ON eGE-ALGEBRAS

R.K. BANDARU

Department of Mathematics
GITAM (Deemed to be University)
Hyderabad Campus, India-502 329

E-mail: ravimaths83@gmail.com

N. RAIFI

Department of Mathematics
Bapatla Engineering College
Bapatla, Andhra Pradesh, India-522 101

E-mail: rafimaths@gmail.com

AND

A. REZAEI

Department of Mathematics
Payame Noor University
P.O. Box 19395-3697, Tehran, Iran

E-mail: rezaei@pnu.ac.ir

Abstract

A new algebraic structure was introduced, called an eGE-algebra, which is a generalisation of a GE-algebra and investigated its properties. We explore the definition of filters and the quotient algebra associated with such filters.

Keywords: BE-algebra, GE-algebra, eGE-algebra, transitive, filter.

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1Corresponding author.
1. Introduction and preliminaries

L. Henkin and T. Skolem developed the idea of Hilbert algebra in the early 50-ties for some investigations of implication in intuitionistic and other nonclassical logics. In 60-ties, these algebras were particularly studied by Horn and Diego [6] from algebraic point of view. Hilbert algebras are a valuable tool for some algebraic logic investigations as they can be regarded as fragments of any propositional logic that contains a logical connective implication ($\rightarrow$) and the constant 1 that is assumed to be the logical meaning “true”. Many researchers have done a significant amount of work on Hilbert algebras [3–5,7–9,13–16]. As a generalization of Hilbert algebras, Bandaru et al. [1] introduce the notion of GE-algebras. They studied the various properties and filter theory of GE-algebras. BCK-algebras and BCI-algebras were introduced by Imai and Iseki [10,11]. H.S. Kim and Y.H. Kim [12] developed the concept of BE-algebra as a generalization of dual BCK-algebra. Many researchers developed theory of BE-algebras [2,17–19]. Rezaei [20] has introduced the notion of eBE-algebra as a generalization of BE-algebra and has studied some of its properties. It is important to make clear the corresponding algebraic structures for the creation of many-valued logical system. As a generalization of GE-algebra, we are inspired to concentrate on a new algebraic structure, called eGE-algebra, and thus to investigate some properties.

Definition 1.1 [1]. Let $X$ be a non-empty set with a constant 1 and $\ast$ a binary operation on $X$. Then an algebraic structure $(X, \ast, 1)$ of type $(2,0)$ is said to be a GE-algebra if it satisfies the following axioms:

(GE1) $u \ast u = 1$,  
(GE2) $1 \ast u = u$,  
(GE3) $u \ast (v \ast w) = u \ast (v \ast (u \ast w))$

for all $u, v, w \in X$.

In a GE-algebra $X$, a binary relation “$\leq$” is defined by

\[(\forall u, v \in X) \ (u \leq v \iff u \ast v = 1).\]

Proposition 1.2 [1]. Every GE-algebra $X$ satisfies the following items.

(2) \[(\forall u \in X) \ (u \ast 1 = 1).\]

(3) \[(\forall u, v \in X) \ (u \ast (u \ast v) = u \ast v).\]

(4) \[(\forall u, v \in X) \ (u \leq v \ast u).\]

Definition 1.3 [1]. A GE-algebra $X$ is said to be transitive, if it satisfies:

(5) \[(\forall x, y, z \in X) \ (x \ast y \leq (z \ast x) \ast (z \ast y)).\]
Proposition 1.4 [1]. Every transitive GE-algebra $X$ satisfies the following assertions.

(6) \[(\forall x, y, z \in X) (x \ast y \leq (y \ast z) \ast (x \ast z)) \].

(7) \[(\forall x, y, z \in X) (x \leq y \Rightarrow z \ast x \leq z \ast y, y \ast z \leq x \ast z) \].

Definition 1.5 [1]. A subset $F$ of a GE-algebra $X$ is called a filter of $X$ if it satisfies:

(8) \[1 \in F,\]

(9) \[(\forall x, y \in X)(x \ast y \in F, x \in F \Rightarrow y \in F)\].

Lemma 1.6 [1]. In a GE-algebra $X$, every filter $F$ of $X$ satisfies:

(10) \[(\forall x, y \in X)(x \leq y, x \in F \Rightarrow y \in F)\].

Definition 1.7 [20]. Let $X$ be a non-empty set. By an eBE-algebra we shall mean an algebra $(X, \ast, A)$ such that “$\ast$” is a binary operation on $X$ and $A$ is a non-empty subset of $X$ satisfying the following axioms:

(eBE1) $x \ast x \in A$,

(eBE2) $x \ast A \subseteq A$,

(eBE3) $A \ast x = \{x\}$,

(eBE4) $x \ast (y \ast z) = y \ast (x \ast z)$

for all $x, y, z \in X$.

Definition 1.8 [20]. An eBE-algebra $X$ is said to be self distributive if it satisfies:

(11) \[(\forall x, y, z \in X)x \ast (y \ast z) = (x \ast y) \ast (x \ast z)\].

2. On eGE-algebras

In this section, we present the notion of eGE-algebra as a generalization of GE-algebra and study its properties.

Definition 2.1. An algebraic structure $(X, \ast, E)$, where $\ast$ is a binary operation on a non-empty set $X$ and $E$ is a non-empty subset of $X$, is said to be an extended GE-algebra (eGE-algebra for short) if it satisfies the following axioms:

(eGE1) $u \ast u \in E$,

(eGE2) $u \ast E \subseteq E$,

(eGE3) $E \ast u = \{u\}$,

(eGE4) $u \ast (v \ast w) = u \ast (v \ast (u \ast w))$

for all $u, v, w \in X$. 
Throughout the paper, \( E \ast u = \{ e \ast u \mid e \in E \} \) and \( u \ast E = \{ u \ast e \mid e \in E \} \). If \( a, b \in E \) then, by (eGE3), we have \( a \ast b = b \in E \) and \( b \ast a = a \in E \). Hence \( E \) is a closed subset of \( X \).

We introduce a relation \( \leq \) on \( X \) by \( u \leq v \) if and only if \( u \ast v \in E \). By (eGE1) the relation \( \leq \) is reflexive.

**Theorem 2.2.** Every GE-algebra is an eGE-algebra.

**Proof.** Put \( E = \{1\} \). Then \((X, \ast, E)\) is an eGE-algebra.

Every eGE-algebra need not be a GE-algebra which is shown in the following example.

**Example 2.3.** Let \( X = \{a, b, c, d, e\} \) be a set and \( \ast \) a binary operation given in the table:

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Then \((X, \ast, E)\), where \( E = \{c, d\} \), is an eGE-algebra, but not a GE-algebra. Since \( b \ast b = d \) and \( c \ast c = c \) and there is no \( 1 \in X \), such that \( u \ast u = 1 \), for all \( u \in X \).

Note that the relation \( \leq \) need not be transitive in an eGE-algebra. From Example 2.3, we can observe that \( e \ast b = c \in E, b \ast a = d \in E \), but \( e \ast a = a \notin E \).

In the following example, we show that the axioms (eGE1) to (eGE4) are independent.

**Example 2.4.** (i) Let \( X = \{a, b, c, d\} \) be a set and \( \ast \) a binary operation on \( X \) given in the following table:

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Then \((X, \ast, E)\), where \( E = \{a, b\} \), satisfies (eGE2), (eGE3) and (eGE4), but it does not satisfy (eGE1), since \( E \ast u \neq \{u\} \) i.e., \( a \ast c \neq c \) and \( a \in E \).
(ii) Let \( X = \{a, b, c, d\} \) be a set and \(*\) a binary operation on \( X \) in the following table:

\[
\begin{array}{c|cccc}
* & a & b & c & d \\
\hline
a & b & b & c & c \\
b & a & b & c & d \\
c & a & b & c & d \\
d & a & a & c & c \\
\end{array}
\]

Then \((X, *, E)\), where \( E = \{b, c\} \), satisfies (eGE1), (eGE3) and (eGE4), but it does not satisfy (eGE2), since \( u * E \not\subseteq E \), i.e., \( d * b = a \not\in E \) and \( b \in E \).

(iii) Let \( X = \{a, b, c, d\} \) be a set and \(*\) a binary operation on \( X \) given in the following table:

\[
\begin{array}{c|cccc}
* & a & b & c & d \\
\hline
a & c & b & c & c \\
b & b & c & c & c \\
c & a & b & c & d \\
d & a & a & c & c \\
\end{array}
\]

Then \((X, *, E)\), where \( E = \{b, c\} \) satisfies (eGE1), (eGE2) and (eGE4), but it does not satisfy (eGE3), since \( d * d = a \not\in E \).

(iv) Let \( X = \{a, b, c, d\} \) be a set and \(*\) a binary operation on \( X \) given in the following table:

\[
\begin{array}{c|cccc}
* & a & b & c & d \\
\hline
a & c & b & c & c \\
b & b & c & c & c \\
c & a & b & c & d \\
d & a & a & c & c \\
\end{array}
\]

Then \((X, *, E)\), where \( E = \{c, d\} \), satisfies (eGE1), (eGE2) and (eGE3), but it does not satisfy (eGE4), since

\[
a * (b * a) = a * b = b \neq c = a * c = a * (b * c) = a * (b * (a * a)).
\]

**Theorem 2.5.** Let \((X, *, E)\) be an eGE-algebra. If \( E \) is a singleton set, then \((X, *, E)\) is a GE-algebra.

**Proof.** Let \( E = \{a\} \) be a singleton set. If we put \( 1 = a \), then \((X, *, 1)\) is a GE-algebra. 

**Theorem 2.6.** Let \((X, *, E_i)\), for \( i = 1, 2 \), be two eGE-algebras. Then \((X, *, E_1 \cap E_2)\) is also an eGE-algebra.
**Proof.** Let $u \in X$. Since $u * u \in E_1$ and $u * u \in E_2$, we have $u * u \in E_1 \cap E_2$, and so (eGE1) holds. Let $a \in u * (E_1 \cap E_2)$. Then, we can find $b \in E_1 \cap E_2$ such that $a = u * b$. Since $b \in E_1$, $u * b \in E_1$ and $b \in E_2$, $u * b \in E_2$, we have $a = u * b \in E_1 \cap E_2$ and so $u * (E_1 \cap E_2) \subseteq E_1 \cap E_2$. Hence (eGE2) holds. Let $a \in (E_1 \cap E_2) * u$. Then, we can find $b \in E_1 \cap E_2$ such that $a = b * u$. Since $b * u = u$, we have $a = u$, and so $(E_1 \cap E_2) * u = \{u\}$. Hence (eGE3) holds. (eGE4) is obvious. Thus $(X, *, E_1 \cap E_2)$ is also an eGE-algebra.

**Corollary 2.7.** If $(X, *, E_i)$, for $i \in \Lambda$, is a family of eGE-algebras, then $(X, *, \bigcap_{i \in \Lambda} E_i)$ is an eGE-algebra.

**Theorem 2.8.** Let $(X, *, E_i)$, for $i = 1, 2$, be two eGE-algebras. Then $(X, *, E_1 \cup E_2)$ is also an eGE-algebra.

**Proof.** Let $u \in X$. Since $u * u \in E_1$ and $u * u \in E_2$, we have $u * u \in E_1 \cup E_2$ and so (eGE1) holds. For (eGE2), let $a \in u * (E_1 \cup E_2)$. Then, we can find $b \in E_1 \cup E_2$ such that $a = u * b$. If $b \in E_1$, then $a \in E_1$. Also, if $b \in E_2$, then $a \in E_2$. Thus $a \in E_1 \cup E_2$ and so $u * (E_1 \cup E_2) \subseteq E_1 \cup E_2$. Let $a \in (E_1 \cup E_2) * u$. Then, we can find $b \in E_1 \cup E_2$ such that $a = b * u$. Since $b * u = u$, we have $a = u$ and so $(E_1 \cup E_2) * u = \{u\}$. Therefore (eGE3) holds. (eGE4) is obvious. Thus $(X, *, E_1 \cup E_2)$ is an eGE-algebra.

**Corollary 2.9.** If $(X, *, E_i)$, for $i \in \Lambda$, is a family of eGE-algebras, then $(X, *, \bigcup_{i \in \Lambda} E_i)$ is also an eGE-algebra.

**Lemma 2.10.** Let $(X, *, E)$ be an eGE-algebra and $u, v \in X$. Then $u * (u * v) = u * v$.

**Proof.** Let $u, v \in X$. Using (eGE1), (eGE3) and (eGE4), we get

$$u * (u * v) = u * ((u * u) * (u * v)) = u * ((u * u) * v) = u * v.$$

**Theorem 2.11.** Every self-distributive eBE-algebra is an eGE-algebra.

**Proof.** Let $(X, *, E)$ be a self-distributive eBE-algebra and $u, v, w \in X$. Then, by (eBE1), (eBE3), (eBE4), and self-distributivity,

$$u * (v * w) = (u * u) * (u * (v * w)) = u * (u * (v * w)) = u * (v * (u * w)).$$

Hence $X$ is an eGE-algebra.

The converse of the Theorem 2.11 does not have to be true. From Example 2.3, we can observe that $X$ is an eGE-algebra, but not a self-distributive eBE-algebra.
Theorem 2.12. Let $(X, *, E)$ be an eBE-algebra having the property $u*(u*v) = u*v$, for all $u, v \in X$. Then $X$ is an eGE-algebra.

Proof. Let $u, v, w \in X$ and $u*(u*v) = u*v$. Then $u*(v*w) = u*(u*(v*w)) = u*(v*(u*w))$. Hence $X$ is an eGE-algebra.

Proposition 2.13. Let $(X, *, E)$ be an eGE-algebra. Then

(i) $(X; *, X \setminus E)$ is not an eGE-algebra,
(ii) $v*w \in E$ implies $u*(v*w) \in E$,
(iii) $u*(v*u) \in E$,
(iv) $u \leq v*w$ implies $v \leq u*w$,
(v) $u*(v*w) \in E$ implies $v*(u*w) \in E$ and $v*(u*(v*w)) \in E$,
(vi) $u*(v*w) \leq v*(u*w)$,
(vii) $u*(v*w) \not\in E$ implies $u*w \not\in E$ for all $u, v, w \in X$.

Proof. (i) (eGE2) does not hold, since $u*E \not\subseteq X \setminus E$ and $u*E \subseteq E$.
(ii) By (eGE2), (ii) is obvious.
(iii) Using (eGE4), (eGE1) and (eGE2), we have

$$u*(v*u) = u*(v*(u*u)) \in u*E \subseteq E.$$  

(iv) Let $u \leq v*w$. Hence $u*(v*w) \in E$. Then, by (eGE4) and (eGE2), we have $v*(u*w) = v*(u*(v*w)) \in v*E \subseteq E$. Therefore $v \leq u*w$.
(v) From (eGE4), (eGE1) and (eGE2) we have

$$u*((u*v)*u) = u*((u*v)*(u*u)) \in u*E \subseteq E.$$ 

Therefore $u \leq (u*v)*u$.
(vi) Applying (iv) and (eGE4), we can prove (vi).
(vii) By routine calculation we can see that

$$(u*(v*w))*(v*(u*w))$$

$$(u*(v*w))*(v*((u*(v*w)))*u*(v*w))) \in E.$$ 

Thus $u*(v*w) \leq v*(u*w)$.
(viii) It is obvious by (ii).

Theorem 2.14. Let $(X, *, E)$ be an eGE-algebra. The following are equivalent.

(i) $u*v \leq (w*u)*(w*v)$,
(ii) $u*v \leq (v*w)*(u*w)$

for all $u, v, w \in X$. 

Proof. (i)⇒(ii) Let \( u, v, w \in X \) and assume (i). Then
\[
(u \ast v) \ast ((w \ast u) \ast (w \ast v)) \in E.
\]
Hence, by (eGE4) and (eGE2), we get
\[
(u \ast v) \ast ((v \ast w) \ast (u \ast w)) = (u \ast v) \ast ((v \ast w) \ast ((u \ast v) \ast (u \ast w))) \in E.
\]
Therefore \( u \ast v \leq (v \ast w) \ast (u \ast w) \).

(ii)⇒(i) Let \( u, v, w \in X \) and assume (ii). Then \((u \ast v) \ast ((v \ast w) \ast (u \ast w)) \in E\). Hence, by (eGE4) and (eGE2), we get
\[
(u \ast v) \ast ((w \ast u) \ast (w \ast v)) = (u \ast v) \ast ((w \ast u) \ast ((u \ast v) \ast (w \ast v))) \in E.
\]
Therefore \( u \ast v \leq (w \ast u) \ast (w \ast v) \).

Definition 2.15. An eGE-algebra \((X, \ast, E)\) is said to be transitive if it satisfies:
\[
(\forall u, v, w \in X) (u \ast v \leq (w \ast u) \ast (w \ast v)).
\]

Example 2.16. Let \( X = \{a, b, c, d\} \) be a set and \( \ast \) a binary operation given in the following table:

\[
\begin{array}{cccc}
  \ast & a & b & c & d \\
  a & d & d & d & d \\
  b & a & c & c & c \\
  c & a & b & c & d \\
  d & a & b & c & d \\
\end{array}
\]

Then \((X, \ast, E)\), where \( E = \{c, d\}\), is a transitive eGE-algebra but not an eBE-algebra, since \( a \ast (b \ast c) = a \ast c = d \neq c = b \ast d = b \ast (a \ast c) \).

Theorem 2.17. Let \((X, \ast, E)\) be a transitive eGE-algebra. The following hold:

1. \( u \leq v \) implies \( w \ast u \leq w \ast v \),
2. \( u \ast v \leq (v \ast w) \ast (u \ast w) \),
3. \( u \leq v \) implies \( v \ast w \leq u \ast w \),
4. \( ((u \ast v) \ast v) \ast w \leq u \ast w \),
5. \( u \leq v \) and \( v \leq w \) imply \( u \leq w \),
6. \( u \ast (v \ast w) \leq (u \ast v) \ast (u \ast w) \)

for all \( u, v, w \in X \).
Theorem 2.18. Let \((X, \ast, E)\) be an eGE-algebra. Consider \(Y := (X \setminus E) \cup \{1\}\) and define the operation \(\triangleright\) on \(Y\) as follows:

\[
u \triangleright v = \begin{cases} 
  u \ast v & \text{if } u, v \neq 1 \text{ and } u \ast v \notin E, \\
  1 & \text{if } u, v \neq 1 \text{ and } u \ast v \in E, \\
  v & \text{if } u = 1, \\
  1 & \text{if } v = 1.
\end{cases}
\]

Then \((Y, \triangleright, 1)\) is a GE-algebra.

Proof. By (eGE1), \(u \ast u \in E\), for all \(u \in X\). Thus \(u \triangleright u = 1\), for all \(u \in Y\), and so (GE1) holds. By definition of \(\triangleright\), (GE2) hold. To prove \((Y; \triangleright, 1)\) is a GE-algebra it is sufficient to prove that \(u \triangleright (v \triangleright w) = u \triangleright (v \triangleright (u \triangleright w))\), for all \(u, v, w \in Y\). If \(u = 1\) or \(v = 1\) or \(w = 1\), then we have \(u \triangleright (v \triangleright w) = u \triangleright (v \triangleright (u \triangleright w))\). Now, let \(u, v, w \neq 1\). If \(v \ast w \in E\), then \(v \ast w = 1\), and so \(u \ast (v \ast w) = 1\). On the other hand, if \(u \ast w \in E\), then \(u \ast w = 1\) and \(u \ast (v \ast (u \ast w)) = u \ast (v \ast u) = u \ast 1 = 1 = u \ast (v \ast w)\).

Thus \(u \ast (v \ast w) = u \ast (v \ast (u \ast w))\), in this case. If \(u \ast (v \ast w) \notin E\) then, by Proposition 2.13(ii) \& viii), \(u \ast w \notin E, v \ast w \notin E\) and \(v \ast (u \ast w) \notin E\).

Example 2.19. Let \(X = \{a, b, c, d\}\) and \(E = \{c, d\}\). A binary operation \(\ast\) on \(X\) is given in the following table:

\[
\begin{array}{c|cccc}
  \ast & a & b & c & d \\
  \hline 
  a & a & b & c & c \\
  b & d & d & d & d \\
  c & a & b & c & d \\
  d & a & b & c & d \\
\end{array}
\]

Then \((X, \ast, E)\) is an eGE-algebra which is not an eBE-algebra.

Now, \(Y = (X \setminus E) \cup \{1\} = \{1, a, b\}\). Define \(\triangleright\) on \(Y\) with the following table:

\[
\begin{array}{c|cc}
  \triangleright & 1 & a \\
  \hline 
  1 & a & b \\
  a & 1 & b \\
  b & 1 & 1 \\
\end{array}
\]
Then \((Y, \triangleright, 1)\) is a GE-algebra.

We conclude this section with the following theorem whose proof is straightforward.

**Theorem 2.20.** Let \((X, \ast, 1)\) be a GE-algebra and \(E_0\) be a set such that \(E_0 \cap X = \emptyset\). If we define \(Y = X \cup E_0\), \(E = E_0 \cup \{1\}\) and define the operation \(\triangleleft\) on \(Y\) as follows:

\[
x \triangleleft y = \begin{cases} 
  x \ast y & \text{if } x, y \notin A_0, \\
  y & \text{otherwise}.
\end{cases}
\]

Then \((Y, \triangleleft, E)\) is an eGE-algebra.

### 3. Quotient eGE-algebras

In this section, we introduce the notion of a filter in an eGE-algebra and study its properties. We construct a quotient eGE-algebra via a filter of an eGE-algebra. Throughout this section, \(X\) means \((X, \ast, E_0)\) is an eGE-algebra, unless specified otherwise.

**Definition 3.1.** A subset \(F\) of \(X\) is called a filter of \(X\) if it satisfies:

\((\text{eGEF1})\) \(E \subseteq F\),

\((\text{eGEF2})\) \(u \in F\) and \(u \ast v \in F\) imply \(v \in F\).

The set of all filters of \(X\) will be denoted by \(\mathcal{F}(X)\). Clearly, \(\mathcal{F}(X) \neq \emptyset\), since \(X \in \mathcal{F}(X)\).

**Example 3.2.** Let \(X = \{a, b, c, d, e\}\) be a set and \(\ast\) a binary operation on \(X\) given in the following table:

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<td>b</td>
<td>c</td>
<td>d</td>
<td>d</td>
</tr>
</tbody>
</table>

Then \((X, \ast, E)\), where \(E = \{c, d\}\), is an eGE-algebra. Let \(F = \{c, d, e\}\). Then \(F \in \mathcal{F}(X)\).

**Proposition 3.3.** Let \(F \in \mathcal{F}(X)\). If \(u \in F\) and \(u \leq v\), then \(v \in F\).

**Proof.** Let \(u \in F\) and \(u \leq v\). Then \(u \ast v \in E\) and \(E \subseteq F\). So that \(u \ast v \in F\). Hence \(v \in F\), since \(u \in F\) and \(F\) is a filter of \(X\). \(\blacksquare\)
Theorem 3.4. In $X$, $E \in \mathcal{F}(X)$.

Proof. Clearly (eGEF1) holds, since $E \subseteq E$. Now we prove (eGEF2). Let $u, u * v \in E$. Now, by (eGE3), we have $v = u * v \in E$. Therefore $E \in \mathcal{F}(X)$.

Proposition 3.5. If $F_i \in \mathcal{F}(X)$, for $i \in \Lambda$, then $\bigcap_{i \in \Lambda} F_i \in \mathcal{F}(X)$.

Theorem 3.6. Let $F \in \mathcal{F}(X)$. Then $F_1 = (F \setminus E) \cup \{1\}$ is a filter of $(Y, \triangleright, 1)$, which is defined in Theorem 2.18.

Proof. Clearly $1 \in F_1$. Let $u \in F_1$ and $u \triangleright v \in F_1$. If $u = 1$, then $v = 1 \triangleright v \in F_1$. Let $u \neq 1$. If $v = 1$, then $v \in F_1$. If $v \neq 1$. Then $u \in F \setminus E$ and $v \in X \setminus E$. If $u \triangleright v = 1$ by definition of $\triangleright$ we get $u * v \in E$. Then $u * v \in F$, since $F$ is a filter of $X$. Hence $v \in F$. Thus $v \in F_1$. If $u \triangleright v \neq 1$, then by definition of $\triangleright$, $u * v \notin F$ and $u \triangleright v = u * v \in F_1$. Thus $u * v \in F$. Since $F \in \mathcal{F}(X)$, we have $v \in F$. Hence $v \in F_1$. Therefore $F_1 \in \mathcal{F}(Y)$.

Example 3.7. From Theorem 2.18 and Example 2.19, we get $Y = \{1, a, b\}$ with the following table:

<table>
<thead>
<tr>
<th>$\triangleright$</th>
<th>1</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td>a</td>
<td>1</td>
<td>1</td>
<td>b</td>
</tr>
<tr>
<td>b</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

which is a GE-algebra. We can observe that $F = \{a, c, d\}$ is a filter of $(X, *, E)$ and $F_1 = (F \setminus E) \cup \{1\} = \{1, a\}$ is a filter of $(Y, \triangleright, 1)$.

The following theorem can be proved easily.

Theorem 3.8. Let $(X, *, 1)$ be a GE-algebra, $F \in \mathcal{F}(X)$ and $E_0$ be a set such that $X \cap E_0 = \emptyset$. Then $F_0 = F \cup E_0$ is a filter of an eGE-algebra $(Y, \triangleleft, E)$, which is defined in Theorem 2.20.

The following example describes the above theorem

Example 3.9. Let $X = \{1, a, b\}$ and $E_0 = \{c, d\}$. According to Example 2.19, $(X, \triangleright, 1)$ is a GE-algebra. We can observe that $F = \{1, a\}$ is a filter of $X$. By Theorem 2.20, we get $Y = \{1, a, b, c, d\}, E = \{1, c, d\}$ and $(Y, \triangleleft, E)$ is an eGE-algebra with the following table:

<table>
<thead>
<tr>
<th>$\triangleleft$</th>
<th>1</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>d</td>
</tr>
<tr>
<td>a</td>
<td>1</td>
<td>1</td>
<td>b</td>
<td>c</td>
<td>d</td>
</tr>
<tr>
<td>b</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>c</td>
<td>d</td>
</tr>
<tr>
<td>c</td>
<td>1</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>d</td>
</tr>
<tr>
<td>d</td>
<td>1</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>d</td>
</tr>
</tbody>
</table>
We can observe that $F_0 = F \cup E_0 = \{1, a, c, d\}$ is a filter of $Y$.

**Proposition 3.10.** A non-empty subset $F$ of an $eGE$-algebra $X$ is a filter of $X$ if and only if it satisfies:

(i) $E \subseteq F$,
(ii) $u \ast (v \ast w) \in F, v \in F$ implies $u \ast w \in F$

for all $u, v, w \in X$.

**Proof.** Suppose $F \in \mathcal{F}(X)$. Then $E \subseteq F$. Let $u, v, w \in X$ be such that $u \ast (v \ast w) \in F$ and $v \in F$. Then, by Theorem 2.13(vii) and Proposition 3.3, we have $v \ast (u \ast w) \in F$. Then $u \ast w \in F$. Conversely, assume that the conditions hold. It is sufficient to prove (eGEF2). Let $x \in F$ and $x \ast y \in F$. Then $x \ast x \in E \subseteq F$ and $(x \ast x) \ast (x \ast y) = x \ast y \in F$. Hence $(x \ast x) \ast y = y \in F$. Thus $F \in \mathcal{F}(X)$. ■

**Theorem 3.11.** Let $F$ be a subset of $X$ satisfying the following conditions:

(eGEF1) $E \subseteq F$,
(eGEF3) $u \in X$ and $r \in F$ imply $u \ast r \in F$,
(eGEF4) $u \in X, r, s \in F$ imply $(r \ast (s \ast u)) \ast u \in F$.

Then $F \in \mathcal{F}(X)$.

**Proof.** It is sufficient to prove (eGEF2). Let $u \in F$ and $u \ast v \in F$. Then, by (eGE1), (eGE3) and (eGEF4), $v = [(u \ast v) \ast (u \ast v)] \ast v \in F$ and hence (eGEF2) holds. Therefore $F \in \mathcal{F}(X)$. ■

**Theorem 3.12.** If $X$ is an $eGE$-algebra and $F$ is a filter of $X$, then $F$ satisfies (eGEF1), (eGEF3) and (eGEF4).

**Proof.** It is sufficient to prove (eGEF3) and (eGEF4). Let $F \in \mathcal{F}(X)$ and $r \in F, u \in X$. Then $r \ast (u \ast r) \in E \subseteq F$ and hence, by (GEF2), $u \ast r \in F$. Let $r, s \in F$. Since $r \ast ((r \ast (s \ast u)) \ast (s \ast u)) \in E \subseteq F$ and $r \in F$, we have $(r \ast (s \ast u)) \ast (s \ast u) \in F$. Hence, by (eGE4) and (eGEF3), $s \ast ((r \ast (s \ast u)) \ast u) = s \ast ((r \ast (s \ast u)) \ast (s \ast u)) \in F$. Thus, by (eGEF2), $(r \ast (s \ast u)) \ast u \in F$. ■

**Theorem 3.13.** Let $F \in \mathcal{F}(X)$. Then $(r \ast u) \ast u \in F$ for all $r \in F$ and $u \in X$.

For a non-empty subset $I$ of $X$, we define the binary relation $\sim_I$ in the following way:

$u \sim_I v$ if and only if $u \ast v \in I$ and $v \ast u \in I$.

The set $\{s \mid r \sim_I s\}$ will be denoted by $[r]_I$.

**Lemma 3.14.** In the above relation $\sim_I$, if $E \subseteq I$ and $r \in E$, then $[r]_I = I$. 
Proof. Let \( u \in I \) and \( r \in E \). By (eGE3), we have \( r \ast u \in E \ast u = \{ u \} \subseteq I \) and so \( r \ast u \in I \). From (eGE2), we have \( u \ast r \in u \ast E \subseteq E \subseteq I \), then \( u \ast r \in I \). Hence \( r \sim_I u \). Therefore \( I \subseteq [r]_I \). Conversely, let \( r \in E \) and \( u \in [r]_I \). Then \( u \sim_I r \) and so \( u \ast r = u \in I \). Hence \([r]_I \subseteq I \). Therefore \([r]_I = I \). \( \square 

Theorem 3.15. Let \((X, \ast, E)\) be a transitive eGE-algebra and \( F \in \mathcal{F}(X) \). Then \( \sim_F \) is a congruence relation on \( X \).

Proof. Since \( u \ast u \in E \subseteq F \), we have \( u \ast u \in F \), and so \( u \sim_F u \). If \( u \sim_F v \), then clearly \( v \sim_F u \). Now, let \( u \sim_F v \) and \( v \sim_F w \). Then \( u \ast v, u \ast u \in F \) and \( v \ast w, u \ast v \in F \). By Proposition 2.17(1), we have \( v \ast w \leq (u \ast v) \ast (u \ast w) \), and so by Proposition 3.3, we have \((u \ast v) \ast (u \ast w) \in F \). Since \( F \) is a filter and \( u \ast v \in F \), we have \( u \ast w \in F \). Similarly, we can prove that \( w \ast u \in F \). Thus \( u \sim_F w \). Therefore \( \sim_F \) is an equivalent relation on \( X \). If \( r \sim_F s \) and \( u \sim_F v \), then \( r \sim s, s \sim r \in F \) and \( u \ast v, v \ast u \in F \). By Proposition 2.17(1), we have \( u \ast v \leq (r \ast u) \ast (r \ast v) \) and \( v \ast u \leq (r \ast v) \ast (r \ast u) \), and so by Proposition 3.3, we have \((r \ast u) \ast (r \ast v) \in F \) and \((r \ast v) \ast (r \ast u) \in F \). Thus \( r \sim u \sim_F r \ast v \). Similarly, we can prove that \( r \sim u \sim_F s \ast v \). Since the relation \( \sim_F \) is transitive, we have \( r \ast u \sim_F s \ast v \) which proves that \( \sim_F \) is a congruence relation on \( X \). \( \square 

Proposition 3.16. Let \( \sim_G \) be a congruence relation on \( X, E \subseteq G \) and \( r \in E \). Then \([r]_G \in \mathcal{F}(X)\).

Proof. By Lemma 3.14, we have \([r]_G = G\). Let \( u, u \ast v \in [r]_G \). Thus \( u \sim_G r \) and \( u \ast v \sim_G r \). Since \( v \sim_G v \) and \( \sim_G \) is a congruence relation, we can observe that \( r \sim_G u \ast v \sim_G r \ast v = v \) (by (eGE3)). Thus \( v \in [r]_G \). Therefore \([r]_G \in \mathcal{F}(X)\). \( \square 

Denote \( \frac{X}{\sim_G} = \{ [u]_G \mid u \in X \} \). Define a binary operation \( \bullet \) on \( \frac{X}{\sim_G} \) by \( [u]_G \bullet [v]_G = [u \ast v]_G \). Then by above theorem, \( \bullet \) is well-defined. The following theorem shows that for a transitive eGE-algebra \((X, \ast, E), r \in E \) and \( F \in \mathcal{F}(X) \), the quotient algebra \( \left( \frac{X}{\sim_F}, \bullet, [r]_F \right) \) is a GE-algebra.

Theorem 3.17. Let \((X, \ast, E)\) be a transitive eGE-algebra, \( F \in \mathcal{F}(X) \) and \( r \in E \). Then \( \left( \frac{X}{\sim_F}, \bullet, [r]_F \right) \) is a GE-algebra.

Proof. Since \( E \subseteq F \), we can observe that \( E \subseteq [r]_F \), for all \( r \in E \). Hence \([r]_F \) is a filter by Proposition 3.16 and so \( \sim_{[r]_F} \) is a congruence relation on \( X \) by Theorem 3.15. Now, we have

\[
(GE1) \quad [u]_F \bullet [u]_F = [u \ast u]_F = [r]_F, \quad \text{since} \quad u \ast u \in E \subseteq [r]_F, \\
(GE2) \quad [r]_F \bullet [u]_F = [r \ast u]_F = [u]_F, \quad \text{since} \quad E \ast u \subseteq \{ u \} \text{ and so } r \ast u = u, \\
(GE3) \quad [u]_F \bullet ([v]_F \bullet [w]_F) = [u]_F \bullet [v \ast w]_F = [u \ast (v \ast w)]_F = [u \ast (v \ast (u \ast w))]_F = [u]_F \bullet ([v \ast (u \ast w)])_F = [u]_F \bullet ([v]_F \bullet [u \ast v]_F) = [u]_F \bullet ([v]_F \bullet ([u]_F \bullet [w]_F)).
\]

Thus \( \left( \frac{X}{\sim_{[r]_F}}, \bullet, [r]_F \right) \) is a GE-algebra. \( \square 

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References


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