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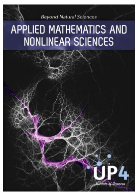


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# Boundary value problems for fractional differential equation with causal operators

Jingfei Jiang†, Dengqing Cao, Huatao Chen.

Division of Dynamics and Control, School of Astronautics, Harbin Institute of Technology, P.O. Box 137, Harbin, 150001, China P R

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**Abstract**

In this paper, we study the two-point boundary value problems for fractional differential equation with causal operator. By lower and upper solution method and the monotone iterative technique, some results for the extremal solution and quasisolutions are obtained. At last, an example is given to demonstrate the validity of assumptions and theoretical results.

**Keywords:** Fractional differential equation; Causal operator; Boundary value problems; Lower and upper solution; Extremal solution and quasisolutions.
**AMS 2010 codes:** 34B15, 34A45.

## 1 Introduction

In this paper, we consider the following two-point boundary value problem :

$$\begin{cases} {}^cD^{\alpha}u(t) = (Qu)(t), t \in J = [0,T], \\ g(u(0), u(T)) = 0, \end{cases} \tag{1}$$

where ${}^cD^{\alpha}$ is the Caputo fractional derivative with $0 < \alpha \leq 1$, $Q$ is a causal operator.

Fractional differential equation have proved to be valuable tools in modeling many phenomena in various fields of engineering, physics and economics, it draws a great application in nonlinear oscillations of earthquakes, seepage flow in porous media, fluid dynamics traffic model, to name but a few. Fractional differential equations have been studied extensively in recently years, for more details,one can see the monographs of [1, 2, 7, 11], and the journal literatures [13, 18, 20–22, 24–26]. In these previous works, Cauchy problems,optimal control problems,Numerical methods and the existence and uniquencess of solutions for various

†Corresponding author.
Email address: jjf860623@yeah.net(J.Jiang); dqcao@hit.edu.cn(D.Cao); htchencn@aliyun.com(H.Chen)

classes of initial and boundary value problems for fractional differential equations are discussed.

On the other hand, causal operators is adopted from the engineering literature and the theory of these operators has the powerful quality of unifying ordinary differential equations, integrodifferential equation, differential equations with finite and infinite delay, Volterra integral equations,neutral differential equations and so on. Recently, functional equations with causal operators are discussed (such as the monographs of [3, 4],and the research papers of [8, 27]). Fractional differential equations with causal operator in Banach spaces also have been studied (one can see [6, 9, 12, 28, 29],). the boundary value problems for integer order differential equation with causal operators have been concerned in [5].

As far as the authors are aware, the boundary value problems for fractional functional differential equations ( in form of Caputo derivative) with causal operator in infinite dimensional spaces have not been studied, it is just our interest in this paper. To get approximate solutions of (1), we can apply the monotone iterative technique,which has been investigated extensively, for detailed see [5, 10, 14–16]. The rest of this paper is organized as follows. In sect.2, Some notations and preparation results are given, some lemmas which are essential parts of the proof of our main results are proved by Schauder's fixed point theorem. Sect.3 is devoted to obtain the main results by monotone iterative technique and upper and lower solutions method to the extremal solutions and quasisolutions of the differential equation. At last, an examples is given to demonstrate the validity of assumptions and theoretical results in sect.4.

## 2 Preliminaries

We introduce some preliminaries which are used throughout the paper in this section. Let $E = C(J,\mathbb{R})$ be the space of all continuous functions $x : J \to \mathbb{R}$ with $J = [0,T]$. $Q \in C(E,E)$ is said to be a causal operator, or nonanticipative if the following property is satisfied: for each couple of elements $x,y$ of $E$ such that $x(s) = y(s)$ for $0 \le s \le t$, we also have $(Qx)(s) = (Qy)(s)$ for $0 \le s \le t,\ \ t < T$; for details see [7].

**Definition 1.** The Riemann-Liouville derivative of order $\alpha$ with the lower limit $t_0$ for a function $f : [t_0,\infty) \to R$ can be written as

$$^{L}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_{t_0}^{t}\frac{f(s)}{(t-s)^{1+\alpha-n}}ds,\ t > t_0,\ n-1 < \alpha < n.$$

**Definition 2.** The Caputo derivative of order $\alpha$ for a function $f : [t_0,\infty) \to R$ can be written as

$$^{c}D^{\alpha} f(t) = {}^{L}D^{\alpha}\left[f(t) - \sum_{k=0}^{n-1}\frac{(t-t_0)^k}{k!}f^{(k)}(t_0)\right],\ t > t_0,\ n-1 < \alpha < n.$$

**Lemma 1.** *[1] Let $R(\alpha) > 0$ and let $n$ be given by $n = [R(\alpha)]+1$ for $\alpha \notin N_0$, $\alpha = n$ for $\alpha \in N_0$. If $y(x) \in C^n[a,b]$, then*

$$(I^{\alpha}_{a+}\ {}^{c}D^{\alpha}y)(x) = y(x) - \sum_{k=0}^{n-1}\frac{y^{(k)}(a)}{k!}(x-a)^k,$$

*in particular, if $0 < R(\alpha) \le 1$ and $y(x) \in C^1[a,b]$, then*

$$(I^{\alpha}_{a+}\ {}^{c}D^{\alpha}y)(x) = y(x) - y(a).$$

**Lemma 2.** *[1]. Let $R(\alpha) \ge 0$ and let $n$ be given by $n = [R(\alpha)]+1$ for $\alpha \notin N_0$, $\alpha = n$ for $\alpha \in N_0$. If $y(x) \in C^n[a,b]$, then the Caputo fractional derivative ${}^{c}D^{\alpha}_{a}y(x)$ is continuous on $[a,b]$.*
*(a) If $\alpha \notin N_0$, then*

$$^{c}D^{\alpha}_{a}y(x) = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{x}(x-t)^{n-\alpha-1}y^{(n)}(t)dt =: (I^{n-\alpha}_{a+}D^n y)(x),$$

*where $D = d/dx$, in particular, if $0 < R(\alpha) < 1$ and $y(x) \in C^1[a,b]$, then*

$$^cD_a^\alpha y(x) =: (I_{a+}^{1-\alpha} Dy)(x).$$

*(b) If $\alpha = n \in N_0$, then the fractional derivative $^cD_a^n y(x) = y^{(n)}(x)$. In particular,*

$$^cD_a^0 y(x) = y(x).$$

It is necessary to state the Schauder's fixed point theorem which would be used in the proof of lemmas.

**Theorem 3.** *[4]. Let $E$ be a Banach space and $B \subset E$ be a convex, closed bounded set. If $T : E \to E$ is a continuous operator such that $TB \subset B$ and $T$ is relatively compact, then $T$ has a fixed point.*

Let us recall the definition of a solution of the fractional BVP (1).

**Definition 3.** A function $y \in C^1(J,\mathbb{R})$ is said to be a solution of the fractional BVP(1.1) if $y$ satisfies:
(i) $^cD^\alpha y(t) = (Qy)(t)$ a.e. on $J$,
(ii) $g(y(0), y(T)) = 0$.

We prove the following differential inequalities with positive linear operator $L$ which are important in obtaining our main results.

**Lemma 4.** *Assume that $L \in C(E,E)$ is a positive linear operator. Let $m \in C^1(J,\mathbb{R})$ satisfy:*

$$\begin{cases} ^cD^\alpha m(t) \le -(Lm)(t), & t \in J, \\ m(0) \le rm(T), & r \in [0,1] \end{cases} \tag{2}$$

*and the condition holds*

$$\sup_{t\in J} \left\{ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (L\mathbf{1})(s)ds \right\} \le 1, \tag{3}$$

*where $\mathbf{1}(t) = 1$, $t \in J$. Then $m(t) \le 0$, $t \in J$.*

*Proof.* Case 1. Suppose $m(0) \le 0$. We need to show that $m(t) \le 0, t \in J$. Moreover, if $r = 0$, then $m(0) \le 0$. Assume the above inequality is not true. Then, there exists $t_0 \in (0,T]$ such that $m(t_0) > 0$. Let

$$m(t_1) = \min_{[0,t_0]} m(t) \le 0.$$

Applying the fractional integration operator $I_{t_1+}^\alpha$ to the both sides of the differential inequality in (2), we can get

$$I_{t_1+}^\alpha(^cD^\alpha m(t)) \le -I_{t_1+}^\alpha(Lm)(t),$$

thus, by Lemma 1 and the condition (3), we have

$$\begin{aligned} m(t_0) - m(t_1) &\le -\frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_0} (t_0-s)^{\alpha-1}(Lm)(s)ds \\ &\le -\frac{m(t_1)}{\Gamma(\alpha)} \int_{t_1}^{t_0} (t_0-s)^{\alpha-1}(L\mathbf{1})(s)ds \\ &\le -m(t_1). \end{aligned}$$

then $m(t_0) \le 0$. This is a contradiction.
Case 2. Assume $m(0) > 0$. Note that $m(T) > 0$. there are two situations:

(2a) $r=1$, (2b) $0<r<1$.
Subcase 2a. Let $r=1$.
Subcase 2a(i).Suppose $m(t)\geq 0$ on $J$ and $m(t)\not\equiv 0$. then

$$m(t)-m(0)\leq -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(Lm)(s)ds$$

take the boundary conditions into account, it can be obtain

$$m(0)\leq m(T)\leq m(0)-\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}(Lm)(s)ds.$$

Hence $\int_0^T (T-s)^{\alpha-1}(Lm)(s)ds\leq 0$ which is a contradiction.
Subcase 2a(ii). Let $m(t)<0,\ \ t\in(0,T]$. Put

$$m(t_1)=\min_{t\in J} m(t)=-\lambda,\ \ \lambda>0.$$

Then ${}^cD^{\alpha}m(t)\leq -(Lm)(t)\leq \lambda(L\mathbf{1})(t), t\in J$.
Taking the fractional integration operator $I^{\alpha}_{t_1+}$ on the both sides of the differential inequality in (2) , we see that

$$\begin{aligned} m(T)-m(t_1) &\leq \frac{\lambda}{\Gamma(\alpha)}\int_{t_1}^T (T-s)^{\alpha-1}(L\mathbf{1})(s)ds \\ m(T)+\lambda &\leq \frac{\lambda}{\Gamma(\alpha)}\int_{t_1}^T (T-s)^{\alpha-1}(L\mathbf{1})(s)ds\leq \lambda. \end{aligned}$$

Then $m(T)\leq 0$. This is a contradiction.
Subcase 2b. Let $0<r<1$.
Subcase 2b(i). Let $m(t)\geq 0$ on J and $m(t)\not\equiv 0$. Then, in view of the boundary conditions, we have

$$\frac{1}{r}m(0)\leq m(T)\leq m(0)-\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}(Lm)(s)ds.$$

So

$$m(0)\leq -\frac{r}{1-r}\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}(Lm)(s)ds.$$

Hence

$$m(t)\leq -\frac{r}{1-r}\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}(Lm)(s)ds-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(Lm)(s)ds\leq 0.$$

Since $m(t)\geq 0,\ \ t\in J$. This means that $m(t)\equiv 0$. This is a contradiction.
Subcase 2b(ii). Let $m(t)<0$. Put

$$m(t_1)=\min_{t\in J} m(t)=-\lambda,\ \ \lambda>0.$$

Then ${}^cD^{\alpha}m(t)\leq -(Lm)(t)\leq \lambda(L\mathbf{1})(t)$. In the same way as before, we see that

$$m(T)+\lambda\leq\lambda.$$

We can obtain $m(T)\leq 0$,this is a contradiction. Thus,the proof is completed.

Define $I^0 f(t)=f(t)$, for $f(t)\in C(J,\mathbb{R}),\ \ t\in J$, the following holds:

**Lemma 5.** *Let $L \in C(E,E)$ be a positive linear operator and $K \in C(J,\mathbb{R})$. Let $m \in C^1(J,\mathbb{R})$ satisfy:*

$$\begin{cases} {}^cD^\alpha m(t) \le -K(t)m(t) - (Lm)(t), & t \in J, \\ m(0) \le rm(T), & r \in [0,1] \end{cases} \tag{4}$$

*with $0 \le r\overline{q}(T) \le 1$ for $\overline{q}(t) = e^{-\int_0^t K(s)ds}$. In addition, we assume that*

$$\sup_{t\in J} \left\{ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} I_{0+}^{1-\alpha}[e^{\int_0^s K(\tau)d\tau}(L\overline{q})](s)ds \right\} \le 1. \tag{5}$$

*Then $m(t) \le 0, \ t \in J$.*

*Proof.* Set $q(t) = e^{\int_0^t K(s)ds} m(t)$, then

$${}^cD^\alpha q(t) = I_{0+}^{1-\alpha} q'(t) \le -I_{0+}^{1-\alpha} e^{\int_0^t K(s)ds}(L\widetilde{q})(t)$$

where, $\widetilde{q} = \overline{q}(t)q(t)$. Thus, system (4) takes the from

$$\begin{cases} {}^cD^\alpha q(t) \le -I_{0+}^{1-\alpha} e^{\int_0^t K(s)ds}(L\widetilde{q})(t), & t \in J, \\ q(0) \le r_1 q(T), \ \ for & r_1 = r\overline{q}(T). \end{cases}$$

According to the Lemma 4, the proof is complete.

**Lemma 6.** *Let $L \in C(E,E)$ be a positive linear operator and $K, \ \sigma \in C(J,\mathbb{R})$. Assume that condition (5) holds, $0 \le r_1 < 1$ with $r_1 = re^{-\int_0^T K(s)ds}$. Then the linear problem*

$$\begin{cases} {}^cD^\alpha v(t) = -K(t)v(t) - (Lv)(t) + \sigma(t), & t \in J, \\ v(0) = rv(T) + \beta, & \beta \in \mathbb{R}, \end{cases} \tag{6}$$

*for $v(t) \in C^1(J,\mathbb{R})$ has a unique solution $u \in C^1(J,\mathbb{R})$.*

*Proof.* To begin with, we prove the problem owns at most one solution.

Assume that it has two different solutions $X, Y \in C^1(J,\mathbb{R})$. let $p = X - Y$,then $p$ satisfies the following problem

$$\begin{cases} {}^cD^\alpha p(t) = -K(t)p(t) - (Lp)(t), \\ p(0) = rp(T). \end{cases} \tag{7}$$

By the Lemma 5, we have $p(t) \le 0$, so $X(t) \le Y(t), \ t \in J$. On the other hand, Let $p = Y - X$, similarly, we can get $Y(t) \le X(t)$, $t \in J$. then $X = Y$.

We will show problem 7 has a solution.Put $u(t) = e^{\int_0^t K(s)ds} v(t)$,then system 7 takes the form

$$\begin{cases} {}^cD^\alpha u(t) = -(B^*u)(t) + \sigma^*(t), & t \in [0,T], \\ u(0) = r_1 u(T) + \beta, & \beta \in \mathbb{R}, \end{cases} \tag{8}$$

where$B^* = I_{0+}^{1-\alpha}[e^{\int_0^t K(s)ds}(L\widetilde{u})](t), \ \ \widetilde{u} = ue^{-\int_0^t K(s)ds}, \ \ \sigma^* = I_{0+}^{1-\alpha}[e^{\int_0^t K(s)ds}\sigma](t)$.
Applying the fractional integration operator $I_{0+}^{\alpha}$ to (8),we have

$$\begin{aligned} u(t) = & -\frac{r_1}{1-r_1}\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}(B^*u)(s)ds - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(B^*u)(s)ds \\ & +\sigma^{**}(t) \equiv (Au)(t) \end{aligned}$$

where

$$\sigma^{**}=\frac{r_1}{1-r_1}\frac{1}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}\sigma^*(s)ds+\frac{\beta}{1-r_1}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\sigma^*(s)ds.$$

Let $v_n\in C(J,\mathbb{R})$ and $v_n\to v$ in $C(J,\mathbb{R})$, then $u_n\in C(J,\mathbb{R}),\ u_n\to u in C(J,\mathbb{R})$,here

$$u_n(t)=e^{\int_0^t K(s)ds}v_n(t),\ \ u(t)=e^{\int_0^t K(s)ds}v(t).$$

Hence

$$\begin{aligned}|(Au_n)(t)-(Au)(t)| &\le \frac{r_1}{1-r_1}\frac{1}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}|(B^*u_n)(s)-(B^*u)(s)|ds\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}|(B^*u_n)(s)-(B^*u)(s)|ds\\ &\le \frac{r_1}{1-r_1}\frac{1}{\Gamma(\alpha)}\sup_{t\in[0,T]}|(B^*u_n)(t)-(B^*u)(t)|\int_0^T(T-s)^{\alpha-1}ds\\ &\quad+\frac{1}{\Gamma(\alpha)}\sup_{t\in[0,T]}|(B^*u_n)(t)-(B^*u)(t)|\int_0^t(t-s)^{\alpha-1}ds\\ &\le \frac{1}{1-r_1}\frac{T^\alpha}{\Gamma(\alpha+1)}\sup_{t\in[0,T]}|(B^*u_n)(t)-(B^*u)(t)|.\end{aligned}$$

Since

$$\begin{aligned}|(B^*u_n)(t)-(B^*u)(t)| &\le |I_{0+}^{1-\alpha}[e^{\int_0^t K(s)ds}(L\widetilde{u_n})](t)-I_{0+}^{1-\alpha}[e^{\int_0^t K(s)ds}(L\widetilde{u})](t)|\\ &\le \frac{1}{\Gamma(1-\alpha)}\int_0^t(t-s)^{-\alpha}e^{\int_0^s K(\tau)d\tau}|(L(\widetilde{u_n}-\widetilde{u}))(s)|ds\\ &\le \frac{1}{\Gamma(1-\alpha)}\sup_{t\in[0,T]}\left\{|(L\widetilde{u_n})(t)-(L\widetilde{u})(t)|e^{\int_0^t K(s)ds}\right\}\int_0^t(t-s)^{-\alpha}ds\\ &\le \frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\sup_{t\in[0,T]}\left\{|(L\widetilde{u_n})(t)-(L\widetilde{u})(t)|e^{\int_0^t K(s)ds}\right\}.\end{aligned}$$

By the condition that $L\in C(E,E)$ is a positive linear operator which implies $(L\widetilde{u_n})(t)\to(L\widetilde{u})(t)$ as $\widetilde{u_n}\to\widetilde{u}$, $n\to\infty$. So $|(B^*u_n)(t)-(B^*u)(t)|\to 0$ as $n\to\infty$ for $t\in J$. Thus, we have

$$\sup_{t\in J}|(Au_n)(t)-(Au)(t)|\to 0\ \ if\ \ n\to\infty,$$

So operator $A$ is continuous. Moreover,

$$\sup_{t\in J}|(Au)(t)|\le\sup_{t\in J}|(Au_n)(t)-(Au)(t)|+\sup_{t\in J}|(Au)(t)|,$$

the operator $A:C(J,\mathbb{R})\to C(J,\mathbb{R})$ is bounded. In addition, $\sup_{t\in[0,T]}|(B^*u)(t)|<\infty$. Moreover, for $t_1,\ t_2\in[0,T]$

with $t_1 \leq t_2$, we have

$$
\begin{aligned}
|(Au)(t_1)-(Au)(t_2)| \leq & \frac{1}{\Gamma(\alpha)}\int_0^{t_1} |(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}||(B^*u)(s)|ds \\
& +\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} |(t_2-s)^{\alpha-1}(B^*u)(s)|ds \\
& +\frac{1}{\Gamma(\alpha)}\int_0^{t_1} |(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}||\sigma^*(s)|ds \\
& +\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} |(t_2-s)^{\alpha-1}\sigma^*(s)|ds \\
\leq & \sup_{t\in[0,T]} |(B^*u)(t)|\frac{2(t_2-t_1)^{\alpha}}{\Gamma(\alpha+1)} + \sup_{t\in[0,T]} |\sigma^*(t)|\frac{2(t_2-t_1)^{\alpha}}{\Gamma(\alpha+1)} \\
\leq & [\sup_{t\in[0,T]} |\sigma^*(t)| + \sup_{t\in[0,T]} |(B^*u)(t)|]\frac{2(t_2-t_1)^{\alpha}}{\Gamma(\alpha+1)}.
\end{aligned}
$$

Since $\sigma(t)$ is continuous, then $\sup_{t\in[0,T]}|\sigma^*(t)| < \infty$. Thus, It is proved that the operator $A$ is equicontinuous on $J$, with Arzela-Ascoli theorem, It can be obtain that $A$ is compact. Hence, by Schauder's fixed point theorem, the operator $A$ has a fixed point $u \in C(J,\mathbb{R})$. On the other hand, $u'$ exists and $u' \in C(J,\mathbb{R})$. The proof is completed.

## 3 Main results

$u \in C^1(J,\mathbb{R})$ is called a lower solution of problem (1) if

$$
\begin{cases} {}^cD^{\alpha}u(t) \leq (Qu)(t), \quad t \in J, \\ g(u(0),u(T)) \leq 0 \end{cases}
$$

and it is an upper solution of problem (1) if the above inequalities are reversed.

A solution $u \in C^1(J,\mathbb{R})$ of problem (1) is called maximal if $x(t) \leq u(t),\ t \in J$, for each solution $x$ of problem (1 ), and minimal if the reverse inequality holds.

The existence results for the extremal solutions of problem (1) presented as following:

**Theorem 7.** *Assume that*
$H_1$*: $Q \in C(E,E)$ is a causal operator, $g \in C(R\times R,R)$,*
$H_2$*: $z_0,\ \ y_0 \in C^1(J,R)$ are lower and upper solutions of problem (1) respectively, and $z_0(t) \leq y_0(t),\ \ t \in J$,*
$H_3$*: there exists $K \in C(J,R)$ such that*

$$(Qu)(t)-(Q\overline{u})(t) \leq K(t)[\overline{u}(t)-u(t)]+(L(\overline{u}-u))(t),$$

*for $z_0(t) \leq u(t) \leq \overline{u}(t) \leq y_0(t),\ \ t \in J$,*
$H_4$*: there exist constants $0 \leq b \leq a,\ \ a > 0$ such that*

$$g(\overline{u},\overline{v})-g(u,v) \leq a(\overline{u}-u)-b(\overline{v}-v),$$

*for $z_0(0) \leq u \leq \overline{u} \leq y_0(0)$ and $z_0(T) \leq v \leq \overline{v} \leq y_0(T)$. Moreover, the condition(5)holds. then, there exists monotone sequences $\{z_n(t)\},\ \ \{y_n(t)\}$ such that $\lim_{n\to\infty} z_n(t) = p(t),\ \lim_{n\to\infty} \beta_n(t) = r(t)$, where $p,\ r$ are minimal and maximal solutions of problem(1), respectively, satisfying $z_0 \leq p(t) \leq r(t) \leq y_0$.*

*Proof.* Consider the linear BVP

$$
\begin{cases} {}^cD^{\alpha}u(t)+K(t)u(t) = -(Lu)(t)+\sigma_{\eta}(t), \\ u(0) = \eta(0)-\frac{1}{a}g(\eta(0),\eta(T))+r[u(T)-\eta(T)], \end{cases} \tag{9}
$$

where $\sigma_\eta(t) = (Q\eta)(t) + K(t)\eta(t) + (L\eta)(t),\ \ \eta \in C[J,R]$ and $z_0(t) \le \eta(t) \le y_0(t)$. By lemma 6 , the linear BVP (9) has a unique solution, we set $[z_0, y_0] = \{w \in C(J,\mathbb{R}) : z_0(t) \le w(t) \le y_0(t)\}$. Now we claim that any solution $u(t)$ of (9) satisfies $u(t) \in [z_0(t), y_0(t)],\ t \in J$. By the conditions $H_2,\ H_3$, we have

$$^cD^\alpha z_0(t) \le -K(t)z_0(t) - (Lz_0)(t) + \sigma_{z_0}(t)$$

and

$$^cD^\alpha u(t) = -K(t)u(t) - (Lu)(t) + \sigma_\eta(t) \ge -K(t)u(t) - (Lu)(t) + \sigma_{z_0}(t).$$

Put $p = z_0 - u$, we have

$$\begin{aligned} ^cD^\alpha p(t) &= {}^cD^\alpha z_0(t) - {}^cD^\alpha u(t) \\ &\le -K(t)z_0(t) - (Lz_0)(t) + \sigma_{z_0}(t) \ge +K(t)u(t) + (Lu)(t) - \sigma_{z_0}(t) \\ &= -K(t)p(t) - (Lp)(t) \end{aligned}$$

and

$$\begin{aligned} p(0) &= z_0(0) - u(0) \\ &= z_0(0) - \eta(0) + \frac{1}{a}g(\eta(0), \eta(T)) - r[u(T) - \eta(T)] \\ &\le rp(T). \end{aligned}$$

The last inequality is got by the condition $H_4$. Thus, from the Lemma 5, we have $p(t) \le 0$ and hence $z_0(t) \le u(t),\ t \in J$. Similarly, we can show that $u(t) \le y_0(t),\ t \in J$. Hence, we have $z_0(t) \le u(t) \le y_0(t)$.
Next consider the boundary value problem

$$\begin{cases} ^cD^\alpha y_{n+1}(t) = (Qy_n)(t) - (L(y_{n+1} - y_n))(t) - K(t)[y_{n+1}(t) - y_n(t)], \\ y_{n+1}(0) = y_n(0) - \frac{1}{a}g(y_n(0), y_n(T)) + r[y_{n+1}(T) - y_n(T)] \end{cases} \tag{10}$$

and

$$\begin{cases} ^cD^\alpha z_{n+1}(t) = (Qz_n)(t) - (L(z_{n+1} - z_n))(t) - K(t)[z_{n+1}(t) - z_n(t)], \\ z_{n+1}(0) = z_n(0) - \frac{1}{a}g(z_n(0), z_n(T)) + r[z_{n+1}(T) - z_n(T)]. \end{cases} \tag{11}$$

From the (9), we know problem (10) and (11) have a solution in the sector $[z_0(t),\ y_0(t)]$.
Next, we will show that

$$z_0(t) \le z_1(t) \le z_2(t) \le \cdots \le z_n(t) \le y_n(t) \le \cdots \le y_2(t) \le y_1(t) \le y_0(t), \quad t \in J.$$

First, we show that $z_0 \le z_1$. Now

$$^cD^\alpha z_0(t) \le (Qz_0)(t) \quad and \quad ^cD^\alpha z_1(t) = (Qz_0)(t) - (L(z_1 - z_0))(t) - K(t)[z_1(t) - z_0(t)].$$

Let $p = z_0 - z_1$. Then

$$\begin{aligned} ^cD^\alpha p(t) &= {}^cD^\alpha z_0(t) - {}^cD^\alpha z_1(t) \\ &\le (Qz_0)(t) - (Lz_0)(t) - (Qz_0)(t) + (Lz_1)(t) + K(t)[z_1(t) - z_0(t)] \\ &= -K(t)p(t) - (Lp)(t) \end{aligned}$$

and

$$p(0) = z_0(0) - z_1(0) = \frac{1}{a}g(z_0(0), z_0(T)) - r[z_1(T) - z_0(T)] \le rp(T).$$

By Lemma 5, we show that $z_0 \le z_1, \ \ t \in J$.
Assume $z_{k-1}(t) \le z_k(t), \ \ t \in J$. Let $p = z_k - z_{k+1}$, using hypothesis $(H_3)$ and simplifying, we obtain

$$\begin{aligned} {}^cD^\alpha p(t) &= {}^cD^\alpha z_k(t) - {}^cD^\alpha z_{k+1}(t) \\ &= (Qz_{k-1})(t) - (L(z_k - z_{k-1}))(t) - K(t)[z_k(t) - z_{k-1}(t)] \\ &\quad -(Qz_k)(t) + (L(z_{k+1} - z_k))(t) + K(t)[z_{k+1}(t) - z_k(t)] \\ &\le K(t)[z_k(t) - z_{k-1}(t)] - L(z_k - z_{k+1})(t) - K(t)[z_k(t) - z_{k-1}(t)] \\ &\quad +K(t)[z_{k+1}(t) - z_k(t)] \\ &= -K(t)p(t) - (Lp)(t) \end{aligned}$$

and

$$\begin{aligned} p(0) &= z_k(0) - z_{k+1}(0) \\ &= z_{k-1}(0) - \frac{1}{a} g(z_{k-1}(0), z_{k-1}(T)) + r[z_k(T) - z_{k-1}(T)] - z_k(0) \\ &\quad + \frac{1}{a} g(z_k(0), z_k(T)) - r[z_{k+1}(T) - z_k(T)] \\ &\le z_{k-1}(0) - z_k(0) + \frac{1}{a}[a(z_k(0) - z_{k-1}(0)) - b(z_k(T) - z_{k-1}(T))] \\ &\quad + r[z_k(T) - z_{k-1}(T)] - r[z_{k+1}(T) - z_k(T)] \\ &= rp(T). \end{aligned}$$

Again, using Lemma 5, we get $z_k(t) \le z_{k+1}(t), \ \ t \in J$, thus, by induction, we have

$$z_0(t) \le z_1(t) \le \cdots \le z_k(t), \ \ t \in J.$$

Similarly, we can show that

$$y_k(t) \le y_{k-1}(t) \le \cdots \le y_1(t) \le y_0(t), \ \ t \in J.$$

We next show that $z_n(t) \le y_n(t), \ \ t \in J, \ \ n = 1, 2, \cdots$.
Put $p = z_n - y_n$ and proceeding as before we arrive at

$${}^cD^\alpha p(t) = ({}^cD^\alpha z_n)(t) - ({}^cD^\alpha y_n)(t) \le -K(t)p(t) - (Lp)(t)$$

and $p(0) \le rp(T)$ which yields $z_n(t) \le y_n(t), \ \ t \in J, \ \ n = 1, 2, \cdots$, from Lemma 2.8. Hence, we have

$$z_0(t) \le z_1(t) \le \cdots \le z_n(t) \le y_n(t) \le \cdots \le y_1(t) \le y_0(t), \ \ t \in J.$$

It then follows, using standard arguments, that $\lim_{n\to\infty} z_n(t) = p(t)$ and $\lim_{n\to\infty} y_n(t) = r(t)$ uniform on $J$, and $p(t)$ and $r(t)$ are solutions of problem (1).
To show that $p(t)$ and $r(t)$ are extremal solutions of problem (1). Let $u(t)$ be any solution of problem (1) such that $u(t) \in [z_0, y_0]$, and suppose for some $k \ge 0, \ \ z_{k-1}(t) \le u(t) \le y_{k-1}(t), \ \ t \in J$.
Let $p(t) = z_k(t) - u(t)$. Then

$$\begin{aligned} {}^cD^\alpha p(t) &= {}^cD^\alpha z_k - {}^cD^\alpha u(t) \\ &= (Qz_{k-1})(t) - (L(z_k - z_{k-1}))(t) - K(t)[z_k(t) - z_{k-1}(t)] - (Qu)(t). \end{aligned}$$

Since $z_{k-1}(t) \le u(t)$, we have from the hypothesis $H_3$ of the theorem that

$${}^cD^\alpha p(t) \le -K(t)p(t) - (Lp)(t).$$

Also $p(0) \leq rp(T)$. Now applying Lemma 5, we get $z_k(t) \leq u(t)$. Similarly, $u(t) \leq y_k(t)$.Thus,from the induction principle, it follows that $z_n(t) \leq u(t) \leq y_n(t), \ \ for\ all\ n,\ t \in J$. Taking limits as $n \to \infty$, we obtain $p(t) \leq u(t) \leq r(t)$,hence $p(t)$ and $r(t)$ are extremal solutions of problem (1). The proof is complete.

We say that $u,\ w \in C^1(J,\mathbb{R})$ are coupled lower and upper solutions of problem (1) if

$$\begin{cases} {}^cD^{\alpha}u(t) \leq (Qu)(t), \quad t \in J \\ g(u(0),w(T)) \leq 0. \end{cases} \tag{12}$$

and

$$\begin{cases} {}^cD^{\alpha}w(t) \geq (Qw)(t), \quad t \in J \\ g(w(0),u(T)) \geq 0. \end{cases} \tag{13}$$

Functions $y,\ \ z \in C^1(J,\mathbb{R})$ are called quasisolutions of problem (1) if $y$ and $z$ satisfy the following system:

$$\begin{cases} {}^cD^{\alpha}y(t) = (Qy)(t), \quad \text{for}\ \ t \in J,\ \ g(y(0),z(T)) = 0, \\ {}^cD^{\alpha}z(t) = (Qz)(t), \quad \text{for}\ \ t \in J,\ \ g(z(0),y(T)) = 0. \end{cases} \tag{14}$$

The next theorem deals with the existence results of quasisolutions for problem (1).

**Theorem 8.** *Suppose that*
$\overline{H_2}$: *$z_0,\ y_0 \in C^1(J,\mathbb{R})$ are coupled lower and upper solutions of problem (1) and $y_0(t) \leq z_0(t),\ \ t \in J$,*
$\overline{H_4}$: *there exists constants $a > 0,\ \ b \geq 0$ such that*

$$\begin{cases} g(\bar{u},v) - g(u,v) \leq a(\bar{u} - u) \quad for\ \ y_0(0) \leq u \leq \bar{u} \leq z_0(0),\ \ t \in J, \\ g(u,v) - g(u,\bar{v}) \leq b(v - \bar{v}) \quad for\ \ y_0(T) \leq v \leq \bar{v} \leq z_0(T),\ \ t \in J. \end{cases} \tag{15}$$

*if assumptions $H_1,\ \overline{H_2},\ H_3, \overline{H_4}$ hold,then there exists a quasisolutions for problem (1) in the sector $[y_0, z_0]$ .*

*Proof.* We define sequences $\{y_n, z_n\}$ as followings:

$$\begin{cases} {}^cD^{\alpha}y_{n+1}(t) = (Qy_n)(t) - (L(y_{n+1} - y_n))(t) - K(t)[y_{n+1}(t) - y_n(t)],\ \ t \in J \\ y_{n+1}(0) = y_n(0) - \frac{1}{a}g(y_n(0), z_n(T)), \end{cases} \tag{16}$$

and

$$\begin{cases} {}^cD^{\alpha}z_{n+1}(t) = (Qz_n)(t) - (L(z_{n+1} - z_n))(t) - K(t)[z_{n+1}(t) - z_n(t)],\ \ t \in J \\ z_{n+1}(0) = z_n(0) - \frac{1}{a}g(z_n(0), y_n(T)), \end{cases} \tag{17}$$

for $n = 0, 1, \cdots$ .
In a way similar to the way we used in the proof of Theorem 7, we have

$$z_0(t) \leq z_1(t) \leq \cdots \leq z_n(t) \leq y_n(t) \leq \cdots \leq y_1(t) \leq y_0(t),\ \ t \in J.$$

Hence, $\{y_n\}$ and $\{z_n\}$ converge uniformly on $J$ to limit functions $y,\ z \in C^1(J,\mathbb{R})$.
Indeed, $y,\ \ z$ are the quasisolutions of problem (1). This ends the proof.

## 4 Illustrative example

In this section, we give the following example to demonstrate the validity of assumptions and theoretical results.

**Example 9.** *Consider the problem*

$$\begin{cases} {}^{c}D^{\alpha}x(t) = -M(t)x(t) - M(t)[1-\cos(x(\frac{1}{2}t))] - t\int_0^t s^2x(s)ds, \\ 0 = e^{x(0)} - x(1) - \frac{3}{2}, \end{cases} \tag{18}$$

*where* $M(t) \in C(J,[0,\infty), t \in J = [0,1]$, *and* $Q(t) = -M(t)x(t) + M(t)[1-\cos(x(\frac{1}{2}t))] - t\int_0^t s^2x(s)ds$. *Put* $y_0(t) = 0,\ \ z_0(t) = \gamma$ *with* $1 \le \gamma \le \frac{\pi}{2}$. *Then*

$$(Qy_0)(t) = 0 = ^{c} D^{\alpha}x(t),$$

$$(Qz_0)(t) = 0 = -M(t)\gamma + M(t)(1-\cos\gamma) - \gamma\frac{t^4}{3} \le M(t)(1-\gamma) \le 0 = ^{c} D^{\alpha}z_0(t),$$

$$g(y_0(0), y_0(1)) = -\frac{1}{2} \le 0,$$

$$g(z_0(0), z_0(1)) = e^{\gamma} - \gamma - \frac{3}{2} \ge 0.$$

*It means that* $y_0$, $z_0$ *are lower and upper solutions of problem (18), respectively. If we also assume that*

$$\sup_{t\in J}\left\{\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} I_0^{1-\alpha}[e^{\int_0^s M(\tau)d\tau} s\int_0^s \xi^2 e^{-\int_0^{\xi} M(\tau)d\tau}d\xi](s)ds\right\} \le 1.$$

*Then, the problem has extremal solutions.*

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