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Determination of robust optimum plot size and shape – a model-based approach

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SUMMARY

Determination of optimum plot size has been regarded as an important and useful area of study for agriculturists and statisticians since the first remarkable contribution on this problem came to light in a paper by Smith (1938). As we explore the scientific literature relating to this problem, we may note a number of contributions, including those of Modjeska and Rawlings (1983), Webster and Burgess (1984), Sethi (1985), Zhang et al. (1990, 1994), Bhatti et al.(1991), Fagroud and Meirvenne (2002), etc. In Pal et al. (2007), a general method was presented by means of which the optimum plot size can be determined through a systematic analytical procedure. The importance of the procedure stems from the fact that even with Fisherian blocking, the correlation among the residuals is not eliminated (as such the residuals remain correlated). The method is based on an application of an empirical variogram constructed on real-life data sets (obtained from uniformity trials) wherein the data are serially correlated. This paper presents a deep and extensive investigation (involving theoretical exploration of the effect of different plot sizes and shapes in discovering the point – actually the minimum radius of curvature of the variogram at that point – beyond which the theoretical variogram assumes stationary values with further increase in lags) in the case of the most commonly employed model (incorporating a correlation structure) assumed to represent real-life data situations (uniformity trial or designed experiments, RBD/LSD).

Key words: non-random data, model-based theoretical variogram, radius of curvature, robust optimum plot sizes and shapes

1. Introduction

The classical Fisherian technique of analysis of variance assumes the independence of observations when applied to real-life data from designed experiments, although it is not an unusual phenomenon that the data obtained from field experiments are often found to be spatially correlated. As mentioned in the abstract, in Pal et al. (2007) the aspect of spatial correlation is taken into account by considering the well-known variogram technique, which is used to discover the spatial heterogeneity structure in a set of data. The definition of variogram is presented in the next paragraph.

Let $\{Y(\mathbf{s}): \mathbf{s} \in D_s \subset \mathbf{R}^2\}$ be a real-valued spatial process defined on a domain D_s of the 2-dimensional Euclidean space $\mathbf{R}^2$, and it is supposed that the variance of the difference of the values of the variable at $\mathbf{s}_1$ and $\mathbf{s}_2$ (displaced h-apart, i.e. $\mathbf{s}_1 = \mathbf{s}$, and $\mathbf{s}_2 = \mathbf{s} + \mathbf{h}$) varies in a way that depends only on $\mathbf{s}_1 - \mathbf{s}_2 = \mathbf{h}, \mathbf{h} > \mathbf{0}$. More specifically, it is assumed that $Var\ [Y(\mathbf{s} + \mathbf{h}) - Y(\mathbf{s})] = 2\gamma_Y(\mathbf{h})\ (= 2\gamma(\mathbf{h}))$, for all $(\mathbf{s}, \mathbf{s} + \mathbf{h}) \in D_s$, the variogram must satisfy the conditional-non-positive-definiteness condition. $\gamma(\mathbf{h})$ is called the semi-variogram. The quantity $2\gamma(\mathbf{h})$, being a function of the difference between the spatial locations $\mathbf{s}$ and $\mathbf{s} + \mathbf{h}$, is called the stationary variogram. When $2\gamma(\mathbf{h})$ becomes independent of $\mathbf{s}$, and is a function of $\|\mathbf{h}\|$ only, for $\mathbf{h} = (h_1, h_2) \in \mathbf{R}^2$, $\|\mathbf{h}\| = (h_1^2 + h_2^2)^{1/2}$, the variogram is said to be isotropic; otherwise, it is said to be anisotropic. For further reading, the paper by Matheron (1963) and the books by Cressie (1993) and by Cressie and Wikle (2011) may be consulted.

The uniformity trial data Y ($\mathbf{s}$) on a spatial location $\mathbf{s}$ is modelled as:

$$Y(\mathbf{s}) = \mu + e(\mathbf{s}),\ V(Y(\mathbf{s})) = (V(e(\mathbf{s})) = \boldsymbol{\sigma}^2;$$

$$Cov(Y(\mathbf{s}), Y(\mathbf{s} + \mathbf{h})) = Cov(e(\mathbf{s}), e(\mathbf{s} + \mathbf{h}) = \rho^{\|\mathbf{h}\|}\ \boldsymbol{\sigma}^2.$$

In the case of data from designed experiments, the variogram of the residuals is to be modelled in the above manner.

2. Method

The expressions of the theoretical variograms, 2γ(**h**) (under the above model) have been obtained for plot sizes, l x k (l = 2, 3, …; k = 2, 3, …), i.e. 2 x 2, 2 x 3 (3 x 2), 2 x 4 (4 x 2), 2 x 5 (5 x 2), 2 x 6 (6 x 2), 2 x 7 (7 x 2), 2 x 8 (8 x 2), 3 x 3, 3 x 4 (4 x 3), 3 x 5 (5 x 3), 4 x 4, etc., respectively (the area of plots being less than or equal to 16 squared units), and some such expressions are presented as follows.

Expressions of variograms in the case of the plot sizes considered above:

1 x 1:

$$2\gamma(h) = 2\sigma^2(1-\rho^h)$$

2 x 2:

$$2\gamma(h) = 8\sigma^2 + 2\sigma^2[(8\rho + 4\rho^{\sqrt{2}}) - (2\rho^{(2h-1)} + 4\rho^{(2h)} + 2\rho^{(2h+1)} + \\ + 2\rho^{\sqrt{(2h-1)^2+1}} + 4\rho^{\sqrt{(2h)^2+1}} + 2\rho^{\sqrt{(2h+1)^2+1}}]$$

2 x 3:

$$2\gamma(h) = 12\sigma^2 + 2\sigma^2[(14\rho + 4\rho^2 + 8\rho^{\sqrt{2}} + 4\rho^{\sqrt{5}}) - (2\rho^{(3h-2)} + 4\rho^{(3h-1)} + \\ + 6\rho^{(3h)} + 4\rho^{(3h+1)} + 2\rho^{(3h+2)} + 2\rho^{\sqrt{(3h-2)^2+1}} + 4\rho^{\sqrt{(3h-1)^2+1}} + 6\rho^{\sqrt{(3h)^2+1}} + \\ + 4\rho^{\sqrt{(3h+1)^2+1}} + 2\rho^{\sqrt{(3h+2)^2+1}})]$$

2 x 4:

$$2\gamma(h) = 16\sigma^2 + 2\sigma^2[(20\rho + 8\rho^2 + 4\rho^3 + 12\rho^{\sqrt{2}} + 8\rho^{\sqrt{5}} + 4\rho^{\sqrt{10}}) + \\ - (2\rho^{(4h-3)} + 4\rho^{(4h-2)} + 6\rho^{(4h-1)} + 8\rho^{(4h)} + 6\rho^{(4h+1)} + 4\rho^{(4h+2)} + \\ + 2\rho^{(4h+3)} + 2\rho^{\sqrt{(4h-3)^2+1}} + 4\rho^{\sqrt{(4h-2)^2+1}} + 6\rho^{\sqrt{(4h-1)^2+1}} + 8\rho^{\sqrt{(4h)^2+1}} + \\ + 6\rho^{\sqrt{(4h+1)^2+1}} + 4\rho^{\sqrt{(4h+2)^2+1}} + 2\rho^{\sqrt{(4h+3)^2+1}})]$$

2 x 5:

$$2\gamma(h) = 20\sigma^2 + 2\sigma^2[(26\rho + 12\rho^2 + 8\rho^3 + 4\rho^4 + 16\rho^{\sqrt{2}} + 12\rho^{\sqrt{5}} + \\ + 8\rho^{\sqrt{10}} + 4\rho^{\sqrt{17}}) - (2\rho^{(5h-4)} + 4\rho^{(5h-3)} + 6\rho^{(5h-2)} + 8\rho^{(5h-1)} + \\ + 10\rho^{(5h)} + 8\rho^{(5h+1)} + 6\rho^{(5h+2)} + 4\rho^{(5h+3)} + 2\rho^{(5h+4)} + 2\rho^{\sqrt{(5h-4)^2+1}} + \\ + 4\rho^{\sqrt{(5h-3)^2+1}} + 6\rho^{\sqrt{(5h-2)^2+1}} + 8\rho^{\sqrt{(5h-1)^2+1}} + 10\rho^{\sqrt{(5h)^2+1}} + \\ + 8\rho^{\sqrt{(5h+1)^2+1}} + 6\rho^{\sqrt{(5h+2)^2+1}} + 4\rho^{\sqrt{(5h+3)^2+1}} + 2\rho^{\sqrt{(5h+4)^2+1}})]$$

3 x 3:

$$2\gamma(\mathrm{h}) = 18\sigma^2 + 2\sigma^2[(246\rho + 12\rho^2 + 16\rho^{\sqrt{2}} + 16\rho^{\sqrt{5}} + \\ + 4\rho^{\sqrt{8}}) - (3\rho^{(3\mathrm{h}-2)} + 6\rho^{(3\mathrm{h}-1)} + 9\rho^{(3\mathrm{h})} + 6\rho^{(3\mathrm{h}+1)} + 3\rho^{(3\mathrm{h}+2)} + \\ + 4\rho^{\sqrt{(3\mathrm{h}-2)^2+1}} + 2\rho^{\sqrt{(3\mathrm{h}-2)^2+4}} + 8\rho^{\sqrt{(3\mathrm{h}-1)^2+1}} + 4\rho^{\sqrt{(3\mathrm{h}-1)^2+4}} + \\ + 12\rho^{\sqrt{(3\mathrm{h})^2+1}} + 6\rho^{\sqrt{(3\mathrm{h})^2+4}} + 8\rho^{\sqrt{(3\mathrm{h}+1)^2+1}} + 4\rho^{\sqrt{(3\mathrm{h}+1)^2+4}} + \\ + 4\rho^{\sqrt{(3\mathrm{h}+2)^2+1}} + 2\rho^{\sqrt{(3\mathrm{h}+2)^2+4}})]$$

3 x 5:

$$2\gamma(h) = 30\sigma^2 + 2\sigma^2[(44\rho + 28\rho^2 + 12\rho^3 + 6\rho^4 + 32\rho^{\sqrt{2}} + 40\rho^{\sqrt{5}} + \\ + 12\rho^{\sqrt{8}} + 16\rho^{\sqrt{10}} + 8\rho^{\sqrt{17}}) - (3\rho^{(5h-4)} + 6\rho^{(5h-3)} + 9\rho^{(5h-2)} + \\ + 12\rho^{(5h-1)} + 15\rho^{(5h)} + 12\rho^{(5h+1)} + 9\rho^{(5h+2)} + 6\rho^{(5h+3)} + 3\rho^{(5h+4)} + \\ + 4\rho^{\sqrt{(5h-4)^2+1}} + 2\rho^{\sqrt{(5h-4)^2+4}} + 8\rho^{\sqrt{(5h-3)^2+1}} + 4\rho^{\sqrt{(5h-3)^2+4}} + \\ + 12\rho^{\sqrt{(5h-2)^2+1}} + 6\rho^{\sqrt{(5h-2)^2+4}} + 16\rho^{\sqrt{(5h-1)^2+1}} + 8\rho^{\sqrt{(5h-1)^2+4}} + \\ + 20\rho^{\sqrt{(5h)^2+1}} + 10\rho^{\sqrt{(5h)^2+4}} + 16\rho^{\sqrt{(5h+1)^2+1}} + 8\rho^{\sqrt{(5h+1)^2+4}} + \\ + 12\rho^{\sqrt{(5h+2)^2+1}} + 6\rho^{\sqrt{(5h+2)^2+4}} + 8\rho^{\sqrt{(5h+3)^2+1}} + 4\rho^{\sqrt{(5h+3)^2+4}} + \\ + 4\rho^{\sqrt{(5h+4)^2+1}} + 2\rho^{\sqrt{(5h+4)^2+4}})]$$

4 x 4:

$$2\gamma(h)=32\sigma^2+2\sigma^2[(48\rho+32\rho^2+16\rho^3+36\rho^{\sqrt{2}}+48\rho^{\sqrt{5}}+16\rho^{\sqrt{8}}+$$
$$+24\rho^{\sqrt{10}}+16\rho^{\sqrt{13}})-(4\rho^{(4h-3)}+8\rho^{(4h-2)}+12\rho^{(4h-1)}+16\rho^{(4h)}+$$
$$+12\rho^{(4h+1)}+8\rho^{(4h+2)}+4\rho^{(4h+3)}+6\rho^{\sqrt{(4h-3)^2+1}}+4\rho^{\sqrt{(4h-3)^2+4}}+$$
$$+2\rho^{\sqrt{(4h-3)^2+9}}+12\rho^{\sqrt{(4h-2)^2+1}}+8\rho^{\sqrt{(4h-2)^2+4}}+4\rho^{\sqrt{(4h-2)^2+9}}+$$
$$+18\rho^{\sqrt{(4h-1)^2+1}}+12\rho^{\sqrt{(4h-1)^2+4}}+6\rho^{\sqrt{(4h-1)^2+9}}+24\rho^{\sqrt{(4h)^2+1}}+16\rho^{\sqrt{(4h)^2+4}}+$$
$$+8\rho^{\sqrt{(4h)^2+9}}+18\rho^{\sqrt{(4h+1)^2+1}}+12\rho^{\sqrt{(4h+1)^2+4}}+6\rho^{\sqrt{(4h+1)^2+9}}+$$
$$+12\rho^{\sqrt{(4h+2)^2+1}}+8\rho^{\sqrt{(4h+2)^2+4}}+4\rho^{\sqrt{(4h+2)^2+9}}+6\rho^{\sqrt{(4h+3)^2+1}}+$$
$$+4\rho^{\sqrt{(4h+3)^2+4}}+2\rho^{\sqrt{(4h+3)^2+9}})]$$

The variogram-graphs corresponding to seven plot sizes under the l = 2 series (i.e. 2 x 2, 2 x 3, 2 x 4, 2 x 5, 2 x 6, 2 x 7 and 2 x 8) are presented in one graph, and the four variogram plots for the remaining four plot sizes (3 x 3, 3 x 4, 3 x 5 and 4 x 4) are included in another graph. Importantly, it is to be noted that the above two graph plots are constructed for each of the values of ρ (= 0.1, 0.2, 0.3, 0.4 and 0.5); values of $\rho > 0.5$ are not very common in real-life field data.

The selection of optimum (best or better) plot sizes is governed by the following criteria:

1. With respect to each plot size the point h_{opt} is determined as the point for which the value of the radius of curvature, r_c, is minimum, the formula for radius of curvature being given below:

$$r_c = (1+\gamma_1(h)^2)^{3/2} / (\gamma_2(h)),\ \text{where}\ \gamma_1(h) = d\gamma/dh\ \text{and}\ \gamma_2(h) = d^2\gamma/dh^2.$$

2. For each value of ρ, say $\rho = 0.1$, the particular plot size is chosen for which the values of the radius of curvature, r_c, are minimum (near to minimum) subject to the restriction that $|l - k| \leq 4$.

3. Plots with unit dimension in any direction (row or column) are not taken into account; also long narrow plots are not recommended (owing to the fact that such plots entail more heterogeneity).

3. Results and discussion

The values of the radius of curvature (r_c) corresponding to 12 different plot sizes are given in Table 1. Also given are the values of h_{opt} with respect to each plot size for five different values of ρ (0.1 to 0.5). The subsequent observations are immediate from the values contained in the Table.

Note: The following observations are valid for the values ρ = 0.1 to ρ = 0.5.

Square plots of sizes 2x2, 3x3, 4x4 have radii of curvature much higher than the desired minimum values (meaning that the curve shapes corresponding to those plot sizes are relatively more flat), thus square plots cannot be taken as optimum plot sizes. Though the radii of curvature corresponding to plots of sizes 2x7 and 2x8 are less than the minimum values of r_c taken into consideration, such plot sizes are still not recommended as optimum plot sizes, as these plots are of long and narrow shape. Plot sizes 5x5 and 6x6 are not taken into account as such plots are of too large size.

Table 1. The values of the radius of curvature (r_c) corresponding to different plot sizes (**σ^2=10**)

	Radius of curvature(r_c)									
ρ →	0.1		0.2		0.3		0.4		0.5	
Plot Size	h_{opt}	r_c	h_{opt}	r_c	h_{opt}	r_c	h_{opt}	r_c	h_{opt}	r_c
2x2	1.85	0.58	1.80	0.82	2.90	1.10	3.80	1.43	4.75	1.90
2x3	1.65	0.40	2.00	0.54	2.50	0.72	3.00	0.95	3.80	1.30
2x4	1.50	0.28	1.80	0.41	2.25	0.54	2.70	0.71	3.25	0.94
2x5	1.45	0.23	1.70	0.32	1.95	0.43	2.45	0.57	2.90	0.75
2x6	1.40	0.20	1.60	0.26	1.85	0.35	2.20	0.47	2.65	0.62
2x7	1.45	0.15	1.70	0.22	1.95	0.30	2.30	0.40	2.80	0.53
2x8	1.30	0.13	1.50	0.20	1.70	0.27	1.95	0.35	2.30	0.47
3x3	1.75	0.39	2.2	0.55	2.70	0.73	3.35	0.95	4.20	1.25
3x4	1.60	0.29	1.95	0.41	2.35	0.54	2.85	0.71	3.55	0.94
3x5	1.50	0.23	1.80	0.33	2.10	0.44	2.55	0.57	3.15	0.75
3x6	1.40	0.19	1.65	0.27	1.95	0.36	2.35	0.48	2.85	0.63
4x4	1.65	0.29	2.00	0.41	2.45	0.55	3.00	0.71	3.75	0.94

- For ρ = 0.1, the following observations are immediate:

Any one of the plot sizes viz. 2x5, 2x6, 3x5 and 3x6 can be taken as the robust optimum plot size, since their radii of curvature lie in the range 0.19 to 0.23 (range of variation = 0.04); thus alternative optimum plot sizes are: 10/12/15/18 (shapes are also given).

- For ρ = 0.2, the following observations are immediate:

Any one of the plot sizes viz. 2x5, 2x6, 3x5 and 3x6 can be taken as the robust optimum plot size, since their radii of curvature lie in the range 0.26 to 0.33 (range of variation = 0.07); thus alternative optimum plot sizes are: 10/12/15/18 (shapes are also given).

- For ρ = 0.3, the following observations are immediate:

Any one of the plot sizes viz. 2x5, 2x6, 3x5 and 3x6 can be taken as the robust optimum plot size, since their radii of curvature lie in the range 0.35 to 0.44 (range of variation = 0.09); thus alternative optimum plot sizes are: 10/12/15/18 (shapes are also given).

- For ρ = 0.4, the following observations are immediate:

Any one of the plot sizes viz. 2x5, 2x6, 3x5 and 3x6 can be taken as the robust optimum plot size, since their radii of curvature lie in the range 0.47 to 0.57 (range of variation = 0.10); thus alternative optimum plot sizes are: 10/12/15/18 (shapes are also given).

- For ρ = 0.5, the following observations are immediate:

Any one of the plot sizes viz. 2x5, 2x6, 3x5 and 3x6 can be taken as the robust optimum plot size, since their radii of curvature lie in the range 0.62 to 0.75 (range of variation = 0.13); thus alternative optimum plot sizes are: 10/12/15/18 (shapes are also given).

In the graph the optimum plots are shown (Figure 1).

The graph plots of ten variograms are presented on the following pages (Figure 2). For ρ = 0.1 to ρ = 0.5, different plot sizes are indicated on the body of the graphs (for each value of ρ, two graphs are presented).

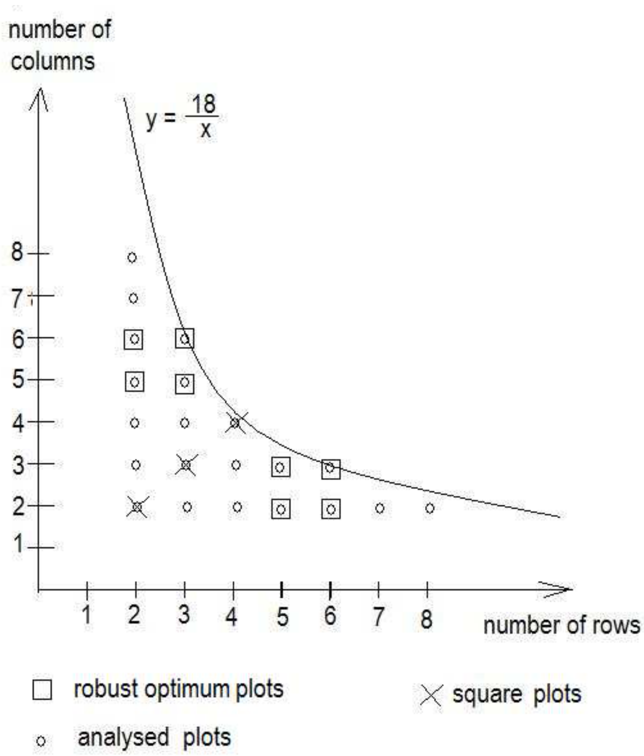
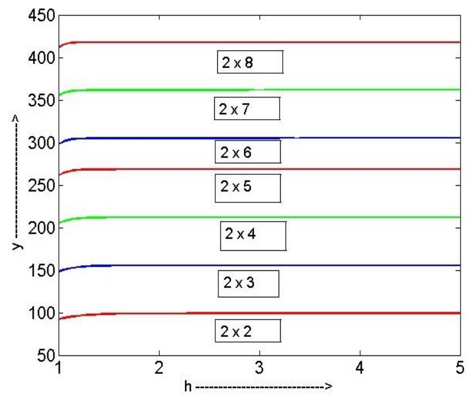
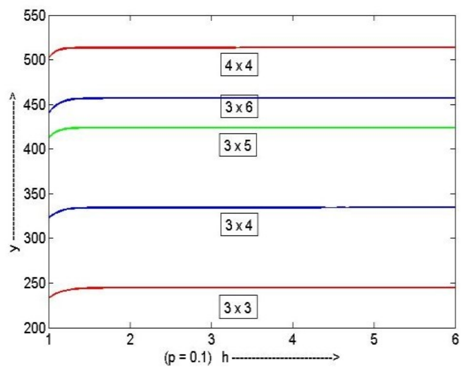
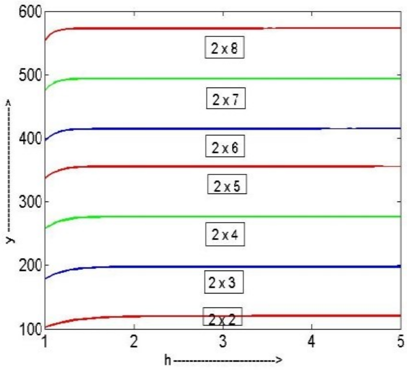


Figure 1. The optimum plots

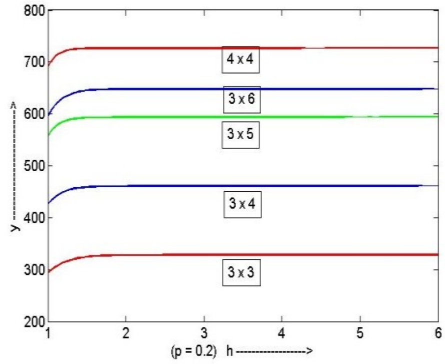
GRAPH

Graph – 1 ρ = 0.1	Graph – 2 ρ = 0.1
	

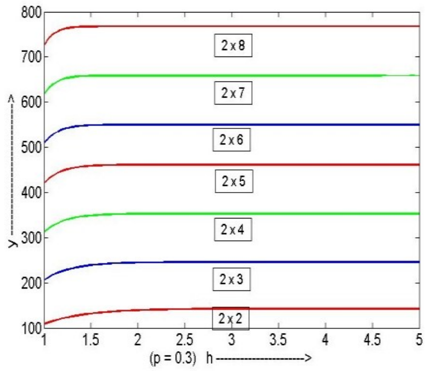
Gragh – 3 ρ = 0.2



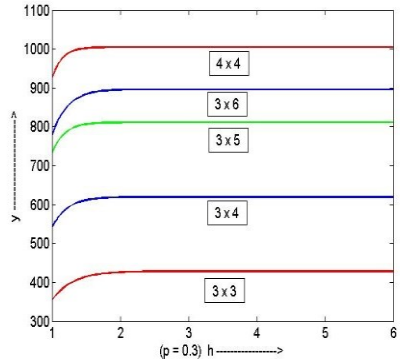
Graph – 4 ρ = 0.2



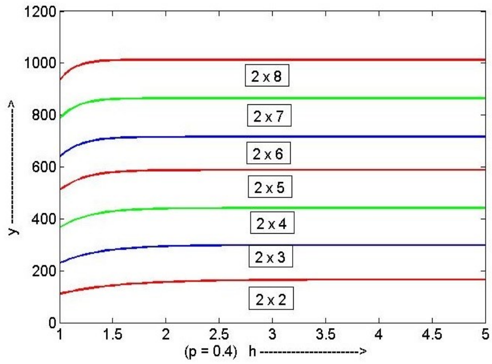
Graph – 5 ρ = 0.3



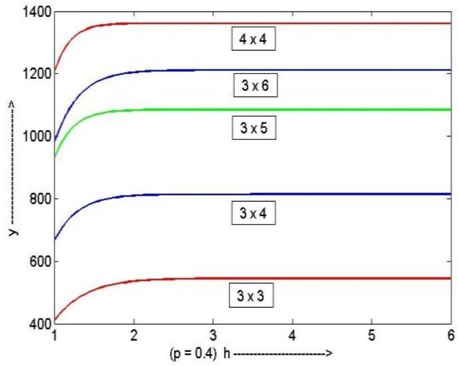
Graph – 6 ρ = 0.3



Graph – 7 ρ = 0.4



Graph – 8 ρ = 0.4



Graph – 9 ρ = 0.5 Graph – 10 ρ = 0.5

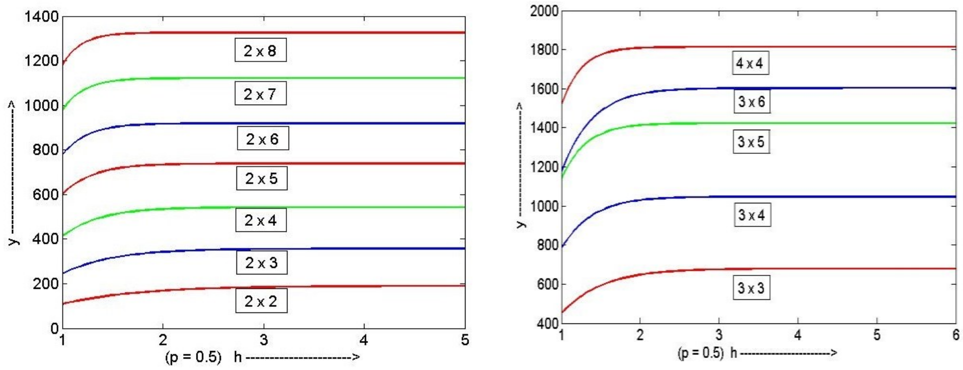


Figure 2. The graph plots of ten variograms

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