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X^{-1}-balance of some partially balanced experimental designs with particular emphasis on block and row-column designs

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SUMMARY

This paper considers block designs and row-column designs where the information matrix **C** has two different nonzero eigenvalues, one of multiplicity 1 and the other of multiplicity h–1, where h is the rank of the matrix **C**. It was found that for each such design there exists a diagonal positive definite matrix **X** such that the design is $\mathbf{X}^{-1}$-balanced.

Key words: block design, row-column design, efficiency balance, variance balance, $\mathbf{X}^{-1}$-balance

1. Introduction and notation

The subject considered here is the problem of balance in experimental designs, especially in block designs or row-column designs. In the literature there exist two main approaches to the notion of balance: "variance balance" (cf. Pearce, 1975) and "efficiency balance" (cf. Nigam, 1976). In the case of equal replications these approaches are equivalent. Otherwise, a variance-balanced design may not be efficiency-balanced and vice versa. These two approaches are special cases of the definition of $\mathbf{X}^{-1}$-balance in block designs; cf. Caliński (1977). This definition can be directly applied to row-column designs also.

Some partially balanced designs with one treatment distinguished will be discussed. It is assumed that the information matrix **C** has two different nonzero eigenvalues, one of multiplicity 1 and the other of multiplicity h–1, where h is the rank of the matrix **C**. The aim is to show that for each such design, there exists a diagonal positive definite matrix **X** such that the design is $\mathbf{X}^{-1}$-balanced. Moreover, it is shown how to introduce a set of base contrasts which can be estimated with the same variance.

1.1. Block designs

Consider a block design with n experimental units grouped in b blocks, in which t treatments are to be compared ($n > t$).

The linear model of observations derived from experiments established in such a design has the form

$$\mathbf{y} = (\mathbf{1}_n, \mathbf{D}', \boldsymbol{\Delta}')\,(\alpha, \boldsymbol{\beta}', \boldsymbol{\gamma}')' + \mathbf{e}\,, \tag{1}$$

where α is a general parameter, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are $b\times 1$ and $t\times 1$ vectors of block and treatment parameters respectively, $\mathbf{1}_n$ is a vector with all components equal to 1, $\mathbf{D}'$ and $\boldsymbol{\Delta}'$ are $n\times b$ and $n\times t$ design matrices for blocks and treatments respectively, and $\mathbf{e}$ is an $n\times 1$ vector of random errors, for which we assume that $\mathrm{E}(\mathbf{e}) = \mathbf{0}$ and $\mathrm{D}^2(\mathbf{e}) = \sigma^2\,\mathbf{I}_n$, where σ^2 denotes the variance of errors.

Making use of the method of least squares estimation, here providing the best linear unbiased estimators for treatment parametric functions, we obtain the following normal equation for the vector $\boldsymbol{\gamma}$:

$$\boldsymbol{\Delta}\mathbf{Q}_{D'}\boldsymbol{\Delta}'\boldsymbol{\gamma} = \boldsymbol{\Delta}\mathbf{Q}_{D'}\mathbf{y}, \tag{2}$$

where $\mathbf{Q}_{D'} = \mathbf{I} - \mathbf{P}_{D'}$ and $\mathbf{P}_{D'}$ denotes the orthogonal projector on the column space of the matrix $\mathbf{D}'$. Commonly, the set of normal equations (2) is expressed as, $\mathbf{C}\boldsymbol{\gamma} = \boldsymbol{\Delta}\mathbf{Q}_{D'}\mathbf{y}$, where $\mathbf{C}$ denotes the information matrix for $\boldsymbol{\gamma}$, $\mathbf{C} = \mathbf{r}^{\delta} - \mathbf{N}\mathbf{k}^{-\delta}\mathbf{N}'$, with $\mathbf{N} = \boldsymbol{\Delta}\mathbf{D}'$ being an incidence matrix, $\mathbf{r}^{\delta} = \boldsymbol{\Delta}\boldsymbol{\Delta}'$, and $\mathbf{k}^{-\delta} = (\mathbf{D}\mathbf{D}')^{-1}$; cf. Caliński (1977) or Pearce et al. (1974).

1.2. Row-column designs

If the outcome of the experiment is influenced by, in addition to the test factor (treatment parameters), one confounding factor, we use a block design. If we must eliminate the effects of two confounding factors, a row-column design should be used. In such an design n experimental units are grouped in b_1 rows (blocks) and b_2 columns. Let $\mathbf{N}_1$, $\mathbf{N}_2$ and $\mathbf{N}_3$ denote the incidence matrices for treatments vs. rows, treatments vs. columns and rows vs. columns.

We shall assume that the observations obtained in an experiment carried out in a row-column design are expressed by the model

$$\mathbf{y} = (\mathbf{1}_n, \mathbf{D}_1', \mathbf{D}_2', \boldsymbol{\Delta}')\,(\alpha, \boldsymbol{\beta}_1', \boldsymbol{\beta}_2', \boldsymbol{\gamma}')' + \mathbf{e}\,, \tag{3}$$

where α is the general parameter, $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2$, and $\boldsymbol{\gamma}$ are the $b_1 \times 1$, $b_2 \times 1$ and $t \times 1$ vectors of row, column and treatment parameters respectively, $\mathbf{1}_n$ is a vector with all components equal to 1, $\mathbf{D}_1'$, $\mathbf{D}_2'$ and $\boldsymbol{\Delta}'$ are the $n \times b_1$, $n \times b_2$ and $n \times t$, design matrices respectively for rows, columns and treatments, and $\mathbf{e}$ is an $n \times 1$ vector of random errors, for which we assume that $E(\mathbf{e}) = \mathbf{0}$ and $D^2(\mathbf{e}) = \sigma^2\, \mathbf{I}_n$. The relevant set of normal equations for the vector $\boldsymbol{\gamma}$ takes the form (2), wherein the matrix $\mathbf{D}'$ is replaced by the partitioned matrix $(\mathbf{D}_1' : \mathbf{D}_2')$ (compare Pearce, 1975).

1.3. X^{-1}-balance of experimental designs.

Let $\mathbf{X}$ be a real diagonal positive definite $t \times t$ matrix. Let λ_i and $\mathbf{w}_i$ denote, respectively, an eigenvalue and the corresponding eigenvector of the matrix $\mathbf{C}$ with respect to $\mathbf{X}$, i.e., let $\mathbf{C}\mathbf{w}_i = \lambda_i \mathbf{X}\mathbf{w}_i$.

There exist h ($h \le t - 1$) nonzero eigenvalues of the matrix $\mathbf{C}$ with respect to $\boldsymbol{X}$ (see Rao and Mitra 1971, Theorems 6.3.1 and 6.3.2). The vectors $\mathbf{w}_i$, $i = 1, 2, \dots, t$, can be chosen such that

$$\mathbf{w}_i'\mathbf{X}\mathbf{w}_i = 1 \text{ and } \mathbf{w}_i'\mathbf{X}\mathbf{w}_j = 0, \text{ where } i \neq j. \tag{4}$$

Because $\mathbf{C}\mathbf{1}_t = \mathbf{0}$, we can assume that the eigenvector $\mathbf{w}_t$ takes the form

$$\mathrm{w}_t = \frac{1}{\sqrt{\mathbf{1}_t'\mathbf{X}\mathbf{1}_t}}\mathbf{1}_t \tag{5}$$

and, consequently, $\mathbf{1}_t'\mathbf{X}\mathbf{w}_i = 0$ for $i = 1, 2, \dots, t\text{-}1$.

According to Lemma 2 in the paper by Caliński (1977), the vectors $\mathbf{X}\mathbf{w}_i$, $i = 1, 2, \dots, h$, span the subspace of all estimable treatment contrasts.

Since each treatment parameter function $\mathbf{s}_i'\boldsymbol{\gamma}$ estimable in the model (1) or (3) is a contrast (i.e. $\mathbf{s}_i'\mathbf{l}_t = 0$), it can be expressed as a linear combination of the basic contrasts

$$\mathbf{w}_i'\mathbf{X}\boldsymbol{\gamma}, \text{ for } i = 1, 2, \ldots, h. \tag{6}$$

The best linear unbiased estimator of the contrast (6) has the form (Caliński, 1977, Theorem 2)

$$\widehat{\mathbf{w}_i'\mathbf{X}\boldsymbol{\gamma}} = \left(\mathbf{w}_i'\boldsymbol{\Delta}\mathbf{Q}_\mathbf{D}\mathbf{y}\right)/\lambda_i,\ i = 1, 2, \ldots, h,$$

and its variance is

$$var(\widehat{\mathbf{w}_i'\mathbf{X}\boldsymbol{\gamma}}) = \sigma^2/\lambda_i,\ i = 1, 2, \ldots, h. \tag{7}$$

With the above considerations, we can formulate a definition of $\mathbf{X}^{-1}$-balance for row-column designs, analogous to that formulated by Caliński (1977, Definition 1) for block designs.

Definition 1. A row-column design is said to be $\mathbf{X}^{-1}$-balanced if all $\mathbf{X}^{-1}$-normalized estimable linear functions of treatment parameters $\mathbf{s}'\boldsymbol{\gamma}$ are estimated with the same variance.

In fact, two commonly used approaches to balance, namely "variance balance" and "efficiency balance", are special cases of Definition 1. The former definition, given by Pearce (1975), corresponds to the case when $\mathbf{X} = \mathbf{I}_t$, while the latter, formulated by Nigam (1976), refers to the case $\mathbf{X} = \mathbf{r}^\delta$.

Theorem 1. An experimental design is $\mathbf{X}^{-1}$-balanced if and only if the nonzero eigenvalues of the matrix $\mathbf{C}$ with respect to $\mathbf{X}$ are all equal.

(cf. Caliński 1977, Theorem 3)

2. Results

Let the experimental design, a block design or row-column design, be partially balanced with respect to the unit matrix, and its information matrix $\mathbf{C}$ have two different nonzero eigenvalues: λ_1 of multiplicity one and λ_2 of multiplicity $h-1$, where h is the rank of the matrix $\mathbf{C}$.

Suppose that the eigenvector $\boldsymbol{\psi}_1$ corresponds to the eigenvalue λ_1, the eigenvectors $\boldsymbol{\psi}_i$, $i = 2, 3, \ldots, h,$ correspond to the eigenvalue λ_2, and the eigenvectors $\boldsymbol{\psi}_{h+1}, \ldots, \boldsymbol{\psi}_t$ correspond to the null eigenvalue, where t is a dimension of the matrix $\mathbf{C}$.

Since $\mathbf{C1}_t = \mathbf{0}$, we can assume that the eigenvector $\boldsymbol{\psi}_t$ takes the form

$$\boldsymbol{\psi}_t = \frac{1}{\sqrt{t}}\mathbf{1}_t \,. \tag{8}$$

Let $\boldsymbol{\Psi}$ denote a matrix whose columns are eigenvectors of the matrix $\mathbf{C}$, that is,

$$\boldsymbol{\Psi} = [\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \ldots, \boldsymbol{\psi}_t], \tag{9}$$

and let the i-th element of the j-th eigenvector of the matrix $\mathbf{C}$ be denoted as $\psi_{ij.}$

The vectors $\boldsymbol{\psi}_i$ are chosen to satisfy the following conditions:

(i) $\boldsymbol{\psi}_i'\boldsymbol{\psi}_i = 1$, $\boldsymbol{\psi}_i'\boldsymbol{\psi}_j = 0$ for $i, j = 1, 2, \ldots, t$ and $i \neq j$, (10)

(ii) for some natural number k, $1 \le k \le t$, it holds that $\psi_{k1} > 0$ and $\psi_{kj} = 0$ for $j = 2, 3, \ldots, h$. (11)

With respect to (8), the condition (10) means that each eigenvector $\boldsymbol{\psi}_i$, $i = 1, 2, \ldots, t-1$, is a contrast. By the condition (11) one can distinguish the k-th treatment in a set of treatments to be compared. If, for example, the first contrast is a comparison of, say, the l-th object with the others, one can choose $k = l$.

Let $\mathbf{X}$ denote the diagonal matrix with $x_{ii} = 1$, for $i = 1, 2, \ldots, k\text{-}1, k\text{+}1, \ldots, t,$ and $x_{kk} = a$, where

$$a = \frac{\lambda_1\psi_{k1}^2}{\lambda_2 - \lambda_1(1-\psi_{k1}^2)}\,; \tag{12}$$

note that the matrix $\mathbf{X}$ can be written as

$$\mathbf{X} = \mathbf{I}_t + (a - 1)\mathbf{e}_k\mathbf{e}_k' \tag{13}$$

where $\mathbf{I}_t$ is the unit matrix of size t, and $\mathbf{e}_k$ is its k-th column.

Matrix $\mathbf{X}$ is positive definite, therefore $a > 0$ and consequently

$$\psi_{k1}^2 > 1 - \frac{\lambda_2}{\lambda_1} \tag{14}$$

Furthermore, let $\mathbf{W}$ denote a matrix of the form

$$\mathbf{W} = [\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{h,} \mathbf{w}_{h+1}, \ldots, \mathbf{w}_t], \tag{15}$$

where

$$\mathbf{w}_1 = \sqrt{\frac{\lambda_1}{\lambda_2}}\mathbf{X}^{-1}\boldsymbol{\psi}_1, \tag{16}$$

$$\mathbf{w}_i = \boldsymbol{\psi}_i \;\; i = 2, 3, \ldots, h, \tag{17}$$

$$\mathbf{w}_t = \frac{1}{\sqrt{\mathbf{1}_t'\mathbf{X}\mathbf{1}_t}}\mathbf{1}_t, \tag{18}$$

The other columns of the matrix $\mathbf{W}$, that is, $\mathbf{w}_{h+1}, \ldots, \mathbf{w}_{t-1}$, are chosen to satisfy the conditions of $\mathbf{X}^{-1}$-ortonormality, i.e. (4).

Theorem 2: Let an experimental design, a block design or row-column design, be partially balanced (with respect to $\mathbf{I}_t$) and such that its information matrix $\mathbf{C}$ has two different nonzero eigenvalues: λ_1 of multiplicity one and λ_2 of multiplicity h–1, where h is the rank of the matrix $\mathbf{C}$. Then the experimental design is $\mathbf{X}^{-1}$-balanced, where the matrix $\mathbf{X}$ is defined in (13) and (14). The only non-zero eigenvalue of the matrix $\mathbf{C}$ with respect to $\mathbf{X}$ is λ_2 of multiplicity h, and a set of corresponding $\mathbf{X}$-orthonormal eigenvectors of this matrix is described by the equations (15)–(18).

Proof.

Note that due to (11), it holds that $\mathbf{e}_k'\boldsymbol{\psi}_i = 0$, for $i = 2, 3, \ldots, h$, and consequently,

$$\mathbf{X}\boldsymbol{\psi}_i = \boldsymbol{\psi}_i \text{ for } i = 2, 3, \ldots, h. \tag{19}$$

Hence, for $i = 2, 3, \ldots, h,$ we have $\mathbf{C}\mathbf{w}_i = \mathbf{C}\boldsymbol{\psi}_i = \lambda_2\boldsymbol{\psi}_i = \lambda_2\mathbf{X}\boldsymbol{\psi}_i = \lambda_2\mathbf{X}\mathbf{w}_i$. It remains to prove the equality $\mathbf{C}\mathbf{w}_1 = \lambda_2\mathbf{X}\mathbf{w}_1$.

In view of (12) and (13), we have

$$\mathbf{X}^{-1} = \mathbf{I}_t + \frac{\lambda_2 - \lambda_1}{\lambda_1\psi_{k1}^2}\mathbf{e}_k\mathbf{e}_k' \tag{20}$$

Furthermore, by (19) and (16), we obtain

$$\mathbf{C}\mathbf{e}_k = \lambda_1\boldsymbol{\psi}_1\boldsymbol{\psi}_1'\mathbf{e}_k + \lambda_2\left(\sum_{i=2}^{h}\boldsymbol{\psi}_i\boldsymbol{\psi}_i'\right)\mathbf{e}_k = \lambda_1\psi_{k1}\boldsymbol{\psi}_1 \tag{21}$$

Thus, using (16), (20), (21) and the equality $\mathbf{e}_k'\boldsymbol{\psi}_1 = \psi_{k1}$ we obtain

$$\mathbf{Cw}_1 = \sqrt{\frac{\lambda_1}{\lambda_2}}\left(\mathbf{C} + \frac{\lambda_2 - \lambda_1}{\lambda_1 \psi_{k1}^2}\mathbf{Ce}_k\mathbf{e}_k'\right)\boldsymbol{\psi}_1$$

$$= \sqrt{\frac{\lambda_1}{\lambda_2}}\left(\lambda_1 \boldsymbol{\psi}_1 + \frac{(\lambda_2 - \lambda_1)\lambda_1 \psi_{k1}^2}{\lambda_1 \psi_{k1}^2}\right)\boldsymbol{\psi}_1 =$$

$$= \sqrt{\frac{\lambda_1}{\lambda_2}}\lambda_2 \boldsymbol{\psi}_1 = \lambda_2 \mathbf{Xw}_1 \cdot$$

Showing that $\mathbf{w}_1$ is the eigenvector of the matrix $\mathbf{C}$ with respect to $\mathbf{X}$ corresponding to the eigenvalue λ_2 completes the proof.

Corollary 1. A set of $\mathbf{X}^{-1}$-orthonormal basic contrasts in an experimental design satisfying the conditions of Theorem 2 coincides with the set of $\mathbf{I}_t$ - orthonormal basic contrasts, where the first contrast is multiplied by $\sqrt{\lambda_1/\lambda_2}$.

Proof :

When $\mathbf{I}_t$ -orthonormal basic contrasts are the columns of the matrix $\boldsymbol{\Psi}$, given in (9), the set of $\mathbf{X}^{-1}$-orthonormal basic contrasts consists of the vectors $\mathbf{Xw}_i$, for $i = 1, 2, \ldots, h$. From (16), (17) and (20), it follows that $\mathbf{Xw}_1 = \sqrt{\lambda_1/\lambda_2}\boldsymbol{\psi}_1$ and $\mathbf{Xw}_i = \mathbf{X}\boldsymbol{\psi}_i = \boldsymbol{\psi}_i$ for $i = 2, 3, \ldots, h$ □.

3. Examples

3.1. Block design

Block designs which satisfy the requirements of this paper include, for example, the S-type block designs introduced by Pearce (1960). In what follows, we refer to the S-type block design with unequal block sizes constructed by Gupta and Kageyama (1993). In this design, four test treatments and one control treatment are arranged in 22 blocks, of which 12 are of size 3 and 10 are of size 2. The test treatments are replicated 11 times, whereas the control treatment is replicated 12 times. The plan of this design is denoted as 3SR1S + R2 in the tables constructed by Clatworthy (1973).

Due to the large number of observations (n = 56) we omit both design and incidence matrices. The information matrix C is of the form

$$\mathbf{C}=\begin{bmatrix} 8 & -2 & -2 & -2 & -2 \\ -2 & 6{,}5 & -1{,}5 & -1{,}5 & -1{,}5 \\ -2 & -1{,}5 & 6{,}5 & -1{,}5 & -1{,}5 \\ -2 & -1{,}5 & -1{,}5 & 6{,}5 & -1{,}5 \\ -2 & -1{,}5 & -1{,}5 & -1{,}5 & 6{,}5 \end{bmatrix},$$

and its eigenvalues are respectively: $\lambda_1 = 10$, $\lambda_2 = \lambda_3 = \lambda_4 = 8$, $\lambda_5 = 0$. The corresponding orthonormal eigenvectors of the matrix $\mathbf{C}$ with respect to $\mathbf{I}_5$ are:

$$\psi_1 = \frac{1}{\sqrt{20}}\begin{bmatrix} -4 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},\ \psi_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix},\ \psi_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix},\ \psi_4 = \frac{1}{2}\begin{bmatrix} 0 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix},\ \psi_5 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}. \quad (22)$$

The condition (14) is satisfied. The first four vectors given in (22) provide a set of basic estimable contrasts. The variance of the contrast's best linear unbiased estimator, according to the formula (7), is

$$var\left(\widehat{\psi_1'\gamma}\right)=\sigma^2/10,\ var\left(\widehat{\psi_i'\gamma}\right)=\sigma^2/8,\ i=2,3,4.$$

The vectors (22) satisfy the conditions (10) and (11), with $k = 1$. Thus the matrix X such that the considered design is $\mathbf{X}^{-1}$-balanced, according to (12) and (13), takes the form

$$\mathbf{X}=\begin{bmatrix} \frac{4}{3} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The eigenvalues of the matrix $\mathbf{C}$ with respect to the matrix $\mathbf{X}$ are, respectively, $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 8$, $\lambda_5 = 0$, and the corresponding orthonormal eigenvectors, by (16), (17) and (18), can be written as:

$$\mathbf{w}_1 = \frac{1}{4}\begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \ \mathbf{w}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{w}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \ \mathbf{w}_4 = \frac{1}{2}\begin{bmatrix} 0 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \ \mathbf{w}_5 = \frac{1}{\sqrt{\frac{16}{3}}}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

The investigated design is X^{-1}-balanced, so the basic estimable contrasts,

$$\mathbf{Xw}_1 = \frac{1}{4}\begin{bmatrix} -4 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \ \mathbf{Xw}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{Xw}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \ \mathbf{Xw}_4 = \frac{1}{2}\begin{bmatrix} 0 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix},$$

are estimated with the same variance. According to the formula (7), we get

$$var\left(\widehat{\mathbf{w}_i'\mathbf{X}\boldsymbol{\gamma}}\right) = \sigma^2/8, \ i = 1, 2, 3, 4.$$

3.2. Row-column design

Consider an experiment performed according to a Latin square plan, where the five treatments are arranged in five rows and five columns. Suppose that one observation, as a result of a random event, is lost. Without loss of generality, we can assume that the plan of this experiment is

Columns

Rows	1	2	3	4	5
1		**B**	**C**	**D**	**E**
2	**E**	**A**	**B**	**C**	**D**
3	**D**	**E**	**A**	**B**	**C**
4	**C**	**D**	**E**	**A**	**B**
5	**B**	**C**	**D**	**E**	**A**

with A, B, C, D and E denoting treatments. In this design, we have $\mathbf{r} = \mathbf{k}_1 = \mathbf{k}_2 = [4, 5, 5, 5, 5]'$, and the relevant information matrix is

$$\mathbf{C} = \frac{1}{16}\begin{bmatrix} 48 & -12 & -12 & -12 & -12 \\ -12 & 63 & -17 & -17 & -17 \\ -12 & -17 & 63 & -17 & -17 \\ -12 & -17 & -17 & 63 & -17 \\ -12 & -17 & -17 & -17 & 63 \end{bmatrix}$$

The eigenvalues of the matrix **C** with respect to the matrix $\mathbf{I}_5$ are: $\lambda_1 = 3.75$, $\lambda_2 = \lambda_3 = \lambda_4 = 5$, and $\lambda_6 = 0$, and the corresponding orthonormal eigenvectors can be written as

$$\boldsymbol{\psi}_1 = \frac{1}{\sqrt{20}}\begin{bmatrix} 4 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \boldsymbol{\psi}_2 = \frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \boldsymbol{\psi}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \boldsymbol{\psi}_4 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \boldsymbol{\psi}_5 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

The condition (14) is satisfied. The vectors $\boldsymbol{\psi}_i$, $i = 1, 2, \ldots, 5$, satisfy the conditions (10) and (11), with $k = 1$, so the treatment of special attention is that marked A. The matrix **X** enabling the design to be $\mathbf{X}^{-1}$-balanced, according to (12) and (13), takes the form

$$\mathbf{X} = \begin{bmatrix} \frac{12}{17} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The eigenvalues and corresponding **X**-orthonormal eigenvectors of the matrix **C** with respect to the matrix **X** are $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 5$, $\lambda_5 = 0$, and

$$\mathbf{w}_1 = \frac{\sqrt{3{,}75}}{10}\begin{bmatrix} \frac{17}{3} \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \mathbf{w}_2 = \frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \mathbf{w}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \mathbf{w}_4 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}\ \mathbf{w}_5 = \sqrt{\frac{17}{80}}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Thus the investigated design is $\mathbf{X}^{-1}$-balanced, and the estimable basic contrasts having the form

$$\mathbf{Xw}_1 = \frac{\sqrt{3{,}75}}{10}\begin{bmatrix} 4 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}\ \mathbf{Xw}_2 = \frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}\ \mathbf{Xw}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}\ \mathbf{Xw}_4 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

are estimated with the same variance, $var\left(\widehat{\mathbf{w}_i'\mathbf{X}\boldsymbol{\gamma}}\right) = \sigma^2/5$, $i = 1, 2, 3, 4$.

4. Conclusions

We have proved that any block design or row-column design whose information matrix **C** has two different nonzero eigenvalues can be seen as being $\mathbf{X}^{-1}$-balanced for some positive definite diagonal matrix **X**.

The results allow one to construct $\mathbf{X}^{-1}$-balanced designs that are not classed as efficiency-balanced or variance-balanced.

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