



On a class of analytic functions governed by subordination

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Abstract. The purpose of this paper is to introduce a class of functions \mathcal{F}_λ , $\lambda \in [0, 1]$, consisting of analytic functions f normalized by $f(0) = f'(0) - 1 = 0$ in the open unit disk \mathbb{U} which satisfies the subordination condition that

$$zf'(z)/\{(1-\lambda)f(z) + \lambda z\} \prec q(z), \quad z \in \mathbb{U},$$

where $q(z) = \sqrt{1+z^2} + z$. Some basic properties (including the radius of convexity) are obtained for this class of functions.

1 Introduction

Let \mathcal{H} denote the class of analytic functions in the open unit disc $\mathbb{U} = \{z : |z| < 1\}$ in the complex plane \mathbb{C} . Also, let \mathcal{A} denote the subclass of \mathcal{H} comprising of functions f normalized by $f(0) = 0$, $f'(0) = 1$, and let $\mathcal{S} \subset \mathcal{A}$ denote the class of functions which are univalent in \mathbb{U} . We say that an analytic function f is subordinate to an analytic function g , and write $f(z) \prec g(z)$, if and only if there exists a function ω , analytic in \mathbb{U} such that $\omega(0) = 0$, $|\omega(z)| < 1$ for

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$|z| < 1$ and $f(z) = g(\omega(z))$. In particular, if g is univalent in \mathbb{U} , then we have the following equivalence:

$$f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(|z| < 1) \subset g(|z| < 1). \tag{1}$$

Let a function f be analytic univalent in the unit disc $\mathbb{U} = \{z : |z| < 1\}$ on the complex plane \mathbb{C} with the normalization $f(0) = 0$, then f maps \mathbb{U} onto a starlike domain with respect to $w_0 = 0$ if and only if

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \quad (z \in \mathbb{U}). \tag{2}$$

It is well known that if an analytic function f satisfies (2) and $f(0) = 0$, $f'(0) \neq 0$, then f is univalent and starlike in \mathbb{U} .

A set E is said to be convex if and only if it is starlike with respect to each of its points, that is if and only if the linear segment joining any two points of E lies entirely in E . Let f be analytic and univalent in $\mathbb{U}_r = \{z : |z| < r \leq 1\}$. Then f maps \mathbb{U}_r onto a convex domain E if and only if

$$\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0 \quad (z \in \mathbb{U}_r). \tag{3}$$

If $r = 1$, then the function f is said to be convex in \mathbb{U} (or briefly convex). The set of all functions $f \in \mathcal{A}$ that are starlike univalent in \mathbb{U} will be denoted by \mathcal{S}^* and the set of all functions $f \in \mathcal{A}$ that are convex univalent in \mathbb{U} by \mathcal{K} .

Definition. For given $\lambda \in [0, 1]$, let \mathcal{F}_λ denote the class of analytic functions f in the unit disc \mathbb{U} normalized by $f(0) = f'(0) - 1 = 0$ and satisfying the condition that

$$\frac{zf'(z)}{(1-\lambda)f(z) + \lambda z} \prec \sqrt{1+z^2} + z =: q(z), \quad z \in \mathbb{U}, \tag{4}$$

where the branch of the square root is chosen to be $q(0) = 1$.

We note that for $\lambda = 0$ in (4), we have the class \mathcal{F}_0 which connects a starlike function with the function $q(z)$ by means of a subordination and is defined by

$$\mathcal{F}_0 = \{f \in \mathcal{A} : zf'(z)/f(z) \prec \sqrt{1+z^2} + z, \quad z \in \mathbb{U}\}. \tag{5}$$

Also, for $\lambda = 1$ in (4), we obtain a class \mathcal{F}_1 which depicts a subordination relationship between the function $f'(z)$ with the function $q(z)$ and this class is defined by

$$\mathcal{F}_1 = \{f \in \mathcal{A} : f'(z) \prec \sqrt{1+z^2} + z, \quad z \in \mathbb{U}\}. \tag{6}$$

The function $w(z) = \sqrt{1+z}$ maps \mathbb{U} onto a set bounded by Bernoulli lemniscate, and the class of functions $f \in \mathcal{A}$ such that $zf'(z)/f(z) \prec \sqrt{1+z}$ was considered in [14], while $zf'(z)/f(z) \prec \sqrt{1+cz}$ was considered in [1]. This way the well known class of k -starlike functions were seen to be connected with certain conic domains. For some recent results for k -starlike functions, we refer to [8, 11, 13, 15]. Certain function classes were also considered in recent papers [2, 3, 4, 5, 7, 12] which were defined by means of the subordination that $zf'(z)/f(z) \prec \widehat{q}(z)$, where $\widehat{q}(z)$ was not univalent. For a unified treatment of some special classes of univalent functions we refer to [10] (see also [16]).

2 Auxiliary results

Lemma 1 *The function*

$$h(z) = \frac{z}{\sqrt{1+z^2}} \tag{7}$$

is convex in \mathbb{U}_r , where $r = \sqrt{2}/2$.

Proof. Using (7), we have

$$1 + \frac{zh''(z)}{h'(z)} = \frac{1-2z^2}{1+z^2},$$

hence

$$\Re \left\{ 1 + \frac{zh''(z)}{h'(z)} \right\} > 0 \text{ for } |z| < \frac{\sqrt{2}}{2}$$

and thus $h(z)$ is convex in \mathbb{U}_r , where $r \leq \sqrt{2}/2$. □

Corollary 1 *If $r \leq \sqrt{2}/2$ and $h(z) = z/\sqrt{1+z^2}$, then we have*

$$\min_{|z| \leq r} \{\Re\{h(z)\}\} = \frac{-r}{\sqrt{1+r^2}}.$$

Proof. By Lemma 1, the function $h(z)$ is convex in \mathbb{U}_r , where $r \leq \sqrt{2}/2$ and $h(\mathbb{U}_r)$ is symmetric with respect to the real axis. Since the function $h(z)$ is real for real z , therefore, $\Re\{h(z)\}$ attains its extremal values at $-r$ and r , which proves the corollary. □

Lemma 2 *The function*

$$q(z) = \sqrt{1+z^2} + z$$

is convex in \mathbb{U}_r , where r is at least $\sqrt{2}/2$.

Proof. By elementary calculations, it can easily be shown that $q(z)$ is univalent in the unit disc. For the proof that $q(z)$ is convex, we use (3). Thus, we obtain

$$\begin{aligned} 1 + \frac{zq''(z)}{q'(z)} &= \frac{1}{1+z^2} + \frac{z}{\sqrt{1+z^2}} \\ &= \frac{1}{1+z^2} + h(z), \end{aligned}$$

where $h(z)$ is given in (7). By Corollary 1, we have

$$\begin{aligned} \min_{|z| \leq \sqrt{2}/2} \left\{ \Re \left\{ 1 + \frac{zq''(z)}{q'(z)} \right\} \right\} &\geq \min_{0 < x \leq \sqrt{2}/2} \left\{ \Re \left\{ \frac{1}{1+x^2} - \frac{x}{\sqrt{1+x^2}} \right\} \right\} \\ &= \frac{2-\sqrt{3}}{3} > 0, \end{aligned} \tag{8}$$

because

$$t(x) = \frac{1}{1+x^2} - \frac{x}{\sqrt{1+x^2}}$$

decreases in $\left[0, \sqrt{(\sqrt{5}-1)/2}\right]$ from $t(0) = 1$ to $t\left(\sqrt{(\sqrt{5}-1)/2}\right) = 0$, so that $t(\sqrt{2}/2) = (2-\sqrt{3})/3$ is the smallest value of $t(x)$ for $0 < x \leq \sqrt{2}/2$. Therefore, in view of (8), the function $q(z) = \sqrt{1+z^2} + z$ is convex in \mathbb{U}_r , where r is at least $\sqrt{2}/2$. □

Corollary 2 *If $r \leq \sqrt{2}/2$ and $q(z) = \sqrt{1+z^2} + z$, then we have*

$$\min_{|z| \leq r} \{\Re\{q(z)\}\} = \sqrt{1+r^2} - r.$$

Proof. By Lemma 2, the function $q(z)$ is convex in \mathbb{U}_r , where $r \leq \sqrt{2}/2$ and $h(\mathbb{U}_r)$ is symmetric with respect to the real axis. Therefore, $q(z)$ is real for real z , and thus, $\Re\{q(z)\}$ attains its extremal values at $-r$ and r . □

Lemma 3 *The function $q(z) = \sqrt{1+z^2} + z$ satisfies*

$$\Re\{q(z)\} > 0 \tag{9}$$

in \mathbb{U} .

Proof. Let $z = e^{it}$, $t \in [0, 2\pi)$. We assume that $\arg\{e^{2it} + 1\} \in (-\pi, \pi]$. It follows that $|e^{2it} + 1| = |2 \cos t|$ and

$$\arg(e^{2it} + 1) = \begin{cases} t & \text{for } t \in [0, \pi/2), \\ t - \pi & \text{for } t \in (\pi/2, 3\pi/2), \\ t - 2\pi & \text{for } t \in (3\pi/2, 2\pi). \end{cases}$$

Therefore, we infer that

$$e^{it} + \sqrt{e^{2it} + 1} = \begin{cases} \cos t + i \sin t + \sqrt{|2 \cos t|}(\cos t/2 + i \sin t/2) & \text{for } t \in [0, \pi/2), \\ i & \text{for } t = \pi/2, \\ \cos t + i \sin t + \sqrt{|2 \cos t|}(\sin t/2 - i \cos t/2) & \text{for } t \in (\pi/2, 3\pi/2), \\ -i & \text{for } t = 3\pi/2, \\ \cos t + i \sin t + \sqrt{|2 \cos t|}(-\cos t/2 - i \sin t/2) & \text{for } t \in (3\pi/2, 2\pi). \end{cases}$$

Now some simple calculations show that $\Re\{e^{it} + \sqrt{e^{2it} + 1}\} = 0$ if and only if $t = \pi/2$ or if $t = 3\pi/2$, which implies that $\Re\{q(z)\} > 0$ in \mathbb{U} (see Fig.1 below). \square

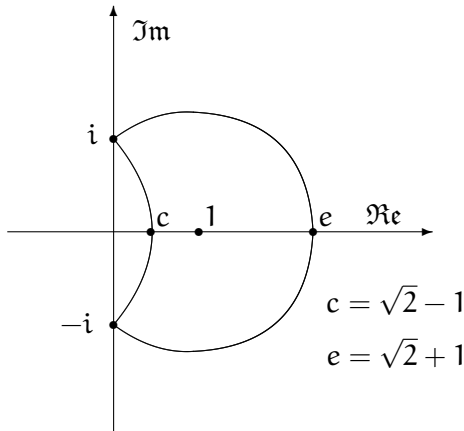


Figure 1. $q(e^{it})$.

3 Basic properties of the class \mathcal{F}_λ

Corollary 3 Let $n \geq 2$ be a given positive integer. Then the function

$$f_{n,a}(z) = z + az^n \quad (z \in \mathbb{U})$$

is in the class \mathcal{F}_λ if and only if

$$|a| \leq \frac{2 - \sqrt{2}}{n + (1 - \sqrt{2})(1 - \lambda)}. \tag{10}$$

Proof. The function

$$F_{n,a}(z) := \frac{zf'_{n,a}(z)}{(1 - \lambda)f_{n,a}(z) + \lambda z} = \frac{1 + na z^{n-1}}{1 + (1 - \lambda)az^{n-1}}$$

maps \mathbb{U} onto the disc $F_{n,a}(\mathbb{U})$ that is symmetric with respect to the real axis. For

$$F_{n,a}(z) \prec \sqrt{1 + z^2} + z, \tag{11}$$

it is necessary that $F_{n,a}(z) \neq 0$, and so we may assume that $|na| < 1$. We have then

$$\frac{1 - n|a|}{1 - (1 - \lambda)|a|} < \Re\{F_{n,a}(z)\} < \frac{1 + n|a|}{1 + (1 - \lambda)|a|}.$$

It follows by applying a geometric interpretation of the subordination condition that (11) is equivalent to

$$\sqrt{2} - 1 \leq \frac{1 - n|a|}{1 - (1 - \lambda)|a|} \quad \text{and} \quad \frac{1 + n|a|}{1 + (1 - \lambda)|a|} \leq \sqrt{2} + 1. \tag{12}$$

Since the second inequality in (12) above is weaker, the desired inequality (10) readily follows from the first inequality of (12). □

Theorem 1 *Let the function f defined by*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{U})$$

belong to the class \mathcal{F}_λ , then

$$|a_2| \leq 1/(1 + \lambda) \tag{13}$$

and

$$|a_3| \leq \begin{cases} \frac{3-\lambda}{2(1+\lambda)(2+\lambda)} & \text{for } \lambda \in [0, 1/3], \\ \frac{1}{2+\lambda} & \text{for } \lambda \in (1/3, 1]. \end{cases} \tag{14}$$

Furthermore,

$$|a_4| \leq \frac{5 + 9\lambda - 2\lambda^2 + 2|2\lambda^2 + 11\lambda - 1|}{2(1 + \lambda)(2 + \lambda)(3 + \lambda)}. \tag{15}$$

Proof. Since the function f defined by (1) belongs to the class \mathcal{F}_λ , therefore from (4), we have

$$zf'(z) - \left\{ z + (1 - \lambda) \sum_{n=2}^{\infty} a_n z^n \right\} \omega(z) = \left\{ z + (1 - \lambda) \sum_{n=2}^{\infty} a_n z^n \right\} \sqrt{\omega^2(z) + 1},$$

where ω is such that $\omega(0) = 0$ and $|\omega(z)| < 1$ for $|z| < 1$. Let us denote the function $\omega(z)$ by

$$\omega(z) = \sum_{k=1}^{\infty} c_k z^k. \quad (16)$$

Thus, (16) readily gives

$$\sqrt{\omega^2(z) + 1} = 1 + \frac{1}{2}c_1^2 z^2 + c_1 c_2 z^3 + \left(c_1 c_3 + \frac{1}{2}c_2^2 - \frac{1}{8}c_1^2 \right) z^4 + \dots.$$

Moreover,

$$\begin{aligned} & \left\{ z + (1 - \lambda) \sum_{n=2}^{\infty} a_n z^n \right\} \sqrt{\omega^2(z) + 1} \\ &= z + (1 - \lambda)a_2 z^2 + \left(\frac{1}{2}c_1^2 + (1 - \lambda)a_3 \right) z^3 \\ & \quad + \left(c_1 c_2 + \frac{1 - \lambda}{2}c_1^2 a_2 + (1 - \lambda)a_4 \right) z^4 + \dots \end{aligned} \quad (17)$$

and

$$\begin{aligned} & zf'(z) - \left\{ z + (1 - \lambda) \sum_{n=2}^{\infty} a_n z^n \right\} \omega(z) \\ &= z + (2a_2 - c_1)z^2 + (3a_3 - (1 - \lambda)c_1 a_2 - c_2)z^3 \\ & \quad + (4a_4 - (1 - \lambda)[c_1 a_3 - c_2 a_2] - c_3)z^4 + \dots \end{aligned} \quad (18)$$

Equating now the second, third and fourth coefficients in (17) and (18), we have

$$(i) \quad (1 - \lambda)a_2 = 2a_2 - c_1,$$

$$(ii) \quad \frac{1}{2}c_1^2 + (1 - \lambda)a_3 = 3a_3 - (1 - \lambda)c_1 a_2 - c_2,$$

$$(iii) \quad c_1 c_2 + \frac{1 - \lambda}{2}c_1^2 a_2 + (1 - \lambda)a_4 = 4a_4 - (1 - \lambda)[c_1 a_3 + c_2 a_2] - c_3.$$

From (i), we get

$$a_2 = \frac{c_1}{1 + \lambda}. \tag{19}$$

It is well known that the coefficients of the bounded function $\omega(z)$ satisfies the inequality that $|c_k| \leq 1$, ($k = 1, 2, 3, \dots$), so from (19), we have the first inequality that $|a_2| \leq 1/(1 + \lambda)$. Now, from (ii) and (13), we obtain that

$$\begin{aligned} (2 + \lambda)a_3 &= \frac{1}{2}c_1^2 + (1 - \lambda)c_1a_2 + c_2 \\ &= \frac{1}{2}c_1^2 + \frac{1 - \lambda}{1 + \lambda}c_1^2 + c_2 \\ &= c_2 + \frac{3 - \lambda}{2(1 + \lambda)}c_1^2. \end{aligned} \tag{20}$$

Also,

$$\lambda \in [0, 1/3] \Rightarrow \left| \frac{3 - \lambda}{2(1 + \lambda)} \right| \geq 1 \text{ and } \lambda \in (1/3, 1] \Rightarrow \left| \frac{3 - \lambda}{2(1 + \lambda)} \right| < 1.$$

Therefore, by using the estimate (see [9]) that if $\omega(z)$ has the form (16), then

$$|c_2 - \mu c_1^2| \leq \max\{1, |\mu|\}, \text{ for all } \mu \in \mathbb{C},$$

we obtain (14). Also, from (i)-(iii) and (19)-(20), we find that

$$\begin{aligned} |(3 + \lambda)a_4| &= \left| (1 - \lambda)[c_1a_3 + c_2a_2] + c_3 + c_1c_2 + \frac{1 - \lambda}{2}c_1^2a_2 \right| \\ &= \left| \frac{5(1 - \lambda)}{2(1 + \lambda)(2 + \lambda)}c_1^3 + \frac{5 + 2\lambda - \lambda^2}{(1 + \lambda)(2 + \lambda)}c_1c_2 + c_3 \right| \\ &= \left| \frac{5(1 - \lambda)}{2(1 + \lambda)(2 + \lambda)}(c_1^3 + 2c_1c_2 + c_3) + \frac{7\lambda - \lambda^2}{(1 + \lambda)(2 + \lambda)}c_1c_2 \right. \\ &\quad \left. + \left(1 - \frac{5(1 - \lambda)}{2(1 + \lambda)(2 + \lambda)}\right)c_3 \right| \\ &\leq \frac{5(1 - \lambda)}{2(1 + \lambda)(2 + \lambda)}|c_1^3 + 2c_1c_2 + c_3| + \frac{(7\lambda - \lambda^2)|c_1c_2|}{(1 + \lambda)(2 + \lambda)} \\ &\quad + \frac{|2\lambda^2 + 11\lambda - 1||c_3|}{2(1 + \lambda)(2 + \lambda)}. \end{aligned} \tag{21}$$

We next use some properties of c_k involved in (16). It is known that the function $p(z)$ given by

$$\frac{1 + \omega(z)}{1 - \omega(z)} = 1 + p_1z + p_2z^2 + \dots =: p(z) \tag{22}$$

defines a Caratheodory function with the property that $\Re\{p(z)\} > 0$ in \mathbb{U} and that $|p_k| \leq 2$ ($k = 1, 2, 3, \dots$). Equating of the coefficients in (22) yields that

$$p_2 = 2(c_1^2 + c_2)$$

and

$$p_3 = 2(c_1^3 + 2c_1c_2 + c_3).$$

Hence $|c_1^2 + c_2| \leq 1$ and

$$|c_1^3 + 2c_1c_2 + c_3| \leq 1. \quad (23)$$

By applying (21) and (23), we find that

$$|(3 + \lambda)a_4| \leq \frac{5(1 - \lambda)}{2(1 + \lambda)(2 + \lambda)} + \frac{7\lambda - \lambda^2}{(1 + \lambda)(2 + \lambda)} + \frac{|2\lambda^2 + 11\lambda - 1|}{2(1 + \lambda)(2 + \lambda)},$$

which gives (15). □

4 Some consequences and special cases

It may be observed from (4), (5) and (9) of Lemma 3 that

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \quad (z \in \mathbb{U})$$

for $f \in \mathcal{F}_0$, hence f is univalent starlike with respect to the origin, and this leads to the following result.

Corollary 4 $\mathcal{F}_0 \subset \mathcal{S}^*$.

In view of (5) and (6), we can deduce the coefficient estimates for functions belonging to the classes \mathcal{F}_0 and \mathcal{F}_1 from Theorem 3.1. These results are easy to obtain and we skip mentioning here their details.

Lastly, we prove the radius of convexity of a function belonging to the class \mathcal{F}_0 .

Theorem 2 *If $f \in \mathcal{F}_0$, then f is convex in \mathbb{U}_r , where r is at least*

$$\sqrt{(5 - \sqrt{13})/2} = 0.482\dots$$

Proof. Assume that $|z| < \sqrt{2}/2$. Let $f \in \mathcal{S}^*(q)$, then in view of (4), we have

$$f'(z)/f(z) = \sqrt{1 + \omega^2(z)} + \omega(z),$$

where ω satisfies $\omega(0) = 0$, $|\omega(z)| < 1$ for $|z| < 1$, and by Schwarz Lemma, ω satisfies $|\omega(re^{i\varphi})| < r$. Let us recall that ([see [6], Vol. II, p. 77])

$$|\omega'(z)| \leq \frac{1 - |\omega(z)|^2}{1 - |z|^2}. \tag{24}$$

Differentiating $zf'(z)/f(z) = \sqrt{1 + \omega^2(z)} + \omega(z)$ and using (24), we obtain

$$\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} = \Re \left\{ \sqrt{1 + \omega^2(z)} + \omega(z) + \frac{z\omega'(z)}{\sqrt{1 + \omega^2(z)}} \right\}. \tag{25}$$

Applying now Corollary 2, we get

$$\min_{|z| < \sqrt{2}/2} \left\{ \Re \left\{ \sqrt{1 + \omega^2(z)} + \omega(z) \right\} \right\} = \sqrt{1 + r^2} - r. \tag{26}$$

Hence, from (25) and (26), we have

$$\begin{aligned} \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} &\geq \sqrt{1 + r^2} - r - \left| \frac{z\omega'(z)}{\sqrt{1 + \omega^2(z)}} \right| \\ &\geq \sqrt{1 + r^2} - r - r \frac{1 - |\omega^2(z)|}{1 - |z|^2} \frac{1}{|\sqrt{1 + \omega^2(z)}|} \\ &\geq \sqrt{1 + r^2} - r - r \frac{1 - |\omega^2(z)|}{1 - |z|^2} \frac{1}{\sqrt{1 - |\omega^2(z)|}} \\ &= \sqrt{1 + r^2} - r - r \frac{\sqrt{1 - |\omega^2(z)|}}{1 - r^2} \\ &> \sqrt{1 + r^2} - r - \frac{r}{1 - r^2}. \end{aligned}$$

Solving in $[0, \sqrt{2}/2]$ the inequality:

$$\sqrt{1 + r^2} - r - \frac{r}{1 - r^2} \geq 0,$$

we obtain that $3r^4 - 5r^2 + 1 \geq 0$, and so if $r \in \left[0, \sqrt{(5 - \sqrt{13})/2} \right]$, then by (3) the function f is convex in \mathbb{U}_r . □

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