size). In the worst case we can use any incomplete design. Then it is difficult to describe the statistical properties of the proposed design.

Additionally, we seek methods enabling the generation of new incomplete split-plot designs by using some known incomplete block designs. The use of resolvable designs and affine α-resolvable designs for the construction of new designs was not a random choice. These designs are often used in series of experiments over many environments in variety testing (see, for example, Caliński et al. 2005, 2009a, 2009b, 2015). The Kronecker product of component incomplete block designs is often used for constructing new designs with split units. The final design possesses optimal properties, but it utilizes many experimental units (high cost of the experiment). To overcome this problem (the size of the experiment) we have proposed the use of the semi-Kronecker product as defined in section 2, instead of the ordinary Kronecker product. The final design is much smaller and also possesses desirable statistical properties. Moreover, using the semi-Kronecker product to generate new designs leads to designs that use a much smaller number of units (see section 2).

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