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Construction of Barnette graphs whose large subgraphs are non-Hamiltonian

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Abstract. Barnette's conjecture states that every three connected cubic bipartite planar graph (CPB3C) is Hamiltonian. In this paper we show the existence of a family of CPB3C Hamiltonian graphs in which large and large subgraphs are non-Hamiltonian.

1 Introduction

Barnette's conjecture states that every three connected cubic bipartite planar graph (CPB3C) is Hamiltonian. Goodey [7] proved that if all the faces of a CPB3C graph are either quadrilaterals or hexagons, then the graph is Hamiltonian. Later Hertel [11] mentioned that if Barnette conjecture is true, then perhaps Goodey's result can be extended to show that successively large and large subgraphs of Barnette graphs are Hamiltonian. We show that there exists a family of CPB3C Hamiltonian graphs in which large and large subgraphs are non-Hamiltonian.

Tait in (1884) conjectured that every cubic polyhedral graph is Hamiltonian, this came to be known as Tait's conjecture. It was disproved by Tutte (1946), who constructed a counter example with 46 vertices. Other researchers

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later found even smaller counterexamples, however none of these counterexamples is bipartite. Tutte himself conjectured that every cubic 3-connected bipartite graph is Hamiltonian, but this was shown to be false by discovery of a counterexample, the Horton graph [1]. David W. Barnett (1969) proposed a weakened combination of Tait's and Tutte's conjecture, stating that every cubic bipartite polyhedral graph is Hamiltonian. This conjecture was first announced in [3] and later in [8]. Tutte [18] proved that all planar 4-connected graphs are Hamiltonian. Thomassen [14] extended this result by showing that every planar 4-connected graph is Hamiltonian connected, that is, for any pair of vertices, there is a Hamiltonian path with those vertices as end vertices. It must be noted that if any one of the property of Barnette graph is deleted, then it is non-Hamiltonian.

McKay et al. [15] confirmed through a combination of clever analysis and computer search that all Barnette graphs with 64 or less vertices are Hamiltonian. In an announcement [2, 5], McKay used computer search to extend this result to 84 vertices. This implies that if Barnette's conjecture is indeed false, then a minimal counterexample must contain at least 86 vertices, and is therefore considerable larger than the minimal counterexample to Tait and Tutte conjecture. This is not all we know about a possible counterexample, another interesting result is that of Fowler, who in an unpublished manuscript [6] provided a list of subgraphs that cannot appear in any minimal counterexample to Barnette's conjecture. For more definitions and notations of graph theory, we refer to [16].

Goody [7] considers proper subgraphs of the Barnette graphs and proved the following.

Theorem 1 *Every Barnette graph which has faces consisting exclusively of quadrilaterals and hexagons is Hamiltonian, and further more in all such graphs, any edge that is common to both a quadrilateral and a hexagon is a part of some Hamiltonian cycle.*

Theorem 2 *Every Barnette graph which has faces consisting of 7 quadrilaterals, 1 octagon and any number of hexagons is Hamiltonian, and any edge that is common to both a quadrilateral and an octagon is a part of some Hamiltonian cycle.*

Jensen and Toft [12] reported that Barnette conjecture is equivalent to the following.

Theorem 3 *Barnette's conjecture is true if and only if for every Barnette* *graph G, it is possible to partition its vertices into two subsets so that each induces an acyclic subgraph of G. (This theorem is not correct)*

Theorem 4 [10] *Barnette's conjecture holds if and only if any arbitrary edge in a Barnette graph is a part of some Hamiltonian cycle.*

Theorem 5 *Barnette's conjecture holds if and only if for any arbitrary face in a Barnette graph there is a Hamiltonian cycle which passes through any two arbitrary edges on that face.*

Theorem 6 [13] *Barnette's conjecture holds if and only if for any arbitrary face in a Barnette graph and for any arbitrary edges e_1 and e_2 on that face there is a Hamiltonian cycle which passes through e_1 and avoids e_2.*

Theorem 7 *Barnette's conjecture holds if and only if for any arbitrary path P of length 3 that lies on a face in a Barnette graph, there is a Hamiltonian cycle which passes through the middle edge in P and avoids both its leading and trailing edges.*

Theorem 8 [9] *Barnette's conjecture is true if and only if there is a constant $c > 0$ such that each Barnette graph G contains a path on at least $c|V(G)|$ vertices.*

Theorem 9 [14] *The edges of any bipartite graph G can be colored with α colors, where α is the minimum degree of vertices in G.*

More than thirty papers have been published to strengthening the Barnette's conjecture, not only this, also several proofs of this conjecture have been put forward so far but none of the proof is yet accepted universally. For references, see [4].

2 Main result

The following theorem is the main result, which shows the existence of a family of CPB3C Hamiltonian graphs in which large and large subgraphs are non-Hamiltonian. The proof is by construction.

Theorem 10 *There exist a family of CPB3C Hamiltonian graphs in which large and large subgraphs are non-Hamiltonian.*

Proof. Suppose G is a graph of such family, see Figure 1. The vertex set of graph G is colored with black and red as G is bipartite. If we remove a vertex x (say) in G, we get a non-Hamiltonian graph $G - x$, see Figure 2. Since G is cubic, let the edges incident on x be $e_1 = xu_1,\ e_2 = xu_2,\ e_3 = xu_3$. Since G is Hamiltonian, so at least two of the three edges e_1, e_2, e_3 are always included in the Hamiltonian cycle of G. If vertex x is colored black, it is always adjacent to red color vertices, as G is bipartite. Assume that the edges e_1 and e_2 are included in the Hamiltonian cycle of G. If there is a cycle starting from x to all other vertices of the graph G, then there exists a Hamiltonian path in $G - x$ beginning and ending with the vertices u_1, u_2 of the graph $G - x$. Thus we get a non-Hamiltonian graph $G - x$. We form a large graph by taking $2n$ copies of $G - x$ (n as large as possible) with vertices of different copies colored alternately, so that the vertices of two in different copies can be joined by edges. The construction is done as follows.

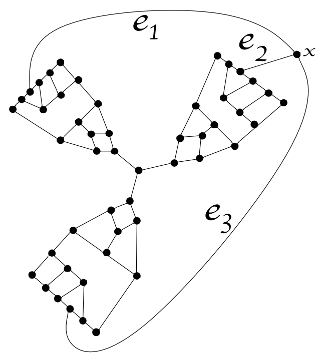

Figure 1: CPB3C Hamiltonian graph G

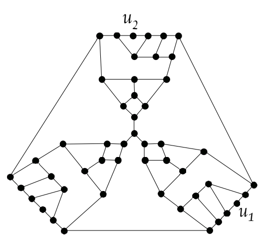

Figure 2: CPB3C non-Hamiltonian graph $G - x$

Take two copies of $G - x$ and we name them as G_1 and H_1. Let u_1^1, u_2^1, u_3^1 be the vertices of degree two in G_1 and v_1^1, v_2^1, v_3^1 be the vertices of degree two in

H_1. We assume u_1^1, u_2^1, u_3^1 are colored red and v_1^1, v_2^1, v_3^1 are colored black. (This is possible since we can choose G_1 and H_1 in such a way so that the color of vertices in G_1 is interchanged with the color of vertices in H_1). Consider the graph $S_1 = G_1 \cup H_1 \cup \{u_1^1v_1^1, u_2^1v_2^1, u_3^1v_3^1\}$, the graph which consists of G_1, H_1 and the edges $u_1^1v_1^1, u_2^1v_2^1, u_3^1v_3^1$. Clearly S_1 is CPB3C. Further S_1 is Hamiltonian. If any two edges from $\{u_1^1v_1^1, u_2^1v_2^1, u_3^1v_3^1\}$ are removed, the resulting graph is non-Hamiltonian. See Figure 3.

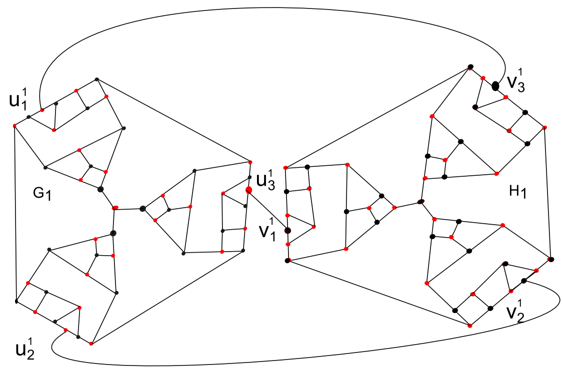

Figure 3: CPB3C Hamiltonian graph

Take four copies of $G - x$ and we name them as G_1, G_2 and H_1, H_2. Let $u_1^1, u_2^1, u_3^1 \in G_1$, $u_1^2, u_2^2, u_3^2 \in G_2$, $v_1^1, v_2^1, v_3^1 \in H_1$ and $v_1^2, v_2^2, v_3^2 \in H_2$ be vertices of degree 2. Further, we let u_1^1, u_2^1, u_3^1, v_1^2, v_2^2, v_3^2 colored red and u_1^2, u_2^2, u_3^2, v_1^1, v_2^1, v_3^1 colored black. Let $C_1 = w_1^1w_2^1w_3^1w_4^1$ be a cycle of length four. Consider the graph

$$S_2 = G_1 \cup G_2 \cup H_1 \cup H_2 \cup C_1 \cup \{u_1^1u_1^2, v_1^1v_1^2, u_2^1v_2^1, u_2^2v_2^2, u_3^1w_1^1, u_3^2w_2^1, v_3^1w_4^1, v_3^2w_3^1\}.$$

By the same argument as above, S_2 is CPB3C Hamiltonian graph. Removal of three edges makes the resulting graph non-Hamiltonian. See Figure 4.

Take $2n$ copies of $G - x$ with n copies named as $G_1, G_2, \dots G_n$ and n copies named as $H_1, H_2, \dots H_n$. Let $u_1^1, u_2^1, u_3^1 \in G_1$, $u_1^2, u_2^2, u_3^2 \in G_2$ and so on $u_1^n, u_2^n, u_3^n \in G_n$ be vertices of degree 2. Similarly let $v_1^1, v_2^1, v_3^1 \in H_1$, $v_1^2, v_2^2, v_3^2 \in H_2$ and so on $v_1^n, v_2^n, v_3^n \in H_n$ be vertices of degree 2. Further consider cycles of length four as $C_1, C_2, \dots C_{n-1}$. Continuing as in Case 2, we form the graph S_n, which is clearly CPB3C Hamiltonian graph. The removal of three edges joining any two adjacent copies of $G - x$ produces a largest non-Hamiltonian subgraph of S_n.

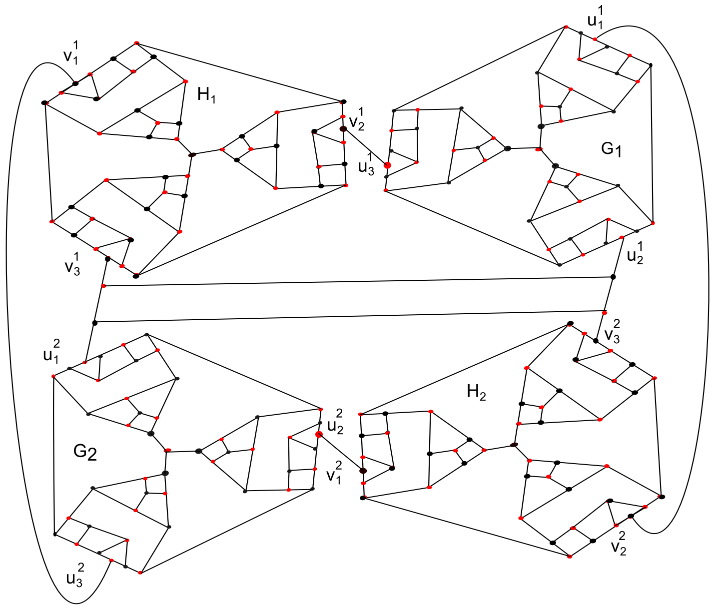

Figure 4: CPB3C Hamiltonian graph

If we remove any of the edges which connects the different components of $G - x$, we get a largest subgraph which is non-Hamiltonian. In this way, if we continuously remove three such edges again and again, we get subgraphs all of which are non-Hamiltonian. Since n is as large as possible, we get $2n$ such non-Hamiltonian graphs. Not only this, if we further remove some edges from all the smallest components of $G - x$), as shown in Figure 5, remaining large subgraphs are non-Hamiltonian. In this way, we get large and large subsets of CPB3C Hamiltonian graphs which are non- Hamiltonian. Thus we conclude that there exist a family of CPB3C Hamiltonian graphs in which large and large subsets are non-Hamiltonian.

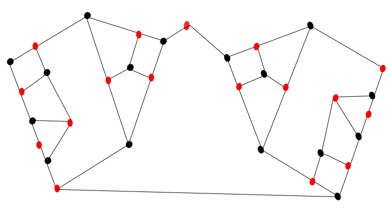

Figure 5: Resulting graph

In this way we construct a CPB3C Hamiltonian graph whose large and large subsets are non-Hamiltonian. In other words, it is possible to construct CPB3C Hamiltonian graph as large as possible, whose large and large subsets are non-Hamiltonian. □

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