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ON APPLICATION OF THE FOURIER TRANSFORM BAND PASS FILTERING TECHNIQUE

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ABSTRACT. In this work theoretical aspects of the Fourier Transform Band Pass Filter (FTBPF) technique are investigated which show that such a procedure is applicable to extraction of irregular monochromatic oscillations with time-varying amplitudes from the analyzed signal data. Considerations concerning the boundary effects occurring in numerical implementation of the FTBPF are included which indicate that the filter transfer function smoothness influences boundary effects magnitude. The possibility of using the studied filtration technique for recognition of elliptical oscillation polarization is envisaged and an estimate of oscillation polarization classification error is derived.

Keywords: Fourier Transform Band Pass Filter, transmittance function, irregular oscillation, boundary effects, oscillation polarization

1. INTRODUCTION

The filtering technique based on the Fourier Transform and appropriately selected transmittance functions has been applied in the spectral analysis of time series for several decades now (Brillinger 1975), (Koopmans 1974). It is implemented in scientific studies in the domain of geophysics (Evans 1985), (Forbes 1988), (Pan 1998), geodesy (Kosek 1995), (Popiński and Kosek 1995) and signal processing (Blackledge 2003), (Koopmans 1974), (Pan 2001), (Speed 1985), since it enables analyzing stationary time series in chosen frequency bands. It can be also used in digital image processing (Hoggar 2006), (Press et al. 1992) and spectral analysis of two-dimensional maps of sea surface topography (Popiński and Kosek 1999). Certain theoretical as well as numerical properties of the considered filtration technique were already examined and described in time series analysis textbooks (Bremaud 2002), (Brillinger 1975), (Koopmans 1974) and articles concerning its applications (Forbes 1988), (Pan 1998), but they were related rather to the case of stationary time series analysis and errors or effects like Gibbs phenomenon or "spectral ringing" connected with numerical implementation of this technique (Evans 1985), (Forbes 1988), (Pan 2001).

Since there is no analytical model of polar motion available yet such a tool is particularly useful in the investigation of oscillations occurring in Earth rotation parameters data (time series of pole coordinates and length-of-day) and in examining the influence of relevant angular momentum excitation functions on them. In consequence, some authors applied it in their studies in this domain of research (Kołaczek 1992), (Kołaczek and Kosek 1993), (Kosek 1995, 2004), (Kosek et al. 1995), (Kosek and Kaczkowski 1994), (Kosek and Popiński 1999), (Nastula et al. 1993), (Popiński and Kosek 1995, 2000). Of course the FTBPF is not the only filtering technique applied in Earth orientation parameters analysis. One can learn about other filters and results of their application in the works of Höpfner (1996) who designed transversal filter, as well as (Kosek 1987) – Ormsby filter, (Zheng and Dong 1986, 1987) – multi-stage filter, (Zheng and Chang 1993) – difference filter.

The present work is a continuation and extension of the author's previous publications (Popiński and Kosek 1995), (Popiński 1997, 2008) on theoretical, statistical, as well as numerical properties of the Fourier Transform Band Pass Filter (FTBPF) technique. It deals with the problem of applicability of such a filtering procedure to extraction of transient irregular monochromatic oscillations with variable amplitudes, present in the analyzed time series (section 2). Investigations on a related subject based on ideas of the Fourier Transform, complex demodulation and tapering were already published by Evans (1985), Hasan (1983) and Park (1992). In section 3 occurrence of boundary effects in the FTBPF results is considered and the assertions of carried out analysis explain why some number of oscillation values at the ends of analyzed series time span must be discarded. Finally, the section 4 is confined to the possibility of application of the studied filtration technique to polarization recognition (occurrence of prograde or retrograde motion) of elliptical oscillations in polar motion.

2. THEORETICAL ASPECTS OF THE FTBPF

The FTBPF is used to filter a narrow-band oscillation with chosen central frequency ω_c from the analyzed signal represented by a complex-valued function $x(t) = \mathrm{Re}[x(t)] + i\,\mathrm{Im}[x(t)]$ of continuous time argument, satisfying $x \in L^2(R)$. According to the definition of this filter such an oscillation is given by the formula (Koopmans 1974):

$$o_x(\omega_c, t) = CFT^{-1}[A(\omega_c, \omega) CFT[x(s)]] \quad , \tag{1}$$

where the CFT operator denotes the Continuous Fourier Transform of the function $x(s)$,

$$CFT[x(s)] = \hat{x}(\omega) = \int_{-\infty}^{+\infty} x(s) \exp(-i\omega s) ds \quad ,$$

and $A(\omega_c, \omega)$ is a real-valued transmittance function of the filter, which is different from zero only in the neighborhood of the central frequency ω_c (pass-band) and satisfies the conditions $0 \le A(\omega_c, \omega) \le 1$, $A(\omega_c, \omega_c) = 1$. The simplest example of the transmittance function is the boxcar one (Speed 1985)

$$A_R(\omega_c, \omega) = \begin{cases} 1 & for \quad |\omega - \omega_c| \le 2\pi\lambda, \\ 0 & for \quad |\omega - \omega_c| > 2\pi\lambda, \end{cases}$$

but better oscillation filtering results can be obtained with the use of smoother transmittance functions like the trapezial one (Popiński and Kosek 1995)

$$A_T(\omega_c, \omega) = \begin{cases} 1 & for \quad |\omega - \omega_c| \le 2\pi\lambda, \\ 1 - \dfrac{|\omega - \omega_c| - 2\pi\lambda}{2\pi\alpha} & for \quad 2\pi\lambda < |\omega - \omega_c| \le 2\pi(\lambda + \alpha), \\ 0 & for \quad |\omega - \omega_c| > 2\pi(\lambda + \alpha), \end{cases}$$

where $\lambda > 0$ and $\alpha > 0$ are the width parameters of the filter pass-band and transition-band. It should be noted that this filtering technique is translation invariant, i.e. oscillation filtered from the translated version of the signal $x_\tau(t) = x(t - \tau)$, where $\tau \in R$, is simply the translated oscillation $o_x(\omega_c, t - \tau)$ filtered from the original signal $x(t)$. This property follows immediately from the formula $\hat{x}_\tau(\omega) = \int_{-\infty}^{+\infty} x(s-\tau)\exp(-i\omega s)ds = \hat{x}(\omega)\exp(-i\omega\tau)$ and from the definition of the inverse CFT (Bremaud 2002).

As it was already remarked in the author's earlier work (Popiński 2008) the FTBPF filtration with the use of such transmittance functions is directly related to the Harmonic Wavelet Transform (HWT) (Newland 1998). The HWT coefficients of the analyzed signal $x(t)$, corresponding to the selected harmonic wavelet function $h \in L^2(R)$ with $CFT[h(s)] = \hat{h}(\omega_c, \omega) = A(\omega_c, \omega)$, are determined as follows

$$H(\omega_c, \tau) = \int_{-\infty}^{+\infty} x(t)\bar{h}(t-\tau)dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{x}(\omega)\bar{\hat{h}}(\omega_c, \omega)\exp(i\tau\omega)d\omega = o_x(\omega_c, \tau) \quad , \qquad (2)$$

where $\tau \in R$ is the translation parameter, and the second equality follows easily from the classical Plancherel identity for the CFT, in which over-bar denotes complex conjugation and $i = \sqrt{-1}$ (Bremaud 2002).

For sufficiently regular transmittance functions like the ones mentioned above we immediately obtain from the definition of the inverse CFT (Bremaud 2002)

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{h}(\omega_c, \omega)\exp(it\omega)d\omega = \frac{1}{2\pi}\int_{-2\pi\lambda}^{2\pi\lambda} \hat{h}(0, \upsilon)\exp(it(\upsilon + \omega_c))d\upsilon = \exp(it\omega_c)h_0(t) \quad ,$$

where $h_0(t)$ is the harmonic wavelet function with $\hat{h}_0(\omega) = \hat{h}(0, \omega) = A(0, \omega)$. Simple analytical integration shows that the harmonic wavelet functions $h_{R0}(t),\ h_{T0}(t)$ with the CFTs $A_R(0, \omega),\ A_T(0, \omega)$, respectively, are given by the following formulae:

$$h_{R0}(t) = \frac{\sin(2\pi\lambda t)}{\pi t} \quad \text{for } t \neq 0, \ h_{R0}(0) = 2\lambda,$$

$$h_{T0}(t) = \frac{1}{2\pi^2 t^2 \alpha}\left[\cos(2\pi\lambda t) - \cos(2\pi(\lambda + \alpha)t)\right] \quad \text{for } t \neq 0, \ h_{T0}(0) = 2\lambda + \alpha .$$

The graphs of the chosen harmonic wavelet CFTs (FTBPF transmittance functions $A_R(0, \omega),\ A_T(0, \omega)$) are presented in Fig. 1 and the graphs of the wavelet functions $h_{R0}(t),\ h_{T0}(t)$ themselves in Fig. 2.

Putting $h(t) = \exp(i\omega_c t)h_0(t)$ in the definition (2) yields the equality

$$H(\omega_c, \tau) = \int_{-\infty}^{+\infty} x(t)\bar{h}(t-\tau)dt = \int_{-\infty}^{+\infty} x(t)\bar{h}_0(t-\tau)\exp(-i\omega_c(t-\tau))dt = C(\omega_c, \tau)\exp(i\omega_c\tau) \quad ,$$

where $C(\omega_c, \tau) = \int_{-\infty}^{+\infty} x(t)\bar{h}_0(t-\tau)\exp(-i\omega_c t)dt$. From the formula defining the function $C(\omega_c, \tau)$ one can see immediately that it is obtained by complex demodulation (Hasan 1983) of the analyzed signal $x(t)$ with application of the low-pass filter of the demodulated signal $x(t)\exp(-i\omega_c t)$ using the transmittance function $\hat{h}_0(\omega) = \hat{h}(0, \omega) = A(0, \omega)$. Hence, one can see the relation of the FTBPF technique to the complex demodulation method (Hasan 1983).

Let us observe now that the above defined filtering technique with rectangular transmittance function $A_R(\omega_c, \omega)$ passes without distortion any function $y \in L^2(R)$ which satisfies the condition $\hat{y}(\omega) = 0$ for $|\omega - \omega_c| > 2\pi\lambda$, i.e. the CFT of which has support included in the interval $|\omega - \omega_c| \le 2\pi\lambda$. Indeed, according to (2) and the inverse CFT formula we have

$$o_y(\omega_c, t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{y}(\omega) A_R(\omega_c, \omega)\exp(it\omega)d\omega = \frac{1}{2\pi}\int_{\omega_c - 2\pi\lambda}^{\omega_c + 2\pi\lambda} \hat{y}(\omega)\exp(it\omega)d\omega = y(t) \ .$$

As it was shown in the author's earlier work on a related subject (Popiński 2008) this property holds also for monochromatic oscillations $o_0(t) = a\exp(i\omega_0 t + i\varphi)$, where $|\omega_0 - \omega_c| \le 2\pi\lambda$, $a > 0$ and φ are constant frequency, amplitude and phase parameters, respectively. Clearly, such oscillations are not square integrable functions (i.e. $o_0 \notin L^2(R)$) so the property of passing without distortion holds for a broader class of functions than described above.

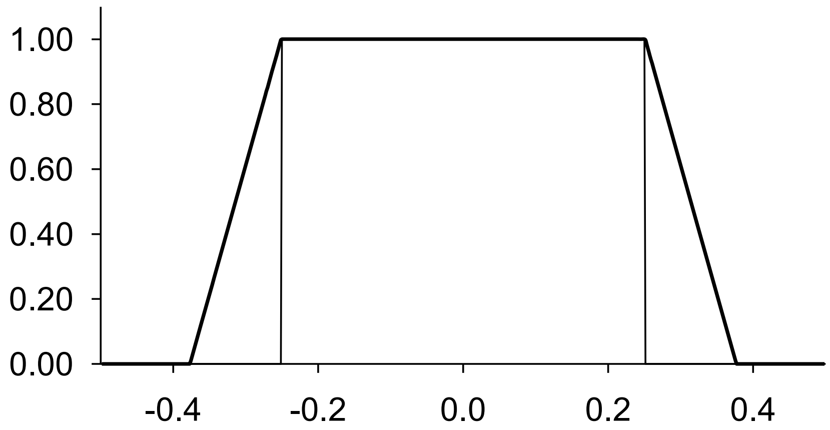

Fig. 1. The FTBPF boxcar ($\lambda = 0.04$, thin line) and trapezial ($\lambda = 0.04$, $\alpha = 0.02$, bold line) transmittance functions (the chosen harmonic wavelet CFTs).

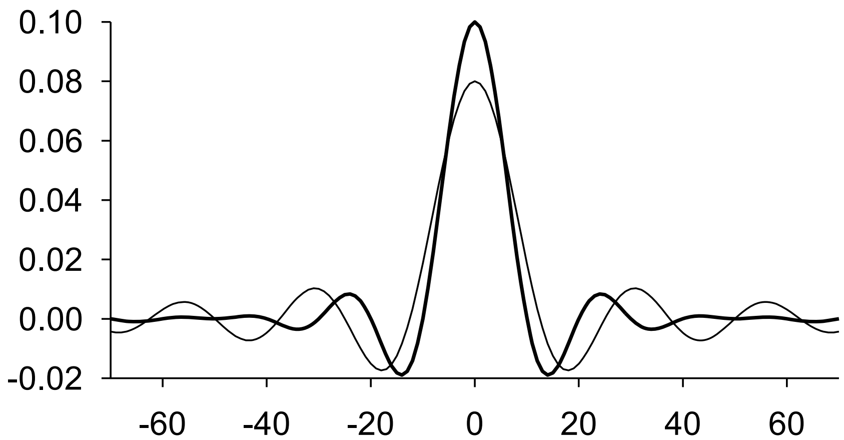

Fig. 2. The harmonic wavelet functions corresponding to the boxcar ($\lambda = 0.04$, thin line) and trapezial ($\lambda = 0.04$, $\alpha = 0.02$, bold line) CFTs.

The simplest examples of square integrable functions which are passed without distortion by the analyzed filter with rectangular transmittance function $A_R(\omega_c, \omega)$ are provided by the functions $h_R(t) = h_{R0}(t)\exp(i\omega_c t)$ and $h_T(t) = h_{T0}(t)\exp(i\omega_c t)$. One can clearly see from the above given formulae that the functions $h_{R0}(t)$ and $h_{T0}(t)$ decrease to zero asymptotically as fast as $|t|^{-1}$ and $|t|^{-2}$, respectively, as $|t|$ tends to infinity. Consequently, the functions $h_R(t)$ and $h_T(t)$ behave like monochromatic oscillations with decaying amplitudes. Other examples of functions with analogous properties can be found in (Popiński 2008). Of course, since the FTBPF technique is linear and translation invariant also time translations of such functions and their finite linear combinations are not distorted by the filter with the boxcar transmittance function. Furthermore, from linear combinations of translations of such functions one can easily construct signals which can clearly have transient, nonstationary character, but are passed undistorted by the chosen filter. The only requirement for this to hold is that they are of the form

$$y(t) = \left[\sum_{k=1}^{K} c_k h_{0k}(t-\tau_k) \exp(-i\omega_c \tau_k) \right] \exp(i\omega_c t) \quad ,$$

where c_k are complex-valued coefficients, $\tau_k \in R$ are translation parameters and the functions $h_{0k}(t)$ have CFTs $\widehat{h}_{0k}(\omega) = A_k(0,\omega)$, which satisfy the condition $\widehat{h}_{0k}(\omega) = 0$ for $|\omega| > 2\pi\lambda$, $k = 1,2,...,K$. Hence, for a function $y \in L^2(R)$ satisfying the above requirement we have $o_y(\omega_c, t) = y(t)$, when we use the boxcar transmittance function in the FTBPF. This implies that the FTBPF technique can be useful in filtering monochromatic oscillations with irregular time variable amplitudes, which is evidently an asset of this technique. Moreover, the class of signals which are passed undistorted by the considered filter contains oscillations of rather unusual and complicated character like the so called superoscillations described in the work of Ferreira and Kempf (2006) .

3. THE BOUNDARY EFFECTS IN THE FTBPF

In practice only time series of the analyzed signal observation values at discrete time moments $x(n\Delta t)$, $n = 0,1,...,N-1$, where $\Delta t > 0$ denotes the sampling interval, is available. Then, we can compute approximately the values of the HWT, or equivalently the values of the oscillation filtered by the FTBPF, at discrete time moments $\tau\Delta t$, $\tau = 0,1,...,N-1$, according to the formula applied in the case of wavelet transforms (Torrence and Compo 1998)

$$H(\omega_c, \tau\Delta t) = o_x(\omega_c, \tau\Delta t) \cong \Delta t \sum_{n=0}^{N-1} x(n\Delta t)\overline{h}((n-\tau)\Delta t) \quad . \tag{3}$$

However, in view of definition (2) the sum in the above formula approximates only the mid integral in the following decomposition of $H(\omega_c, \tau\Delta t)$:

$$H(\omega_c, \tau\Delta t) = \int_{-\infty}^{0} x(t)\overline{h}(t-\tau\Delta t)dt + \int_{0}^{N\Delta t} x(t)\overline{h}(t-\tau\Delta t)dt + \int_{N\Delta t}^{+\infty} x(t)\overline{h}(t-\tau\Delta t)dt \quad . \tag{4}$$

We can nevertheless try to estimate the values of the two remaining integrals which are not comprised in the approximate formula (3). Namely, since $x, h \in L^2(R)$ Schwartz inequality (Bremaud 2002) and changing the integration variables yields

$$\left| \int_{-\infty}^{0} x(t)\overline{h}(t-\tau\Delta t)dt \right|^2 \le \int_{-\infty}^{0} |x(t)|^2 dt \int_{-\infty}^{0} |h(t-\tau\Delta t)|^2 dt = \int_{-\infty}^{0} |x(t)|^2 dt \int_{-\infty}^{-\tau\Delta t} |h(s)|^2 ds \ ,$$

$$\left| \int_{N\Delta t}^{\infty} x(t)\overline{h}(t-\tau\Delta t)dt \right|^2 \le \int_{N\Delta t}^{\infty} |x(t)|^2 dt \int_{N\Delta t}^{\infty} |h(t-\tau\Delta t)|^2 dt = \int_{N\Delta t}^{\infty} |x(t)|^2 dt \int_{(N-\tau)\Delta t}^{\infty} |h(s)|^2 ds \ .$$

If we use in our computations the above introduced harmonic wavelet function corresponding to the boxcar transmittance function $h_R(t) = h_{R0}(t)\exp(i\omega_c t)$, where $h_{R0}(t) = \frac{\sin(2\pi\lambda t)}{\pi t}$ for $t \neq 0$, then we have $|h_R(t)| \leq \pi^{-1} |t|^{-1}$. Consequently, we easily obtain the inequalities

$$\int_{-\infty}^{-\tau\Delta t} | h_R(s) |^2 ds \leq \frac{1}{\pi^2} \int_{-\infty}^{-\tau\Delta t} \frac{1}{s^2} ds = \frac{1}{\pi^2 \tau\Delta t} \quad ,$$

$$\int_{(N-\tau)\Delta t}^{\infty} | h_R(s) |^2 ds \leq \frac{1}{\pi^2} \int_{(N-\tau)\Delta t}^{\infty} \frac{1}{s^2} ds = \frac{1}{\pi^2 (N-\tau)\Delta t} \quad .$$

Thus, for the chosen harmonic wavelet function we finally obtain the estimates

$$\left| \int_{-\infty}^{0} x(t)\bar{h}_R(t-\tau\Delta t)dt \right| \leq \left(\int_{-\infty}^{0} | x(t) |^2 dt \right)^{1/2} \frac{1}{\pi\sqrt{\tau\Delta t}} \quad ,$$

$$\left| \int_{N\Delta t}^{\infty} x(t)\bar{h}_R(t-\tau\Delta t)dt \right| \leq \left(\int_{N\Delta t}^{\infty} | x(t) |^2 dt \right)^{1/2} \frac{1}{\pi\sqrt{(N-\tau)\Delta t}} \quad .$$

From the above estimates it can be seen that the errors related to neglecting the integrals over the two infinite intervals in (4) can have small values for $\tau = M, M+1, ..., N-M$, where $M < N/2$ is a fixed positive integer depending on the signal and the harmonic wavelet function used. On the other hand it is evident that they can be much more significant for $\tau = 0, 1, ..., M-1$, and $\tau = N-M+1, N-M+2, ..., N-1$, which explains the occurrence of boundary effects in FTBPF results, observed by Popiński and Kosek (1995). This fact is also in agreement with conclusions concerning filtration of stationary time series of stochastic character using the FTBPF technique, obtained by Koopmans (1974). Furthermore, using the smoother harmonic wavelet function corresponding to the trapezial transmittance function $h_T(t) = h_{T0}(t)\exp(i\omega_c t)$, where $h_{T0}(t) = \frac{1}{2\pi^2 t^2 \alpha}\left[\cos(2\pi\lambda t) - \cos(2\pi(\lambda+\alpha)t)\right]$ for $t \neq 0$, in the above derivation gives bounds on the neglected integrals values, in which the right hand side depends on $c(\alpha)(\tau\Delta t)^{-3/2}$ and $c(\alpha)((N-\tau)\Delta t)^{-3/2}$, respectively, where $c(\alpha) = (\sqrt{3}\pi^2\alpha)^{-1}$ depends only on α. Thus, as one can plainly see, using smoother transmittance functions can reduce the described boundary effects. This conjecture confirms conclusions of empirical studies concerning the FTBPF characteristics (Forbes 1988), (Popiński and Kosek 1995). Deformation at the ends of analyzed series time span must be also taken into account when assessing results obtained by some other commonly used filters like Vondrak filter (Dong and Zheng 1985), (Liao and Liao 2001).

One may argue that deformation of filtered oscillations at the ends of analyzed series time span can be different when we implement the frequency domain formula using the Discrete Fourier Transform (DFT) for computing oscillations (Popiński 2008) instead of time domain convolution used in (3) above. In fact, Evans (1985) demonstrated that filtering based on the DFT transform should result in exactly the same amount of data loss at the ends as the equivalent time domain filter.

Similar problem of boundary effects occurrence must be dealt with in the case of the classical wavelet transform implementation (Torrence and Compo 1998), (Zheng et al. 2000). For example, the time domain formula for the Continuous Wavelet Transform coefficients with respect to the Morlet wavelet reads (Gasquet and Witomski 1999)

$$W(a,\tau) = \frac{1}{\sqrt{|a|}} \int_{-\infty}^{+\infty} x(t)\overline{\psi}_M((t-\tau)/a)dt \quad ,$$

where $a \neq 0$ is the period parameter, $\tau \in R$ is the translation parameter of the transform and $\psi_M(t) \cong (\sqrt{2\pi}\sigma)^{-1} \exp(-t^2/2\sigma^2)\exp(i2\pi t)$ is the Morlet wavelet function with parameter $\sigma > 0$ which controls its decay in time and frequency domain (Schmitz-Hübsch and Schuh 1999). Now, derivation analogous to the one presented above in the case of the HWT yields the following estimates

$$\left| \frac{1}{\sqrt{|a|}} \int_{-\infty}^{0} x(t)\overline{\psi}_M((t-\tau\Delta t)/a)dt \right| \leq \left(\int_{-\infty}^{0} | x(t) |^2 dt \right)^{1/2} \left(\frac{1}{2\sqrt{\pi}\sigma} erf\left(-\frac{\sqrt{2}\tau\Delta t}{\sigma|a|} \right) \right)^{1/2} ,$$

$$\left| \frac{1}{\sqrt{|a|}} \int_{N\Delta t}^{\infty} x(t)\overline{\psi}_M((t-\tau\Delta t)/a)dt \right| \leq \left(\int_{N\Delta t}^{\infty} | x(t) |^2 dt \right)^{1/2} \left(\frac{1}{2\sqrt{\pi}\sigma} erf\left(-\frac{\sqrt{2}(N-\tau)\Delta t}{\sigma|a|} \right) \right)^{1/2} ,$$

where $erf(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-t^2/2)dt$ is the standard error function, $\tau = 0,1,...,N-1$, and N is the number of available data $x(n\Delta t)$, $n = 0,1,...,N-1$, used to compute the discrete version of the Morlet Wavelet Transform (Torrence and Compo 1998). One can clearly see from the above formulae that the smaller are the values of the parameter $a \neq 0$ the smaller is magnitude of integrals related to boundary effects. The same holds also in the case of σ parameter, which follows easily from known estimates of the standard error function values (Johnson et al. 1994).

4. RECOGNITION OF OSCILLATION POLARIZATION

A periodic term of polar motion with frequency ω reads in complex notation (Jochmann and Felsmann 2001)

$$m_\omega(t) = A_+ \exp(i\omega t + i\varphi_+) + A_- \exp(-i\omega t - i\varphi_-) \quad , \tag{5}$$

which is a sum of prograde and retrograde components with constant amplitudes $A_+ \geq 0$, $A_- \geq 0$ and phases φ_+, $\varphi_- \in R$, respectively, and represents elliptical motion of the pole. Putting $\varphi_+ = \alpha + \beta$ and $\varphi_- = \beta - \alpha$, which yields immediately $\alpha = (\varphi_+ - \varphi_-)/2$ and $\beta = (\varphi_+ + \varphi_-)/2$, we can rewrite the equation (5) in the form

$$m_\omega(t) = \exp(i\alpha)\left[(A_+ + A_-)\cos(\omega t + \beta) + i(A_+ - A_-)\sin(\omega t + \beta)\right] \ ,$$

and consequently for $a = A_+ + A_-$ (semi-major axis) and $b =| A_+ - A_- |$ (semi-minor axis) this equation describes periodical motion on the ellipse rotated by the angle α. However, for $A_+ > A_-$ this motion is prograde (i.e. $\omega > 0$ - positive polarization) and for $A_+ < A_-$ it is retrograde (i.e. $\omega < 0$ - negative polarization). The aim of this section is to propose an estimator which can help us recognize the selected oscillation polarization on the basis of available finite duration time series observations.

In section 2 it was shown that for a signal $x \in L^2(R)$ the theoretical values of the HWT coefficients, or equivalently the filtered oscillation values, given by equation (2) can be expressed as $H(\omega_c,\tau) = o_x(\omega_c,\tau) = C(\omega_c,\tau)\exp(i\tau\omega_c)$, where $C(\omega_c,\tau)$ is a complex valued function. Thus, if we use the FTBPF technique with the same transmittance function to filter two oscillations corresponding to central frequencies $\omega_c > 0$ and $-\omega_c < 0$, then we can say that $|o_x(\omega_c,\tau)|$ and $|o_x(-\omega_c,\tau)|$ approximate the instantaneous amplitudes of the prograde and retrograde components.

Assume now that time series of the analyzed signal observations y_n, $n = 0,1,...,N-1$, at discrete time moments is available, according to the model $y_n = x(n\Delta t) + \eta_n$, where $\Delta t > 0$ denotes the sampling interval, η_n, $n = 0,1,...,N-1$, are uncorrelated complex valued random variables (observation errors) having zero-mean $E\eta_n = 0$ and finite variance $E|\eta_n|^2 = \sigma_\eta^2 > 0$. Then, we can compute the values of the discretized HWT (or equivalently the discretized FTBPF oscillation values) of the series y_n, $n = 0,1,...,N-1$, according to the formula (Popiński 2008)

$$\breve{o}_y(\pm\omega_c,\tau) = \frac{1}{N}\sum_{n=0}^{N-1} y_n \sum_{\nu=-N/2+1}^{N/2} \overline{\hat{h}}(\pm\omega_c, 2\pi\nu/N)\exp(i2\pi\nu(\tau-n)/N)$$

for the chosen central frequencies $\omega_c > 0$ and $-\omega_c < 0$, at the time moments $\tau\Delta t$, $\tau = 0,1,...,N-1$. Clearly, linearity of the FTBPF assures that $\breve{o}_y(\pm\omega_c,\tau\Delta t) = \breve{o}_x(\pm\omega_c,\tau\Delta t) + \breve{o}_\eta(\pm\omega_c,\tau\Delta t)$, i.e. oscillation values determined from observations are sums of the oscillations values $\breve{o}_x(\pm\omega_c,\tau\Delta t)$ corresponding to the series of exact signal values $x(n\Delta t)$, $n = 0,1,...,N-1$, and the oscillation values $\breve{o}_\eta(\pm\omega_c,\tau\Delta t)$ corresponding to the noise series η_n, $n = 0,1,...,N-1$. In the work (Popiński 1997) it was proved that $E\breve{o}_\eta(\pm\omega_c,\tau\Delta t) = 0$ and $E|\breve{o}_\eta(\pm\omega_c,\tau\Delta t)|^2 \le \sigma_\eta^2(2N\lambda+1)^2/N$ for $\tau = 0,1,...,N-1$, provided the above formulated assumptions concerning the observation errors are satisfied. Since the values $\breve{o}_x(\pm\omega_c,\tau\Delta t)$ approximate the theoretical values $o_x(\pm\omega_c,\tau\Delta t)$ and $\breve{o}_y(\pm\omega_c,\tau\Delta t)$ are unbiased estimators of $\breve{o}_x(\pm\omega_c,\tau\Delta t)$, we can use them to estimate the analyzed elliptical oscillation instantaneous prograde and retrograde component amplitude values. Namely, we can put

$$\breve{A}_+(\tau\Delta t) = |\breve{o}_y(\omega_c,\tau\Delta t)| \quad , \quad \breve{A}_-(\tau\Delta t) = |\breve{o}_y(-\omega_c,\tau\Delta t)|$$

and consequently $\Delta\breve{A}(\tau\Delta t) = \breve{A}_+(\tau\Delta t) - \breve{A}_-(\tau\Delta t)$ is an estimator of the difference between instantaneous prograde and retrograde component amplitudes $\Delta A(\tau\Delta t) = A_+(\tau\Delta t) - A_-(\tau\Delta t)$, where $A_+(\tau\Delta t) = |\breve{o}_x(\omega_c,\tau\Delta t)|$ and $A_-(\tau\Delta t) = |\breve{o}_x(-\omega_c,\tau\Delta t)|$, respectively. Moreover, we can estimate the mean-square error of $\Delta\breve{A}(\tau\Delta t)$ as follows

$$\sqrt{E|\Delta\breve{A}(\tau\Delta t) - \Delta A(\tau\Delta t)|^2} = \sqrt{E|(\breve{A}_+(\tau\Delta t) - A_+(\tau\Delta t)) - (\breve{A}_-(\tau\Delta t) - A_-(\tau\Delta t))|^2}$$

$$\le \sqrt{E|\breve{A}_+(\tau\Delta t) - A_+(\tau\Delta t)|^2} + \sqrt{E|\breve{A}_-(\tau\Delta t) - A_-(\tau\Delta t)|^2} \quad ,$$

where the triangle inequality is applied (Bremaud 2002), and further since

$$|\breve{A}_+(\tau\Delta t) - A_+(\tau\Delta t)| = ||\breve{o}_y(\omega_c,\tau\Delta t)| - |\breve{o}_x(\omega_c,\tau\Delta t)|| \le |\breve{o}_y(\omega_c,\tau\Delta t) - \breve{o}_x(\omega_c,\tau\Delta t)| = |\breve{o}_\eta(\omega_c,\tau\Delta t)| ,$$

$$|\breve{A}_-(\tau\Delta t) - A_-(\tau\Delta t)| = ||\breve{o}_y(-\omega_c,\tau\Delta t)| - |\breve{o}_x(-\omega_c,\tau\Delta t)|| \le |\breve{o}_y(-\omega_c,\tau\Delta t) - \breve{o}_x(-\omega_c,\tau\Delta t)| = |\breve{o}_\eta(-\omega_c,\tau\Delta t)|$$

we finally obtain

$$\sqrt{E\,|\,\Delta\breve{A}(\tau\Delta t)-\Delta A(\tau\Delta t)\,|^2} \le \sqrt{E\,|\,o_\eta(\omega_c,\tau\Delta t)\,|^2}+\sqrt{E\,|\,o_\eta(-\omega_c,\tau\Delta t)\,|^2} \le 2\sigma_\eta(2N\lambda+1)/\sqrt{N} \ .$$

In accordance with the discussion at the beginning of this section we recognize the elliptic oscillation as instantaneously positively polarized if $\Delta A(\tau\Delta t)>0$ and negatively polarized if $\Delta A(\tau\Delta t)<0$. Of course, since we can not compute $\Delta A(\tau\Delta t)$ from the available data because of observation errors, we must use the estimator $\Delta\breve{A}(\tau\Delta t)$ instead of $\Delta A(\tau\Delta t)$ for oscillation polarization recognition. Thus, we base our polarization recognition method on the sign of $\Delta\breve{A}(\tau\Delta t)$. Let us observe that the proposed oscillation polarization recognition will be erroneous only when $\Delta\breve{A}(\tau\Delta t)>0$ and $\Delta A(\tau\Delta t)<0$ or $\Delta\breve{A}(\tau\Delta t)<0$ and $\Delta A(\tau\Delta t)>0$, i.e. when $\Delta\breve{A}(\tau\Delta t)$ and $\Delta A(\tau\Delta t)$ have different signs. Furthermore, the probability of erroneous oscillation polarization recognition can be estimated as follows

$$P(sign\Delta\breve{A}(\tau\Delta t)\neq sign\Delta A(\tau\Delta t)) \le P(|\,\Delta\breve{A}(\tau\Delta t)-\Delta A(\tau\Delta t)\,| \ge |\,\Delta A(\tau\Delta t)\,|) \ ,$$

and using the Chebyshev inequality (Devroye et al. 1996) yields

$$P(|\,\Delta\breve{A}(\tau\Delta t)-\Delta A(\tau\Delta t)\,| \ \ge \ |\,\Delta A(\tau\Delta t)\,|) \le \frac{E\,|\,\Delta\breve{A}(\tau\Delta t)-\Delta A(\tau\Delta t)\,|^2}{|\,\Delta A(\tau\Delta t)\,|^2} \le \frac{4\sigma_\eta^2(2N\lambda+1)^2}{N\,|\,\Delta A(\tau\Delta t)\,|^2} \ .$$

Hence, if we use the FTBPF with pass-band width parameter $\lambda=m/N$, where $m>0$ is a small integer, then the probability of erroneous oscillation polarization recognition based on the proposed method will tend to zero as the number of observation N grows, provided $|\,\Delta A(\tau\Delta t)\,|$ will not tend to zero simultaneously. In such case our method of oscillation polarization recognition is statistically consistent (Devroye et al. 1996). It is easy to see that our recognition method may not be appropriate when $|\,\Delta A(\tau\Delta t)\,|$ is close to zero, which indicates that the instantaneous oscillation is nearly linear ($A_+ \cong A_-$).

5. SUMMARY AN OUTLOOK

The aspects of the FTBPF technique examined in this work seem to be crucial for understanding its applicability to extraction of oscillations having transient, nonstationary character. In the present study the FTBPF technique is related to the Harmonic Wavelet Transform (Newland 1998) so it is also connected with research on using wavelet transform approach to filtering of irregular oscillations (Fabert 2004), (Fabert and Schmidt 2003).

The method of obtaining upper bounds on the values of neglected integrals over infinite intervals, that are related to boundary effects in numerical implementation of the FTBPF, which is outlined in the section 3, can be applied to estimate values of analogous integrals in the case of the Continuous Wavelet Transform implementation (Torrence and Compo 1998). In view of the earlier investigations of Koopmans (1974) concerning boundary effects occurrence in the case of stochastic stationary time series filtration, the outcome of our analysis seems to extend his conclusions to the case of filtering deterministic signals of nonstationary character.

Polarization recognition method proposed in this work is statistically consistent and appropriate for recognition of elliptical oscillation polarization in the case of oscillations which occur in e.g. polar motion and which are not close to linear oscillations.

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