

ACTUATOR FAULT DIAGNOSIS FOR FLAT SYSTEMS: A CONSTRAINT SATISFACTION APPROACH

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This paper describes a robust set-membership-based Fault Detection and Isolation (FDI) technique for a particular class of nonlinear systems, the so-called flat systems. The proposed strategy consists in checking if the expected input value belongs to an estimated feasible set computed using the system model and the derivatives of the measured output vector. The output derivatives are computed using a numerical differentiator. The set-membership estimator design for the input vector takes into account the measurement noise thereby making the consistency test robust. The performances of the proposed strategy are illustrated through a three-tank system simulation affected by actuator faults.

Keywords: fault detection, input observer, flat systems, consistency techniques.

1. Introduction and problem setting

The theory related to model-based FDI has been developed since the 1970s and can be considered today a mature and well-structured field of research within the control community (for a survey, see Chen and Patton, 1999; Ding, 2008; Korbicz *et al.*, 2004). The role of the FDI unit is to detect, isolate and estimate the severity of a fault. The IFAC SAFEPROCESS Technical Committee defines a fault as an unpermitted deviation of at least one characteristic property or parameter of the system from the standard conditions (Isermann and Ball, 1997). Such malfunctions may occur in sensors, actuators or other devices and affect adversely the local or global behavior of the system.

Generally, the main desirable characteristics of an FDI system are

- early detection and identification of abnormal situations, i.e., detection delay should be minimized;
- good ability to discriminate between different

failures (isolability);

- good robustness to various noise and sources of uncertainty, and their propagation through the system;
- high sensitivity and performance, i.e., high detection rate and low false alarm rate.

The basic structure of a classical model-based FDI technique is depicted in Fig. 1.

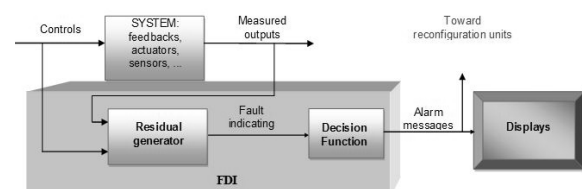


Fig. 1. Basic structure of FDI.

The core element is residual generation and many

approaches concentrate on this step (Fig. 1). Note that if only fault detection is of interest, reconstructing the fault rather than detecting its presence through a residual signal can be a nice alternative solution. Residual evaluation and decision making consist in checking the residuals and triggering alarm messages if the tolerances are exceeded. The thresholds can be set into different kinds.

The simplest way is to use a constant threshold. The main advantage with fixed thresholds is their simplicity and reliability. To enhance the robustness of FDI schemes against small parameter variations and disturbances during residual generation, various design and evaluation tools have been proposed (Hwang *et al.*, 2010). Robust FDI can be achieved if the residual signals maintain these sensitivity properties over a suitable range of system dynamics.

Most of the well known approaches assume the availability of an LTI model or specific nonlinearities. For systems with strong nonlinear behavior, several FDI methodologies exist in the literature, including LPV transformations (Bokor and Balas, 2004; Henry and Zolghadri, 2004; Grenaille *et al.*, 2008), adaptive observers (Ding and Frank, 1993; Wang and Daley, 1996; Wang *et al.*, 1997; Chen *et al.*, 2000; Besançon and Zhang, 2002; Caccavale and Villani, 2004; Liu, 2009), stochastic techniques (Wang and Noriega, 2001; Wang and Lin, 2000; Wang, 2003; Poulsen and Niemann, 2008), algebraic approaches (Zhang *et al.*, 1998; Berdjag *et al.*, 2006) or sliding mode-based methods (Edwards and Spurgeon, 2000; Hakiki *et al.*, 2006; Yan and Edwards, 2008). For a good survey, see the work of Bokor and Szab (2009) and the references therein.

Most of the proposed approaches make use of particular structural characteristics of a given nonlinear system to generate output residuals. Typically, the observer design problem is solvable if the system model can be transformed into a canonical form, which may be an assumption difficult to satisfy in many applications. Another approach is based on input observer design. In this approach, the input observer is a dynamical system with an output that converges asymptotically or in finite time to a fault to be detected (Corless and Tu, 1998; Ha and Trinh, 2004). More recently, set-membership fault detection techniques have been investigated by several authors (Puig *et al.*, 2003; Stancu *et al.*, 2005; Fagarasana *et al.*, 2004; Ragot *et al.*, 2006; Lalami and Combastel, 2006; Raïssi *et al.*, 2010). Many publications are based on interval observers (Gouzé *et al.*, 2000; Bernard and Gouzé, 2004; Moisan and Bernard, 2006; Moisan *et al.*, 2009), where the idea was to design two-point observers to compute an upper and a lower bound for the state vector domain under both stability and cooperativity conditions. An interval residual is given as the difference between the predicted set from the model and the real value measured by the sensors. Another approach reported by

Ingimundarson *et al.* (2009) and Puig (2010) is based on state/parametric consistency checks.

This paper investigates a set-membership fault detection technique for a particular class of nonlinear systems, the so-called flat ones (Fliess *et al.*, 1992). The proposed approach belongs to the class of input fault detection techniques. It consists in formulating the input estimation issue using a Constraint Satisfaction Problem (CSP) (Neumaier, 2004; Arangú and Salido, 2011). A CSP input estimator is then designed using the relation between the input vector and the flat outputs. This observer requires the computation of a finite number of flat output derivatives. The robustness is achieved by taking into account the whole domain of the uncertainties (measurement noise) and without any linearization. This technique is well suited to deal with actuator faults as well as parametric or sensor faults. However, in this paper, we choose to focus on actuator faults. The methodology includes a fault identification stage where interval residual quantities are generated by calculating the gap between the estimated input set and the measured input value at each time instant.

The paper is structured as follows. Section 2 recalls some basic notations which will be used throughout the paper. The problem is formulated in Sections 3 and 4. Then, the CSP input estimator is presented and used in the FDI procedure (Section 5). Finally, the methodology is illustrated through a three-tank system application, and some concluding remarks are given.

2. Preliminaries

This section recalls some basic and useful definitions.

2.1. Flatness. Consider a nonlinear system described by

$$\begin{cases} \dot{x} = f(x) + g(x)u, \\ y = h(x), \end{cases} \quad (1)$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $u \in \mathbb{R}^m$, and assume $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $g : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ are two smooth vector fields while $h : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a differentiable smooth function. The system (1) is said to be flat with a flat output vector y if and only if one can describe the system states and inputs x, u only from the flat outputs and a finite number of their derivatives, i.e.

$$\begin{cases} x = \theta(y, \dot{y}, \dots, y^{(p)}), \\ u = s(y, \dot{y}, \dots, y^{(p+1)}), \end{cases} \quad (2)$$

where θ and s are two smooth vector fields.

2.2. Nonlinear local observability. The matrix

$$H(x) = \begin{pmatrix} dh(x) \\ dL_f h(x) \\ \vdots \\ dL_f^{n-1} h(x) \end{pmatrix} \quad (3)$$

is called the local observability matrix for (1) where $L_f h(x)$ is the Lie derivative of the function h along the vector field f and $dL_f h(x)$ is the gradient $\frac{\partial L_f h(x)}{\partial x}$ (Hermann and Krener, 1977). The system (1) satisfies the observability rank condition at the point x if the observability matrix has full rank, i.e. $\text{rank}(H(x)) = n$.

2.3. Interval tools. A real interval $[a] = [\underline{a}, \bar{a}]$ is a connected and closed subset of \mathbb{R} . The set of all real intervals of \mathbb{R} is denoted by \mathbb{IR} . Real arithmetic operations are extended to intervals (Moore, 1966; Hansen, 2004). Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$. The range of the function f over an interval vector $[x]$ is given by $f([x]) = \{f(x) \mid x \in [x]\}$. An interval function $[f] : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is an inclusion function for f if $\forall [x] \in \mathbb{R}^n, f([x]) \subseteq [f]([x])$.

2.4. Constraint satisfaction problems. A constraint satisfaction problem (CSP) is defined by a set of variables, X_1, X_2, \dots, X_n , and a set of constraints, C_1, C_2, \dots, C_m . Each variable X_i has a nonempty domain D_i of possible values. Each constraint C_i involves some subset of the variables and specifies the allowable combinations of values for that subset. A state of the problem is defined by an assignment of values to some or all of the variables, $\{X_i = v_i, X_j = v_j, \dots\}$ (Neumaier, 2004).

Constraint propagation is a way to solve CSPs. The aim of propagation techniques is to contract as much as possible the domains for the variables without losing any solution (Waltz, 1975). When interval uncertainties are considered, consistency methods combining interval and constraint satisfaction techniques can be used to deal with problems such as parameter/state estimation and further fault detection problems.

In the case of flat systems, CSP variables correspond to the states and inputs related to the flat output by a specific and unique map. This map defines the constraints. In order to retrieve the state and input vectors satisfying the constraints, consistency techniques are applied through flatness equations.

3. Problem statement

Consider a system described by (1) and assume that the measurement error e is bounded with a known bound \bar{e} .

The output vector y belongs to

$$y \in [y_m - \bar{e}, y_m + \bar{e}], \quad (4)$$

where y_m is the measured signal.

Assumption 1. The actuator fault f_a is modeled as follows:

$$u = u_0 + f_a, \quad (5)$$

where u_0 is the nominal input.

This assumption corresponds to additive fault modeling. Note that multiplicative faults can be modeled as additive faults considering the transformation described by Ding (2008). Actually, additive faults do not affect system stability under feedback control whereas multiplicative ones could. The additive fault assumption is made throughout the paper since it is easy to derive the fault f_a from Eqn. (5). Nevertheless, this assumption can be relaxed without any additional theoretical developments since the interval technique used to solve the CSP (13) does not need such assumptions.

It is required to estimate the fault using the output vector and its derivatives. To solve the problem, it is proposed to rewrite the second equation of (2) expressing the input u as a polynomial function of the flat output y and a finite number of its derivatives. This step is based on the parity space approach for polynomial nonlinear systems. The resulting relations between the variables (input, flat outputs and their derivatives, fault) constitute the constraints for the CSP input estimator used in the FDI procedure (Sections 5 and 6). The numerical output differentiation for y is performed using a Higher Order Sliding Modes (HOSM) differentiator. Both fault detection and fault identification phases are based on input estimation through a CSP input estimator.

The common point between the presented approach and the geometric methodology proposed by De Persis and Isidori (2001; 2002) is a kind of model invalidation. In this paper, the model is invalidated through the use of a consistency test whereas the geometric approach consists mainly in finding an unobservable subspace due to the occurrence of a certain fault among others. Note that this geometric approach for fault detection and isolation relies on stochastic assumptions.

4. Parity space and flat systems

The aim of this section is to derive relations between u_0 , y and f_a for the system (1). Taking the derivatives of the

output vector up to an order $n - 1$, we obtain

$$\begin{cases} y &= y^{(0)} = h(x), \\ y^{(1)} &= \frac{\partial h(x)}{\partial t} = \frac{\partial h(x)}{\partial x} \cdot \frac{\partial x}{\partial t} \\ &= \frac{\partial h(x)}{\partial x} \cdot [f(x) + g(x)u] \\ &= L_f h(x) + L_g h(x) \cdot u, \\ y^{(2)} &= L_f^2 h(x) + L_g L_f h(x) \cdot u + L_f L_g h(x) \cdot u \\ &\quad + L_g^2 h(x) \cdot uu^T + L_g h(x) \cdot u^{(1)}, \\ &\vdots \end{cases}$$

Equation (4) can be rewritten as

$$Y = H + GW, \tag{6}$$

where

$$Y = \begin{pmatrix} y \\ y^{(1)} \\ y^{(2)} \\ \vdots \end{pmatrix}, \quad H = \begin{pmatrix} L_f h(x) \\ L_f^2 h(x) \\ L_f^3 h(x) \\ \vdots \end{pmatrix},$$

$$W = \begin{pmatrix} u \\ u \cdot u^T \\ u^{(1)} \\ \vdots \end{pmatrix}$$

and

$$G = \begin{pmatrix} 0 & 0 & 0 \\ L_g h(x) & 0 & 0 \\ L_g L_f h(x) + L_f L_g h(x) & L_g^2 h(x) & L_g h(x) \\ \vdots & \vdots & \vdots \\ & 0 & \dots \\ & \dots & \dots \\ & 0 & \dots \\ & \vdots & \vdots \end{pmatrix}.$$

Equation (6) shows that the consecutive derivatives of the measured output vector are polynomial functions of the derivatives of the input vector u (Sontag and Wang, 1991). The relation (6) can be formulated as

$$\begin{aligned} \phi_i(x, u, \dots, u^{(i-2)}, y^{(i-1)}) \\ = y^{(i-1)} - L_F^{i-1} h(x) = 0, \quad i = 1, \dots, n, \end{aligned} \tag{7}$$

where ϕ_i is a nonlinear function of its arguments and $F(x, u) = f(x) + g(x)u$.

Under the observability rank condition (3) and using the implicit functions theorem (Krantz and Parks, 2002), Eqn. (7) has a solution,

$$x = P(u^{(0)}, y^{(0)}, \dots, u^{(n-1)}, y^{(n-1)}). \tag{8}$$

Thus, the state x is a function of the input and output vector derivatives, transforming (7) into

$$\begin{aligned} \Psi_i(u, y, u^{(1)}, \dots, y^{(1)}, \dots) \\ = \phi_i(P(u, y, \dots, u^{(n-1)}, y^{(n-1)}), u, \\ \dots, u^{(n-2)}, y^{(n-1)}) = 0, \\ i = 1, 2, \dots, \end{aligned} \tag{9}$$

where Ψ_i is a nonlinear function of its arguments. Taking into account the expression of u given by (5), the relation (9) can be transformed into

$$\Psi_i((u_0 + f_a), y, \dots, (u_0 + f_a)^{(\alpha_i)}, y^{(\alpha_i)}) = 0. \tag{10}$$

The relation (10) is established for a general class of nonlinear systems. Nevertheless, it is well known that for flat systems the input is related to the output and its derivatives as

$$u = s(y, y^{(1)}, \dots, y^{(p+1)}). \tag{11}$$

Thus, the relations (9) and (11) lead to

$$\begin{aligned} \Psi_l((u_0 + f_a), y, y^{(1)}, \dots, y^{(p+1)}) = 0, \\ l = 1, \dots, m. \end{aligned} \tag{12}$$

Remark 1. For simplicity, only additive faults are considered in this paper. Nevertheless, multiplicative faults modeled by $u = u_0(1 + \alpha)$, where α is the fault effect, can also be studied without any additional investigations.

5. Input estimation and fault diagnosis

Given a flat system described by (1) and admitting the decoupling relation (12), the input estimation problem is formulated as a CSP given by

$$\begin{cases} \Psi_l(u, y, y^{(1)}, \dots, y^{(p+1)}) = 0, \quad l = 1, \dots, m, \\ y \in [y], y^{(1)} \in [y^{(1)}], \dots, y^{(p+1)} \in [y^{(p+1)}], \\ u \in U, \end{cases} \tag{13}$$

which can be solved using interval analysis. Note that a similar formulation has been performed for state estimation by Jaulin (2013), according to whom the output derivatives are supposed to be measured. This is not the case in our work. Finding solutions to (13) needs an evaluation of the measurement derivatives. In the following, the measurement derivatives are estimated using HOSM differentiators (Levant, 1998; 2003).

5.1. HOSM differentiator for derivative estimation.

Let y_i be the signal to be differentiated and z_0, z_1, \dots, z_n be some estimates for the signal y_i and its derivatives.

Note that y_i is the i -th output vector component and $y_i = y_{i_0} + e_i$; e_i is a bounded Lebesgue-measurable noise with unknown features and an unknown base signal y_{i_0} with the n -th derivative having a known Lipschitz constant $C_i > 0$. The n -th-order HOSM differentiator is given by

$$\left\{ \begin{array}{l} \dot{z}_0 = \nu_0, \\ \nu_0 = -\alpha_0 |z_0 - y_i|^{\frac{n}{n+1}} \text{sign}(z_0 - y_i) + z_1, \\ \dot{z}_1 = \nu_1, \\ \nu_1 = -\alpha_1 |z_1 - \nu_0|^{\frac{n-1}{n}} \text{sign}(z_1 - \nu_0) + z_2, \\ \vdots \\ \dot{z}_k = \nu_k, \\ \nu_k = -\alpha_k |z_k - \nu_{k-1}|^{\frac{n-k}{n-k+1}} \text{sign}(z_k - \nu_{k-1}) \\ + z_{k+1} \\ \vdots \\ \dot{z}_n = -\alpha_n \text{sign}(z_n - \nu_{n-1}). \end{array} \right. \quad (14)$$

It has been proved (Levant, 2001) that the best estimate accuracy of the k -th derivative is proportional to

$$acc_{i_k} = \mu_{i_k} C_i^{\frac{k}{n+1}} \bar{e}_i^{\frac{n+1-k}{n+1}}, \quad k = 0, \dots, n,$$

when the Lipschitz constant of the n -th derivative of the clear-off-noise signal is bounded by a certain constant C_i and $\mu_{i_k} \geq 1$ (μ_{i_k} depends only on α_k , $k = 0, \dots, n$). Please refer to Proposition 1 of Levant (2001) for more details). Hence, the derivative domain is $y_i^{(k)} \in [y_{i_{est}}^{(k)} - acc_{i_k}, y_{i_{est}}^{(k)} + acc_{i_k}]$ where $y_{i_{est}}^{(k)} = z_k$ is the estimate of the k -th derivative for y_i .

5.1.1. Input estimation. At each time instant t_j , denote by U_j the domain of u at t_j . If no prior information about the domain of u is available, we can select the domain $U_j =] -\infty, +\infty[$. Thus, the basics of the input estimation method consist in computing all the values of u satisfying (13). The idea is to remove parts of the searching domain U_j for the model input that are inconsistent with the measured data y_j and their derivatives up to an order $p + 1$. For easy reference, the main steps of the input estimation are summarized as Algorithm 1.

Note that U is the initial search domain for the input vector and SIVIAP is a set inversion algorithm which makes it possible to solve CSPs in a guaranteed way using interval analysis (Jaulin *et al.*, 2001).

Remark 2. The vectors \bar{e} , acc_k and $y^{(k)}$ should be respectively understood as $\bar{e} = (\bar{e}_1, \dots, \bar{e}_m)^T$, $acc_k = (acc_{1_k}, \dots, acc_{m_k})^T$, and $y^{(k)} = (y_1^{(k)}, \dots, y_m^{(k)})^T$, where m represents the flat output vector dimension.

5.1.2. Residual generation and fault diagnosis. The residual generation and fault detection procedures are

Algorithm 1. CSP Estimator.

Step 0. (Inputs: $y(t_j)$, $j = 1, \dots, N, U$), Output: $[u(t_j)]$)

Step 1. Flatness modelling as a CSP (13).

Step 2. For $j = 1, \dots, N$, do

Step 2.1. Estimate the derivatives $y^{(k)}$, $k = 1, \dots, p + 1$ using (14).

Step 2.2. Estimate the bound acc_k and construct the domains of $y(t_j)$ and its derivatives.

Step 2.3 Solve the CSP using the SIVIAP algorithm to obtain $[u(t_j)]$.

based on input estimation. At each time instant t_j , we consider the algorithm output $[u(t_j)]$ and an inclusion test is run over an interval residual. This residual denotes the gap between the estimated input set and the expected input value. The lower (respectively, upper) bound of the residual corresponds to the difference between the estimated set lower (respectively, upper) bound input u and the value of the fault free model input u_0 . The residual is defined by

$$[r(t_j)] = [\underline{u}(t_j) - u_0(t_j), \bar{u}(t_j) - u_0(t_j)]. \quad (15)$$

The consistency test aim is to check whether the expected (fault-free case or controller output) input value $u_0(t_j)$ belongs to the estimated domain $[u(t_j)]$, i.e.,

$$\det_j = \begin{cases} 0, & 0 \in [r(t_j)], \\ 1, & 0 \notin [r(t_j)], \end{cases} \quad (16)$$

which is equivalent to checking

$$\det_j = \begin{cases} 0, & u_0(t_j) \in [u(t_j)], \\ 1, & u_0(t_j) \notin [u(t_j)]. \end{cases}$$

The algorithm used for input estimation could involve several bisections over the initial input domain for the admissible value research. Actually, the computing time increases exponentially with the input dimension (Jaulin *et al.*, 2001). In the fault detection phase, we only check whether the actual input u_0 is consistent with the measurements. If a fault is detected, the algorithm *CSP Estimator* will be applied to estimate its amplitude.

5.1.3. Fault identification. Fault identification is the characterization of the fault besides its occurrence time. This task is here restricted to the determination of the fault amplitude. Since we consider additive actuator faults, the fault amplitude or, more precisely, the estimated fault domain is given by the residual expression

$$[\underline{f}_a, \bar{f}_a] = [r] = [\underline{u} - u_0, \bar{u} - u_0]. \quad (17)$$

Note that the approaches developed by Jiang *et al.* (2004), Akhenak *et al.* (2003), Kabore and Wang (2001)

as well as Orani *et al.* (2009) propose alternative solutions for actuator failures and simultaneous faults diagnosis. These techniques are based on a robust sliding mode observer design whereas the sliding mode concept is used in this paper for robust numerical differentiation of bounded noisy measurements. Most of these methodologies require system linearizations (Akhenak *et al.*, 2003; Kabore and Wang, 2001) or a canonical form (Jiang *et al.*, 2004). The technique proposed in this paper considers the original nonlinear model without any linearization and the results are guaranteed by the fact that the variables are modeled by means of intervals containing the actual value which cannot be known exactly. This is one of the main advantages of set-membership techniques. However, the computational burden remains a drawback when dealing with high dimensional systems.

5.1.4. Minimum detectable fault. The minimum detectable fault corresponds to the smallest amplitude for a fault to be detectable with respect to the measurement error bounds \bar{e}_i and output derivatives error bounds $acc_{i,k}$. Recall that the bound acc_k is given by

$$acc_k = \mu_k \cdot C^{\frac{k}{n+1}} \cdot e^{\left(\frac{n+1-k}{n+1}\right)}, \quad k = 0, \dots, n.$$

When no fault occurs, the residual $[r]$ is centered around the value 0 and its upper and lower bounds (\bar{r} and \underline{r}) can be computed from the residual expression and the bounds acc_k :

$$\begin{cases} [r] = [\underline{r}, \bar{r}] = [\underline{u} - u_0, \bar{u} - u_0], \\ [\underline{r}, \bar{r}] = [s] ([y], [y^{(1)}], \dots, [y^{(p+1)}]) - u_0. \end{cases} \quad (18)$$

In a fault-free case, we have $\bar{r} \geq 0$ (respectively $\underline{r} \leq 0$). Thus, the minimum detectable fault is the fault that makes \bar{r} (respectively, \underline{r}) cross the null value towards negative values (respectively, positive values). The minimum detectable fault depends on the nominal input and output amplitudes.

In the following, we assume that the system is subject to null valued inputs and the flat outputs are subsequently null ($y_m = 0$ under null initial conditions or in a steady state case). Moreover, we assume that a fault f_a occurs at the same time on the actuators. The output domain is then

$$y \in [y_m - \bar{e}, y_m + \bar{e}] \Rightarrow y \in [-\bar{e}, +\bar{e}] = [-acc_0, acc_0]. \quad (19)$$

The residual domain becomes

$$[\underline{r}, \bar{r}] = [s] ([-acc_0, acc_0], \dots, [-acc_{p+1}, acc_{p+1}]) - u_0. \quad (20)$$

Since u_0 is the nominal input value and is assumed to be null, the residual domain is

$$[\underline{r}, \bar{r}] = [s] ([-acc_0, acc_0], \dots, [-acc_{p+1}, acc_{p+1}]). \quad (21)$$

Actually, this domain gives roughly the minimum detectable fault f_{amin} amplitude:

$$f_{amin} = \min(|\underline{r}|, |\bar{r}|). \quad (22)$$

For the general case, the nominal inputs and outputs should be taken into account.

6. Application: A three tank hydraulic system

The proposed methodology is illustrated on a well-known hydraulic laboratory system (Fig. 2) which is modeled by

$$\dot{x}_1 = -\frac{a_{13}}{S} \sqrt{x_1 - x_3} + \frac{1}{S} u_1, \quad (23a)$$

$$\dot{x}_2 = \frac{a_{32}}{S} \sqrt{x_3 - x_2} - \frac{a_{20}}{S} \sqrt{x_2} + \frac{1}{S} u_2, \quad (23b)$$

$$\dot{x}_3 = \frac{a_{13}}{S} \sqrt{x_1 - x_3} - \frac{a_{32}}{S} \sqrt{x_3 - x_2}, \quad (23c)$$

$$y_1 = x_1, \quad (23d)$$

$$y_2 = x_3, \quad (23e)$$

where y_1 and y_2 are the flat outputs of the system and $x = (x_1, x_2, x_3)^T = (h_1, h_2, h_3)^T$ represents the state vector (under the condition $h_1 > h_3 > h_2$ for all times), $u = (u_1, u_2)^T = (Q_1, Q_2)^T$ is the control vector.

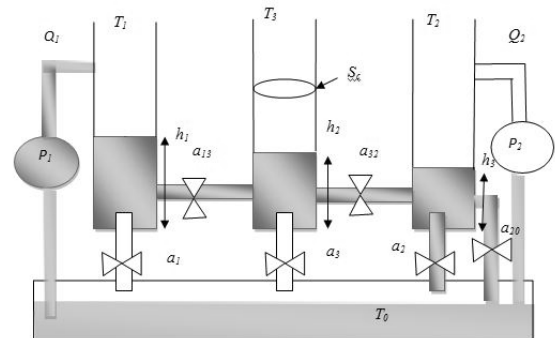


Fig. 2. Three-tank system.

The three-tank system is a popular benchmark used in many published works. Moreover, this example is fully nonlinear and flat, which is important for our approach. This example has been addressed in other papers (Join *et al.*, 2005; Theilliol *et al.*, 2002; Zolghadri *et al.*, 1996), where the authors adopted the assumption that the whole state vector of the plant is available for measurement. The technique proposed in this paper only requires two measurable flat outputs.

The noise magnitude corrupting the measurements y_m is bounded by $[\pm 2]$ mm. The system is under feedback control and two PI controllers have been implemented for

the flat output x_1 and the non-flat output x_2 . It is easy to prove the system flatness (and subsequently its local observability) through the following equations:

$$\begin{cases} x_1 = y_1, \\ x_2 = y_2 - \left(\frac{a_{13}\sqrt{y_1 - y_2} - S_c \dot{y}_2}{a_{32}} \right)^2, \\ x_3 = y_2, \end{cases} \quad (24)$$

and

$$\begin{cases} u_1 = S_c \dot{y}_1 + a_{13}\sqrt{y_1 - y_2}, \\ u_2 = -\frac{2S_c}{a_{32}} (a_{13}\sqrt{y_1 - y_2} - S_c \dot{y}_2) \\ \quad \times \left(\frac{a_{32}}{2\sqrt{y_1 - y_2}} (\dot{y}_1 - \dot{y}_2) - S_c \ddot{y}_2 \right) \\ \quad - a_{13}\sqrt{y_1 - y_2} \\ \quad + a_{20}\sqrt{y_2 - \left(\frac{a_{13}\sqrt{y_1 - y_2} - S_c \dot{y}_2}{a_{32}} \right)^2}. \end{cases} \quad (25)$$

Fault detection and isolation. Additive actuator faults (bias) with different amplitudes are applied to both actuators and the consistency test (16) is used here. Equations (25) expressing the inputs in function of the flat outputs are required for detection. To control the conservatism induced by interval analysis due to the several occurrences of the variables y_1 , y_2 , \dot{y}_1 and \dot{y}_2 in the second equation of (25), we propose to use first (24) to estimate x_2 . In addition, the derivative of x_2 is computed through the HOSM differentiator (14). Then, Eqn. (23b) is used to perform the inclusion test for u_2 ; thus only the first derivative of y_1 and x_2 is required to estimate u_1 and u_2 .

As mentioned in Section 5, a robust differentiator built on the high order sliding modes technique provides the successive derivatives of the flat outputs, which yields

$$\begin{cases} \dot{y}_1 \in [\dot{y}_{1_est} - acc_{\dot{y}_{1_est}}, \dot{y}_{1_est} + acc_{\dot{y}_{1_est}}], \\ \dot{x}_2 \in [\dot{x}_{2_est} - acc_{\dot{x}_{2_est}}, \dot{x}_{2_est} + acc_{\dot{x}_{2_est}}], \end{cases} \quad (26)$$

where

$$acc_{\dot{y}_{1_est}} = (0.002)^{\frac{1}{2}} \times (25 \cdot 10^{-6})^{\frac{1}{2}} \times 1 = 0.000224$$

and

$$acc_{\dot{x}_{2_est}} = (0.002)^{1/2} \times (80e-8)^{1/2} \times 1 = 13e-6$$

with a first order sliding-mode differentiator in both cases.

Table 1 sums up fault detection performances for the algorithm *CSP Estimator* applied to the three-tank system with the following faults:

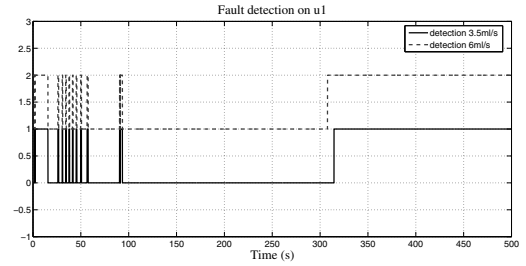


Fig. 3. Fault detection on u_1 .

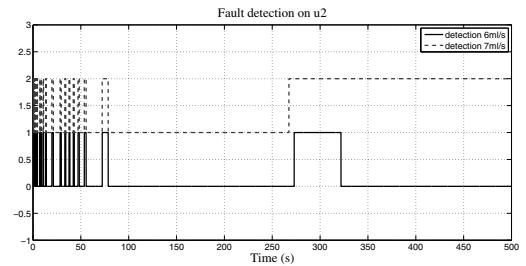


Fig. 4. Fault detection on u_2 .

- additive bias on the first actuator (u_1) appearing at $t = 300$ s with amplitudes 3.5 ml/s and 6 ml/s;
- additive bias on the second actuator (u_2) appearing at $t = 250$ s with amplitudes 6 ml/s and 7 ml/s.

As we can see, the detection time decreases with respect to the fault amplitude growth. The detectability is stronger for the Actuator 1 and this could be explained by the fact that differential equation (23a) leading it is relatively less complex than (23b) for Actuator 2 in the sense that for the variables x_1 , x_2 and x_3 the occurrence number is lower than in (23b), which leads to less conservatism in the interval computations. Actually, in the case of a fault with an amplitude of 6 ml/s for the second actuator, the fault effects are not persistent. Note that the time interval $[0, 80]$ s corresponds to system transient behavior.

Fault identification. We can simultaneously estimate the admissible domain for the inputs u_1 and u_2 and the

Table 1. Fault detection performances.

Actuator	Fault	Amplitude (%) $\times u_{i \max}$	Amplitude	Delay
No. 1	Bias	8%	3.5 ml/s	14.5 s
		14%	6ml/s	8 s
No. 2	Bias	11%	6 ml/s	23 s
		13%	7 ml/s	17 s

fault (residual) estimated domain. Figures 5 and 6

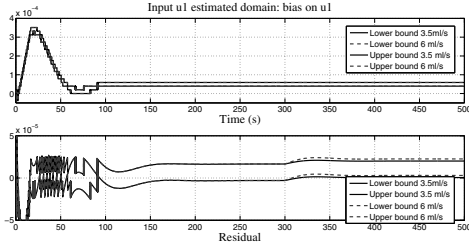


Fig. 5. Fault identification for u_1 .

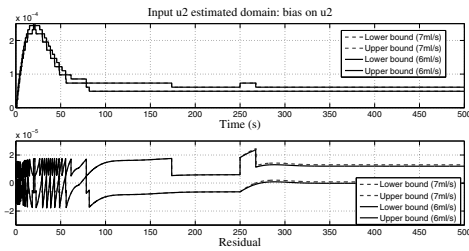


Fig. 6. Fault identification for u_2 .

depict the kind of faults affecting the actuators. Moreover, the fault occurrence time is clearly shown ($t_1 = 300$ s for u_1 and $t_2 = 250$ s for u_2) despite the existence of some detection delay time in both cases. The residual bounds, beyond the detection instant, give estimation for the domain of the fault amplitude. Depending on the differential equation leading to the actuator and the measurement derivative estimation, the pessimism induced around the fault amplitude may change.

Minimum detectable fault.

Actuator 1: The expression of the input is given by $u_1 = S\dot{y}_1 + a_{13}\sqrt{y_1 - y_2}$. Given the bounds $acc_{y_1} = acc_{y_2} = \bar{e} = 0.002$ and $acc_{\dot{y}_{1est}} = 0.000224$, we have

$$f_{a_{1min}} = S \cdot |acc_{\dot{y}_{1est}}| + a_{13}\sqrt{acc_{y_1} - acc_{y_2}} = 3.44 \cdot 10^{-6} \text{ mls}^{-1}. \tag{27}$$

Actuator 2: The expression of u_2 is given by

$$u_2 = -\frac{2S_c}{a_{32}}(a_{13}\sqrt{y_1 - y_2} - S_c\dot{y}_2) - a_{13}\sqrt{y_1 - y_2} + a_{20}\sqrt{y_2 - \left(\frac{a_{13}\sqrt{y_1 - y_2} - S_c\dot{y}_2}{a_{32}}\right)^2},$$

which leads to

$$f_{a_{2min}} = -\frac{2S_c}{a_{32}}(a_{13}\sqrt{acc_{y_1} - acc_{y_2}} - S_c acc_{\dot{y}_{2est}}) - a_{13}\sqrt{acc_{y_1} - acc_{y_2}} + a_{20}\sqrt{acc_{y_2} - \frac{a_{13}\sqrt{acc_{y_1} - acc_{y_2}} - S_c acc_{\dot{y}_{2est}}}{a_{32}}}$$

Both flat outputs y_1 and y_2 are bounded by the value 0.002 m ($acc_{y_1} = acc_{y_2} = 2$ mm). These uncertainties lead to a null denominator ($acc_{y_1} - acc_{y_2} = 0$) in the expression of $f_{a_{2min}}$. Since we consider that both outputs y_1 and y_2 are bounded by acc_{y_1} and acc_{y_2} , we can extrapolate and consider x_2 as an output that would also be bounded by the same value ($acc_{x_2} = 2$ mm), Then, from (23b), we have

$$u_2 = S * \dot{x}_2 - a_{32}\sqrt{x_3 - x_2} + a_{20}\sqrt{x_2},$$

and the expression $f_{a_{2min}}$ of the smallest detectable fault on the second actuator is given by

$$f_{a_{2min}} = S \cdot acc_{\dot{x}_{2est}} - a_{32}\sqrt{acc_{y_2} - acc_{x_2}} + a_{20}\sqrt{acc_{x_2}} = 7.91 \text{ mls}^{-1}. \tag{28}$$

Figure 4 shows that some faults with an amplitude lower than (28) can also be detected, which means that the expression given by (28) is conservative due to the estimation of the derivatives error bounds.

The drawback of the proposed technique is the computation time due to bisections in the set inversion procedure. It has been shown by Jaulin and Walter (1993) that the complexity of set inversion algorithms is exponential with respect to the unknown vector dimension m (here, the dimension of the input vector):

$$N = \left(\frac{w(U)}{\varepsilon} + 1\right)^m,$$

where U is the initial searching domain for the input u , $w(U)$ is the domain width and ε is the bisection tolerance threshold. This technique can be implemented for systems with a low dimension for the input (or fault) vector.

7. Concluding remarks

In this paper, a new model-based fault diagnosis methodology with interval consistency techniques has been proposed for a class of nonlinear systems. The objective is to provide an original solution to actuator fault diagnosis for flat systems. The main important advantage

of the technique is the possibility offered to generate a residual that gives the fault amplitude and satisfies robustness conditions under valid modeling hypotheses. Finally, the performance of the proposed strategy has been illustrated through a numerical example where no false alarm occurred during the detection phase. However, concerning an on-line implementation, the computational burden remains a drawback when dealing with high dimensional systems. An appealing direction for further investigations is the impact of parametric uncertainties and component faults that could lead to flatness property loss for a system. This is a topic of our current research.

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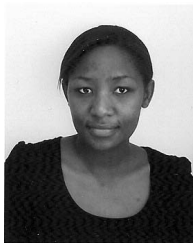
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Appendix

Table A1. Three-tank system parameters.

Parameter	Description	Units	Nominal values
S	Tank surface area	m^2	0.0154
S_n	Pipe surface area	m^2	$5 \cdot 10^{-5}$
a_{13}	$a_{z13} S_n \sqrt{2g}$	–	
a_{20}	$a_{z20} S_n \sqrt{2g}$	–	
a_{32}	$a_{z32} S_n \sqrt{2g}$	–	
a_{z13}		–	0.6
a_{z20}		–	0.8
a_{z32}		–	0.6
g	Gravity constant	N/m^2	9.81
$h_{1 \max}$	Maximum levels in tank 1, 2 and 3	m	0.5
$h_{2 \max}$			0.3
$h_{3 \max}$			0.4
$Q_{1 \max}$	Input flow rates (steady state)	mls^{-1}	42
$Q_{2 \max}$			55

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