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COMPARISON OF DIRECT AND PERTURBATION APPROACHES TO ANALYSIS OF INFINITE–DIMENSIONAL FEEDBACK CONTROL SYSTEMS

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For general boundary control systems in factor form some necessary and sufficient conditions for generation of an analytic exponentially stable semigroup are proposed in both direct and perturbation forms for comparison. The direct approach is applicable to operators with the numerical range satisfying certain additional conditions. In particular, it applies to operators similar to convexoids and therefore it generalizes previous results known for hyponormal operators. The perturbation result (indirect approach) is derived and formulated as an exponential stability robustness result using the frequence-domain considerations. It is expressed in terms of some estimates of the resolvent growth over the open right complex half-plane and compared with some recent results. The analysis is illustrated in detail with examples of an unloaded and loaded electric RC-transmission line with proportional negative feedback.

Keywords: perturbation, semigroups, systems in factor form, spectral operators.

1. Introduction

Well-posedness and stability are usually a starting point for the analysis of infinite-dimensional systems. Exponential stability is the most important kind of stability in engineering applications. After building an abstract model of dynamics, well-posedness is reduced to examination whether the state operator is generating a C_0 or analytic semigroup. Standard tools for that are the Hille–Phillips–Yosida, Lumer–Phillips or Hille theorems. In some cases verification of their assumptions may turn to be difficult and one has to apply indirect approaches such as spectral of perturbation methods. The same concerns verification of the stability examination. Both problems are much more subtle when feedback is introduced because generation of the semigroup property as well as exponential stability might then be lost. Robustness of semigroup generation as well as exponential stability with respect to a prescribed type of feedback is the most desired case.

The problem of spectral characterizations of semigroup generators has a long history (Röh, 1982b; 1982a).

Application of Riesz bases of eigenvectors and generalized eigenvectors has been a topic of several survey papers addressing *control theory* problems (Curtain, 1984; Curtain and Zwart, 1995; Grabowski, 1990; 1999; 2006). It is emphasized in last two references that we have numerous effective criteria of recognizing whether a given linear differential operator has a Riesz basis of eigenvectors and generalized eigenvectors (Kesel'man, 1982; Mikhajlov, 1962; Katsnel'son, 1967; Shkalikov, 1982; 1986; Mennicken andMöller, 2003). Spectral properties of the second order differentiation operator are fully analyzed by Lang and Locker (1989; 1990).

As a rule, the strict regularity of boundary conditions leads to the existence of such a Riesz basis. Regular boundary conditions lead to either Riesz bases of eigensubspaces or to nonspectral Dunford–Schwartz operators.

The direct approach to the strongly continuous or analytic semigroup generation problem employing Riesz bases becomes complicated when explicit calculation of spectra or eigenvectors/Riesz projectors are required. In such cases one can try to apply perturbation methods.

Those are quite natural in control theory since a feedback state operator can be regarded as a perturbation of an open-loop operator, perturbed by a feedback of a linear and nonlinear type.

The aim of this paper is twofold. Firstly, we present a basis theory of boundary control systems in factor form; within the frames of it, the spectral approach and perturbation attempts aiming at semigroup generation will be the given. Secondly, we compare the efficiencies of spectral and perturbation analyses.

Earlier foundations of perturbation methods in semigroup generation aspects can be found, e.g., in the works of Pazy (1983) or Engel and Nagel (2000). A recent paper by Adler *et al*. (2017) brought the sharpening of several previous perturbation criteria including the Weiss–Staffans perturbation theorem to its contemporary form. The authors proved two results. The first one concerns C_0-semigroup generators.

Theorem 1. *Let $\mathcal{A}$ generate a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on a Banach space X, $B \in \mathcal{L}(U, X_{-1}^{\mathcal{A}})$ and $\mathcal{C} \in \mathcal{L}(Z, Y)$; U, Z and Y be Banach spaces, $X_{-1}^{\mathcal{A}}$ be a Sobolev extrapolated space. Moreover, assume that there exist $1 \leq p < \infty$, $t > 0$ and $M \geq 0$ such that*

(i) $\operatorname{range}((\lambda I - \mathcal{A}_{-1})^{-1}B) \subseteq Z$ *for some* $\lambda \in \rho(\mathcal{A})$,

(ii) $\displaystyle\int_0^t T_{-1}(t-s)Bu(s)\,\mathrm{d}s \in X$ *for all* $u \in L^p(0, t; U)$,

(iii) $\displaystyle\int_0^t \|\mathcal{C}T(s)x\|_U^p\,\mathrm{d}t \leq M\|x\|_X^p$ *for all* $x \in D(\mathcal{A})$,

(iv) $\displaystyle\int_0^t \|\mathcal{C}\int_0^r T_{-1}(r-s)Bu(s)\,\mathrm{d}(s)\|_U^p\,\mathrm{d}r \leq M\|u\|_U^p$ *for all* $u \in W_0^{2,p}([0,t]; U)$,

(v) $1 \in \rho(\mathbb{F}_t^{(\mathcal{A},B,\mathcal{C})})$,

$$\mathbb{F}_t^{(\mathcal{A},B,\mathcal{C})} := \mathcal{C}\int_0^r T_{-1}(r-s)Bu(s)\,\mathrm{d}s,$$

Then the operator $\mathcal{A}_{B\mathcal{C}}$ generates a C_0-semigroup on X,

$$\mathcal{A}_{B\mathcal{C}} = (\mathcal{A}_{-1} + B\mathcal{C})|_X,$$
$$D(\mathcal{A}_{B\mathcal{C}}) = \{x \in Z : (\mathcal{A}_{-1} + B\mathcal{C})x \in X\}.$$

Here $X_{-1}^{\mathcal{A}}$ is defined as completion of X with respect to the norm $\|x\|_{X_{-1}^{\mathcal{A}}} := \|\mathcal{A}^{-1}x\|_X$; $\mathcal{A} : (D(\mathcal{A}) \subset X) \longrightarrow X$ as the generator of a C_0-semigroup $\{T(t)\}_{t\geq 0} \subset \mathbf{L}(X)$ extends to $\mathcal{A}_{-1} : (D(\mathcal{A}_{-1}) = X \subset X_{-1}^{\mathcal{A}}) \longrightarrow X_{-1}^{\mathcal{A}}$, generating a C_0-semigroup $\{T_{-1}(t)\}_{t\geq 0} \subset \mathbf{L}(X_{-1}^{\mathcal{A}})$, with $T_{-1}(t)$ being the extension of $T(t)$.

The second result of Adler *et al*. (2017) concerns the robustness of analytic semigroup generation.

Theorem 2. *Let $\mathcal{A}$ generates an analytic semigroup of angle $\theta \in (0, \pi/2]$ on X. If there are $\beta \in [0,1]$ and $\gamma \in (0,1]$ such that*

(i) $\operatorname{range}((\lambda I - \mathcal{A}_{-1})^{-1}B) \subseteq F_{1-\beta}^{\mathcal{A}}$ *for some* $\lambda \in \rho(\mathcal{A})$, *where* $F_\alpha^{\mathcal{A}}$ *is the* Favard space *of order* α,

(ii) $[D((\lambda I - \mathcal{A})^\gamma] \hookrightarrow Z$ *for some* $\lambda > \omega_0(\mathcal{A})$, *where* $\omega_0(\mathcal{A})$ *denotes the semigroup type,*

(iii) $\beta + \gamma < 1$,

then A_{BC} generates an analytic semigroup of angle θ on X.

Theorem 1 has to be compared with the result of Grabowski (2017), repeated for convenience in the present paper as Theorem 8. Here $\mathrm{H} = X$, U and Y are assumed to be Hilbert spaces and $\mathcal{A}$ generates an exponentially stable **EXS** C_0-semigroup on H. Since $\mathcal{A}^{-1} \in \mathbf{L}(\mathrm{H})$, one has $\mathcal{A}_{B\mathcal{C}} = \mathcal{A}_c$, $\mathcal{A}_c$ is given by (10) with $\mathcal{K} = -I$ and $\mathcal{D} = \mathcal{A}_{-1}B$ (factor control dynamics do not involve $X_{-1}^{\mathcal{A}}$). Our condition $R(\mathcal{D}) \subset D(\mathcal{C})$, $\mathcal{C}\mathcal{D} \in \mathbf{L}(\mathrm{U}, \mathrm{Y})$ implies (i) with $\lambda = 0$. The assumptions (ii), (iii) hold with $p = 2$ as $\mathcal{C}$ and $\mathcal{D}$ are infinite-time admissible. By the Paley–Wiener theory the assumption (iv) holds as (5) is satisfied. Finally, **(v)** is fulfilled thanks to our assumption (8) with $\mathcal{K} = -I$.

In terms of factor control systems with $\mathcal{A}$ generating an **EXS** analytic semigroup, the condition (i) of Theorem 2 requires that

$$\operatorname{range}(\mathcal{A}(\lambda I - \mathcal{A})^{-1}\mathcal{D}) \subseteq F_{1-\beta}^{\mathcal{A}},$$

$$F_\alpha^{\mathcal{A}} = \{x \in X : \lim_{\lambda\to\infty, \lambda\in\mathbb{R}} \|\lambda^\alpha \mathcal{A}(\lambda I - \mathcal{A})^{-1}x\|_X = 0\}$$

for $\alpha \in (0,1)$, while (ii) means that $\operatorname{range}(H^*) \subseteq D((-\mathcal{A})^\gamma)$, and $(-\mathcal{A})^\gamma$ is the fractional power of $(-\mathcal{A})$ of order γ.

Those facts show that Theorem 1 ensures robustness of a semigroup generation while Theorem 8 yields robustness of **EXS** semigroup generation. Similarly, Theorem 2 concerns stability of generation of an analytic semigroup of angle θ while Theorem 9 establishes robustness of analytic **EXS** semigroup generation, with indication for which $\mathcal{K}$ this is true. Thus, in the case of Hilbert spaces, the perturbation results of the present paper yield stronger statements under stronger assumptions. Theorems 1 and 2 are time-domain results on a finite horizon, while Theorems 8 and 9 are frequency-domain results. Let us mention that the finite time admissibility and the infinite one are equivalent under **EXS**. Though Theorem 9, in comparison with Theorem 2, seems to be a simple result, a deeper fact is that in Theorem 9 the system-theoretic operators are treated to act in the balance aspect, contrary to Theorem 2, where their actions are treated separately. It should be stressed that in the case where the open-loop system is statically exponentially stabilizable the assumption that open-loop system operator generates an exponentially stable semigroup is not too restrictive. There are also open-loop systems which do not generate any semigroup, but the closed-loop state operator generates an exponentially stable analytic semigroup. An example of such a system is given in Section 6.

Example 1 of the present paper corresponds to an SISO control system with $\mathcal{D} = d \in F^{\mathcal{A}}_{1/4-\varepsilon}$ (by (32)) and $H^* = h \in D((-\mathcal{A})^{3/4-\varepsilon})$, $\varepsilon > 0$ ($\langle \mathcal{A}x, h\rangle_{\mathrm{L}^2(0,1)} = \langle (-\mathcal{A})^{1/4+\varepsilon}x, (-\mathcal{A})^{3/4-\varepsilon}h\rangle_{\mathrm{L}^2(0,1)}$ (see (27)). Thus the assumption (i) of Theorem 2 is satisfied with $\beta = 3/4 + \varepsilon$ and the assumption (ii) holds for $\gamma = 1/4 + \varepsilon$, so $\beta + \gamma > 1$ and the assumption (iii) is not satisfied. Thus this example is beyond the scope of Theorem 2, but it is captured by Theorem 9 as elucidated in Section 4.1. Moreover, as shown in Sections 4.2 and 4.3, spectral methods can also be applied to establishing **EXS** analytic semigroup generation under feedback perturbation, although with tiresome consideration.

A relation with the theory of *robustness of strong stability* (Paunonen, 2014) is commented in Remark 3.

Example 2 of Section 5 demonstrates that Theorem 9 still remains an effective tool for obtaining analytic **EXS** semigroup generation, contrary to spectral methods (its state operator is not a Dunford–Schwartz spectral operator).

The paper is structured as follows. In Section 2 an overview of the theory of boundary controlled systems in factor form is presented. The main aspects of direct and perturbation attempts are stated in Section 3. Examples are widely discussed in Sections 4 and 5. Complementary results and relationships with some related problems are given in Section 6.

2. Overview of control systems in factor form

Consider a class of control systems with observation governed by the model in factor form,

$$\begin{cases} \dot{x}(t) = \mathcal{A}\left[x(t) + \mathcal{D}u(t)\right], \\ x(0) = x_0, \\ y(t) = \mathcal{C}x(t), \end{cases} \tag{1}$$

where the linear *state operator* $\mathcal{A} : (D(\mathcal{A}) \subset \mathrm{H}) \longrightarrow \mathrm{H}$ acts on a Hilbert *state space* H with scalar product $\langle\cdot,\cdot\rangle_{\mathrm{H}}$ and is invertible with $\mathcal{A}^{-1} \in \mathbf{L}(\mathrm{H})$.

$\mathcal{C} : (D(\mathcal{C}) \subset \mathrm{H}) \longrightarrow \mathrm{Y}$ is an *observation (output) operator*, such that $D(\mathcal{A}) \subset D(\mathcal{C})$ and $H := \mathcal{C}\mathcal{A}^{-1} \in \mathbf{L}(\mathrm{H}, \mathrm{Y})$. Here Y denotes an *output space* which is a Hilbert space with scalar product $\langle\cdot,\cdot\rangle_{\mathrm{Y}}$.

$\mathcal{D} \in \mathbf{L}(\mathrm{U}, \mathrm{H})$ with range $R(\mathcal{D}) \subset D(\mathcal{C})$, $\mathcal{C}\mathcal{D} \in \mathbf{L}(\mathrm{U}, \mathrm{Y})$ is a *factor control operator* and U stands for a space of controls which is also a Hilbert space with scalar product $\langle\cdot,\cdot\rangle_{\mathrm{U}}$.

Two cases where $\mathcal{C}\mathcal{D} \in \mathbf{L}(\mathrm{U}, \mathrm{Y})$ are indicated in Section 6.

2.1. Semigroups and state operators.

Definition 1. A family $\{S(t)\}_{t\geq 0} \subset \mathbf{L}(\mathrm{H})$ is a C_0-*semigroup* on H if **(i)** $S(0) = I$, $S(t+\tau) = S(t)S(\tau)$ for $t, \tau \geq 0$ and **(ii)** $S(t)x_0 \to x_0$ as $t \to 0$ for every $x_0 \in \mathrm{H}$.

$\{S(t)\}_{t\geq 0}$ is *exponentially stable* (**EXS**) if there exist $M \geq 1$, $\alpha > 0$ such that

$$\|S(t)\|_{\mathbf{L}(\mathrm{H})} \leq Me^{-\alpha t} \qquad \forall t \geq 0. \tag{2}$$

The *generator* of a C_0 semigroup $\{S(t)\}_{t\geq 0}$ is defined by

$$\mathcal{A}x_0 = \lim_{h\to 0} \frac{1}{h}[S(h)x_0 - x_0],$$
$$D(\mathcal{A}) = \{x_0 \in \mathrm{H} : \exists \lim_{h\to 0} \frac{1}{h}[S(h)x_0 - x_0]\}.$$

Theorem 3. (Hille–Phillips–Yosida) *A linear operator $\mathcal{A} : (D(\mathcal{A}) \subset H) \longrightarrow H$ generates a C_0-semigroup $\{S(t)\}_{t\geq 0}$ satisfying the growth estimate $\|S(t)\|_{\mathbf{L}(H)} \leq Me^{\omega t}$ for $t \geq 0$ and some $M \geq 1$, $\omega \in \mathbb{R}$ (by the principle of boundedness every C_0-semigroups satisfies this estimate) iff $\mathcal{A}$ is closed densely defined and its resolvent $(sI - \mathcal{A})^{-1}$ satisfies the estimate*

$$\|(sI - \mathcal{A})^{-n}\|_{\mathbf{L}(H)} \leq \frac{M}{(s-\omega)^n} \qquad \forall s > \omega, \quad \forall n \in \mathbb{N}.$$

Theorem 4. (Prüss–Huang–Weiss) *A C_0-semigroup generated by $\mathcal{A}$ is* **EXS** *iff $s \longmapsto (sI - \mathcal{A})^{-1}$ is in the Hardy class $H^\infty(\mathbb{C}^+, \mathbf{L}(H))$, $\mathbb{C}^+ = \{s \in \mathbb{C} : \mathrm{Re}\, s > 0\}$.*

The main contribution is due to Prüss (1984). A new short proof is given by Curtain and Zwart (1995, Theorem 5.1.5, p. 222). A yet another proof is given by Engel and Nagel (2000, pp.302-303).

In what follows we shall assume that $\mathcal{A}$ generates **EXS** semigroup $\{S(t)\}_{t\geq 0}$ on H. As H is a Hilbert space, this implies that $\mathcal{A}^*$ generates **EXS** semigroup $\{S^*(t)\}_{t\geq 0}$ on H. Since the resolvent $s \mapsto (sI - \mathcal{A})^{-1}x_0$ is the *Laplace transform* of $t \mapsto S(t)x_0$, by (2), the half-plane $\{s \in \mathbb{C} : \mathrm{Re}\, s > -\alpha\}$ is contained in $\rho(\mathcal{A})$, the resolvent set of $\mathcal{A}$, which, in particular, implies that $\mathcal{A}$ is invertible with $\mathcal{A}^{-1} \in \mathbf{L}(\mathrm{H})$.

Definition 2. Let $x_0 \in \mathrm{H}$ and $u \in \mathrm{L}^2(0,\infty;\mathrm{U})$. A continuous vector valued function $t \mapsto x(t) \in \mathrm{H}$ is called a *weak solution* of (1) if $x(0) = x_0$ and x satisfies (1) in a *weak sense*, i.e., the function $t \mapsto \langle x(t), w\rangle_{\mathrm{H}}$ is absolutely continuous and, for almost all $t \geq 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle x(t), w\rangle_{\mathrm{H}} = \langle x(t), \mathcal{A}^* w\rangle_{\mathrm{H}} + \langle \mathcal{D}u(t), \mathcal{A}^* w\rangle_{\mathrm{H}}, \quad w \in D(\mathcal{A}^*).$$

Theorem 5. (Ball, 1977) *A linear operator $\mathcal{A}$ generates a C_0-semigroup $\{S(t)\}_{t\geq 0}$ on H iff $\mathcal{A}$ is closed densely defined and for each $x_0 \in H$ there exists a unique weak solution of (1) with $\mathcal{D} = 0$ and $\mathcal{C} = 0$.*

It is known that, if X is a Hilbert space, then

$$\begin{aligned}\mathcal{L}_{\mathrm{X}} f &= f',\\ D(\mathcal{L}_{\mathrm{X}}) &= \mathrm{W}^{1,2}([0,\infty);\mathrm{X})\\ &:= \left\{f \in \mathrm{L}^2(0,\infty;\mathrm{X}) : f' \in \mathrm{L}^2(0,\infty;\mathrm{X})\right\}\\ &\subset \mathrm{C}([0,\infty);\mathrm{X})\end{aligned}$$

generates C_0-semigroup $\{T_{\mathrm{X}}(t)\}_{t\geq 0}$ of *left-shifts* on $\mathrm{L}^2(0,\infty;\mathrm{X})$,

$$(T_{\mathrm{X}}(t)f)(\tau) := f(t+\tau), \quad t \geq 0,$$

whilst its adjoint $\mathcal{L}_{\mathrm{X}}^* := \mathcal{R}_{\mathrm{X}}$,

$$\mathcal{R}_{\mathrm{X}} f = -f', \quad D(\mathcal{R}_{\mathrm{X}}) = \mathrm{W}_0^{1,2}([0,\infty);\mathrm{X}),$$

$$\mathrm{W}_0^{1,2}([0,\infty);\mathrm{X}) := \left\{f \in \mathrm{W}^{1,2}([0,\infty);\mathrm{X}) : f(0) = 0\right\},$$

generates an adjoint C_0-semigroup of *right-shifts* on $\mathrm{L}^2(0,\infty;\mathrm{X})$,

$$(T_{\mathrm{X}}^*(t)f)(\tau) := \begin{cases} f(\tau - t) & \text{if } \tau \geq t,\\ 0 & \text{if } 0 \leq \tau < t,\end{cases}$$

$t \geq 0$.

2.2. Admissible observation operators. Define $\mathcal{Z} \in \mathbf{L}(\mathrm{H}, \mathrm{L}^2(0,\infty;\mathrm{Y}))$,

$$(\mathcal{Z}x_0)(t) := HS(t)x_0 \quad \left[\Leftrightarrow \mathcal{Z}^* f = \int_0^\infty S^*(t)H^* f(t)\,\mathrm{d}t\right].$$

The operator, called the *observability map*,

$$\boldsymbol{\Psi} := \mathcal{L}_{\mathrm{Y}}\mathcal{Z}, \qquad D(\boldsymbol{\Psi}) = \{x \in \mathrm{H} : \mathcal{Z}x \in D(\mathcal{L}_{\mathrm{Y}})\},$$

is closed and densely defined, with $\boldsymbol{\Psi}|_{D(\mathcal{A})} = \mathcal{Z}\mathcal{A}$, and therefore it has a closed and densely defined adjoint operator,

$$\begin{aligned}\boldsymbol{\Psi}^* &= \mathcal{A}^*\mathcal{Z}^*,\\ D(\boldsymbol{\Psi}^*) &= \{y \in \mathrm{L}^2(0,\infty;\mathrm{Y}) : \mathcal{Z}^* y \in D(\mathcal{A}^*)\},\end{aligned}$$

and $\boldsymbol{\Psi}^*|_{D(\mathcal{R}_{\mathrm{Y}})} = \mathcal{Z}^*\mathcal{R}_{\mathrm{Y}}$.

Definition 3. $\mathcal{C}$ is an admissible *observation (output) operator* if $\boldsymbol{\Psi} \in \mathbf{L}(\mathrm{H}, \mathrm{L}^2(0,\infty;\mathrm{Y}))$.

In Definition 3 one can alternatively assume that $\boldsymbol{\Psi}$ is bounded or, by the closed graph theorem, $R(\mathcal{Z}) \subset D(\mathcal{L}_{\mathrm{Y}})$.

Lemma 1. *If $\mathcal{C}$ is admissible, then $\boldsymbol{\Psi}$ is also a linear densely defined and* bounded *operator from H into $L^1(0,\infty;Y)$.*

This result is proven by Grabowski (2017, Lemma 2.1).

2.3. Admissible control operators. Define $\mathcal{W} \in \mathbf{L}(\mathrm{L}^2(0,\infty;\mathrm{U}), \mathrm{H})$,

$$\mathcal{W}f := \int_0^\infty S(t)\mathcal{D}f(t)\,\mathrm{d}t \quad \left[\Leftrightarrow (\mathcal{W}^* x_0)(t) = \mathcal{D}^* S^*(t)x_0\right].$$

The operator, called the *reachability map*,

$$\begin{aligned}\boldsymbol{\Phi} &:= \mathcal{A}\mathcal{W},\\ D(\boldsymbol{\Phi}) &= \{u \in \mathrm{L}^2(0,\infty;\mathrm{U}) : \mathcal{W}u \in D(\mathcal{A})\},\end{aligned}$$

is closed and densely defined, with $\boldsymbol{\Phi}|_{D(\mathcal{R}_{\mathrm{U}})} = \mathcal{W}\mathcal{R}_{\mathrm{U}}$, and therefore it has a closed and densely defined adjoint operator,

$$\begin{aligned}\boldsymbol{\Phi}^* &= \mathcal{L}_{\mathrm{Y}}\mathcal{W}^*,\\ D(\boldsymbol{\Phi}) &= \{x \in \mathrm{H} :\ \mathcal{W}^* x \in D(\mathcal{L}_{\mathrm{U}})\},\end{aligned}$$

with $\boldsymbol{\Phi}^*|_{D(\mathcal{A}^*)} = \mathcal{W}^*\mathcal{A}^*$.

Definition 4. $\mathcal{D}$ is an admissible *factor control operator* if $\boldsymbol{\Phi} \in \mathbf{L}(\mathrm{L}^2(0,\infty;\mathrm{U}), \mathrm{H})$.

In Definition 4 one can alternatively assume that $\boldsymbol{\Phi}$ is bounded or, by the closed graph theorem, $R(\mathcal{W}) \subset D(\mathcal{A})$.

Using duality arguments, we can state the following result (Grabowski and Callier, 1999).

Lemma 2. *$\mathcal{D}$ is an admissible factor control operator iff $\mathcal{D}^*\mathcal{A}^*$ is an admissible observation operator with respect to the semigroup $\{S^*(t)\}_{t\geq 0}$.*

2.4. Representation of the state. In what follows $\mathrm{BUC}([0,\infty);\mathrm{Z})$ denotes the Banach space of *bounded, uniformly continuous* functions defined on $[0,\infty)$ and taking values in a Hilbert space Z, equipped with standard norm

$$\|f\|_{\mathrm{BUC}([0,\infty);\mathrm{Z})} := \sup_{t\geq 0} \|f(t)\|_{\mathrm{Z}},$$

$$f \in \mathrm{BUC}([0,\infty);\mathrm{Z}),$$

whilst $\mathrm{BUC}_0([0,\infty);\mathrm{Z})$ will stand for its closed subspace consisting of functions that have zero limit at infinity.

Theorem 6. *If $\mathcal{A}$ generates an* **EXS** *C_0-semigroup and $\mathcal{D}$ is an admissible factor control operator, then, for every $x_0 \in H$ and $u \in L^2(0,\infty;U)$,*

$$\begin{aligned} x(t) &:= S(t)x_0 + \mathbf{\Phi} R_t u, \\ (R_t u)(\tau) &:= \begin{cases} u(t-\tau) & \text{if } \tau \leq t, \\ 0 & \text{if } \tau > t, \end{cases} \end{aligned} \tag{3}$$

where $R_t \in \mathbf{L}(L^2(0,\infty;U))$, $\|R_t\|_{\mathbf{L}(L^2(0,\infty;U))} = 1$, $R_t = R_t^*$ is called the operator of reflection *at t, there holds $x \in BUC_0([0,\infty),H)$ and x is a unique weak solution of (1).*

Furthermore, for every $z \in H$ the function $t \longmapsto \langle x(t), z\rangle_H$ is in $L^2(0,\infty)$.

For a proof, see the work of Grabowski (2017, Thm. 2.1).

2.5. Representation of the output. Now we pass to the *construction of the system output* in operator form. For that we assume that the system *transfer function*

$$\begin{aligned} \widehat{G}(s) :=& s\mathcal{C}(sI-\mathcal{A})^{-1}\mathcal{D} - \mathcal{C}\mathcal{D} \\ =& s^2 H(sI-\mathcal{A})^{-1}\mathcal{D} - sH\mathcal{D} - \mathcal{C}\mathcal{D} \end{aligned} \tag{4}$$

(thus $\widehat{G}$ is well-defined for $\mathrm{Re}\, s > -\alpha$) satisfies

$$\widehat{G} \in \mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{U},\mathrm{Y})). \tag{5}$$

Let us remark that $\widehat{G}$ is analytic on a set containing $\overline{\mathbb{C}^+}$ and (4), jointly with **EXS**, implies that $\widehat{G}$ grows no faster than quadratically on $\mathbb{C}^+$,

$$\begin{aligned} \|\hat{G}(s)\|_{\mathbf{L}(\mathrm{U},\mathrm{Y})} \leq\, & |s|^2 \|H\| \|(sI-\mathcal{A})^{-1}\| \|\mathcal{D}\| \\ & + |s| \|H\| \|\mathcal{D}\| + \|\mathcal{C}\mathcal{D}\|, \end{aligned}$$

whence, by the *Phragmén–Lindelöf theorem* (5) is met if $\widehat{G}$ is bounded on $i\mathbb{R}$, $i^2 = -1$ (Arendt *et al.*, 2011, Thm. 3.9.8, p. 176).

Theorem 7. *Let $\mathcal{C}$ and $\mathcal{D}$ be admissible and let (5) hold. Then, for every $x_0 \in H$ and $u \in L^2(0,\infty;U)$,*

$$y = \mathbf{\Psi} x_0 + \mathbb{F}u, \tag{6}$$

where $\mathbb{F} \in \mathbf{L}(L^2(0,\infty;U), L^2(0,\infty;Y))$ constitutes the input–output operator*,*

$$(\mathbb{F}u)(t) := \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t (\mathbf{\Psi}[\mathcal{D}u(\tau)])(t-\tau)\,\mathrm{d}\tau - (\mathcal{C}\mathcal{D})u(t).$$

Moreover,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}[Hx(t)] &= y(t) + \mathcal{C}\mathcal{D}u(t) \\ &= \mathcal{C}x(t) + \mathcal{C}\mathcal{D}u(t). \end{aligned} \tag{7}$$

For a proof, see the work of Grabowski (2017, Thm. 2.2).

3. Perturbation theorem for boundary control systems in factor form

3.1. Problem statement. Consider the *Lur'e system* of automatic feedback control having the structure depicted in Fig. 1. Under the standing assumptions: $\mathcal{A}$ generates an **EXS** C_0-semigroup, $\mathcal{C}\mathcal{D} \in \mathbf{L}(\mathrm{U},\mathrm{Y})$ and $\widehat{G} \in \mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{U},\mathrm{Y}))$ we formulate the following task.

Problem 1. Characterize those static feedback operators $\mathcal{K} \in \mathbf{L}(\mathrm{Y},\mathrm{U})$, which gives rise to a closed-loop state operator $\mathcal{A}_c$ that generates an **EXS** C_0-semigroup $\{S_c(t)\}_{t\geq 0}$.

3.2. Operator description of the closed-loop system. Substituting the feedback equation $u = -\mathcal{K}y$ into (6), we get

$$\begin{aligned} y = \mathbf{\Psi}x_0 + \mathbb{F}u = \mathbf{\Psi}x_0 - \mathbb{F}\mathcal{K}y& \\ \iff (I + \mathbb{F}\mathcal{K})\, y = \mathbf{\Psi}x_0&, \end{aligned}$$

or, in the frequency-domain

$$(I + \widehat{G}\mathcal{K})\hat{y} = \widehat{\mathbf{\Psi}x_0}\ .$$

The complete operator-theoretic description of the closed-loop system is depicted in Fig. 2.

The following result is clear (Weidmann, 1980, Problem 5.26, p. 106, with $\lambda = 1$; $A = \mathcal{K}$, $B = \widehat{G}(s)$ or conversely).

Lemma 3. *If $\mathcal{C}$, $\mathcal{D}$ are admissible, (5) holds and*

$$\begin{aligned} &[I + \widehat{G}\mathcal{K}]^{-1} \in H^\infty(\mathbb{C}^+, \mathbf{L}(Y)) \\ &\qquad \iff [I + \mathcal{K}\widehat{G}]^{-1} \in H^\infty(\mathbb{C}^+, \mathbf{L}(U)), \end{aligned} \tag{8}$$

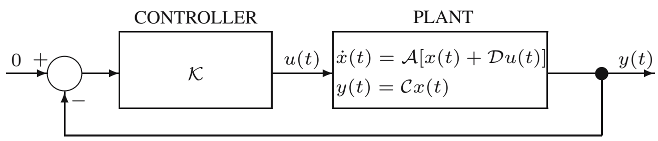

Fig. 1. Lur'e control system with negative feedback.

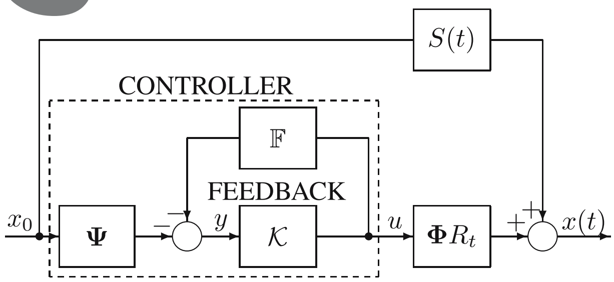

Fig. 2. Operator-theoretic diagram of the Lur'e control system.

then $u \in L^2(0,\infty;U)$*, or equivalently* $\hat{u} = -\mathcal{K}(I + \widehat{G}\mathcal{K})^{-1}\widehat{\mathbf{\Psi}x_0} \in H^2(\mathbb{C}^+, U)$*. Moreover,* $\mathcal{K}(I+\widehat{G}\mathcal{K})^{-1} = (I+\mathcal{K}\widehat{G})^{-1}\mathcal{K}$.

Remark 1. $L, L_0 \in \mathbf{L}(\mathrm{H})$. If $\|L^{-1}(L-L_0)\| < 1$, then, by the *Neumann series theorem*, $\left[I - L^{-1}(L-L_0)\right]^{-1} \in \mathbf{L}(\mathrm{H})$, whence $\left[I - L^{-1}(L-L_0)\right]^{-1} L^{-1} = L_0^{-1} \in \mathbf{L}(\mathrm{H})$. Taking $L_0 = I + \widehat{G}(j\omega)\mathcal{K}$, $L = I + \widehat{G}(s)\mathcal{K}$, $s \in \mathbb{C}^+$, $|s - i\omega|$ sufficiently small, and we conclude that (8) implies

$$[I + \widehat{G}(i\omega)\mathcal{K}]^{-1} \in \mathbf{L}(\mathrm{Y}), \quad \omega \in \mathbb{R}.$$

3.3. Abstract differential model.

Inserting the static controller equation $u = -\mathcal{K}y$ into (1), we get the closed-loop system equation

$$\begin{cases} \dot{x}(t) = \mathcal{A}_c x(t), \\ x(0) = x_0, \end{cases} \tag{9}$$

$$\begin{aligned} \mathcal{A}_c x &= \mathcal{A}(x - \mathcal{D}\mathcal{K}\mathcal{C}x), \\ D(\mathcal{A}_c) &= \{x \in D(\mathcal{C}) :\ x - \mathcal{D}\mathcal{K}\mathcal{C}x \in D(\mathcal{A})\}. \end{aligned} \tag{10}$$

Lemma 4. *If $\mathcal{D}^*\mathcal{A}^*$ extends from $D(\mathcal{A}^*)$ to an operator $\mathcal{D}^\#$ with a larger domain $D(\mathcal{D}^\#)$ such that $R(H^*) \subset D(\mathcal{D}^\#)$, $\mathcal{D}^\# H^* = (\mathcal{C}\mathcal{D})^*$ and $(I + \widehat{G}(0)\mathcal{K})$ is boundedly invertible, then the closed-loop operator (10) is densely defined closed and has a densely defined closed adjoint operator,*

$$\begin{aligned} \mathcal{A}_c^* w &= \mathcal{A}^*\left(w - H^*\mathcal{K}^*\mathcal{D}^\# w\right), \\ D\left(\mathcal{A}_c^*\right) &= \Big\{w \in D(\mathcal{D}^\#) : \\ &\qquad w - H^*\mathcal{K}^*\mathcal{D}^\# w \in D(\mathcal{A}^*)\Big\}. \end{aligned} \tag{11}$$

Proof. We start by demonstrating that

$$\mathcal{A}_c^{-1} = \mathcal{A}^{-1} + \mathcal{D}\mathcal{K}[I - \mathcal{C}\mathcal{D}\mathcal{K}]^{-1}H^*. \tag{12}$$

Clearly, $\mathcal{A}_c^{-1} \in \mathbf{L}(\mathrm{H})$, which implies that $\mathcal{A}_c$ is closed. Next, we find $\mathcal{A}_c^{-*}$ and verify that $\ker \mathcal{A}_c^{-*} = \{0\}$. Hence $\overline{R(\mathcal{A}_c^{-1})} = \overline{D(\mathcal{A}_c)} = \mathrm{H}$, so $\mathcal{A}_c$ is densely defined. The last step is to get (11). Details are given by Grabowski (2017, Lemma 3.2). ■

3.4. Perturbations of C_0-semigroup generators.

Theorem 8. *Let $\mathcal{A}$ generate an **EXS** C_0-semigroup $\{S(t)\}_{t\geq 0}$ on H, $R(\mathcal{D}) \subset D(\mathcal{C})$, $\mathcal{C}\mathcal{D} \in \mathbf{L}(U, Y)$, $\mathcal{C}$ and $\mathcal{D}$ are admissible operators, and $\mathcal{D}^*\mathcal{A}^*$ extend from $D(\mathcal{A}^*)$ to an operator $\mathcal{D}^\#$ with domain $D(\mathcal{D}^\#)$ such that $R(H^*) \subset D(\mathcal{D}^\#)$ and $\mathcal{D}^\# H^* = (\mathcal{C}\mathcal{D})^*$. Assume that (5) and (8) hold. Then, the closed-loop operator $\mathcal{A}_c$, defined by (10), generates an **EXS** C_0-semigroup $\{S_c(t)\}_{t\geq 0}$ on H.*

Proof. By Lemma 4, $\mathcal{A}_c$ is closed and densely defined. From Theorem 6 we know that for each $x_0 \in \mathrm{H}$ the Cauchy problem (10) has a unique weak solution and the assertion follows from Theorem 5 with $\mathcal{A}$ replaced by $\mathcal{A}_c$. ■

Theorem 8 appeared in the work of Grabowski (2017), where $\mathcal{C}\mathcal{A}^{-1} = H^*$, rather than H, so the role of H and H^*, reversed in comparison with the present paper.

3.5. Generators of analytic semigroups.

Definition 5. The C_0-semigroup $\{S(t)\}_{t\geq 0}$ is *analytic* if the semigroup operator $[0,\infty) \ni t \longmapsto S(t) \in \mathbf{L}(\mathrm{H})$ extends analytically in some sector

$$S_\theta := \{z \in \mathbb{C} : |\arg z| < \theta, |z| > 0\}, \qquad \theta \in (0, \pi)$$

to an analytic function $S_\theta \ni z \longmapsto S(z) \in \mathbf{L}(\mathrm{H})$ and, for every $x \in \mathrm{H}$, we have $S(z)x \longrightarrow x$ as $z \to 0$, $z \in S_\theta$.

The extended function necessarily satisfies the semigroup property

$$S(z)S(w) = S(z+w), \qquad \forall w, z \in S_\theta,$$

and is referred to as an *analytic semigroup*. The basic theory of analytic semigroups is presented, e.g., by Pazy (1983), Engel and Nagel (2000), Kantorovitz (2000) and Arendt *et al.* (2011), and we mainly follow them below.

The next result will be of paramount importance.

Proposition 1. *$\mathcal{A}$ generates an **EXS** analytic semigroup $\{S(t)\}_{t\geq 0}$ on H iff (**i**) $\mathbb{C}^+ \cup \{0\} \subset \rho(\mathcal{A})$ and (**ii**) $s(sI - \mathcal{A})^{-1} \in H^\infty(\mathbb{C}^+, \mathbf{L}(H))$.*

Proof. If $\mathcal{A}$ generates an analytic semigroup, then, by the result of Arendt *et al.* (2011, Cor. 3.7.17, p. 157, with $a = 0$), there exists $r > 0$ such that $s(sI - \mathcal{A})^{-1}$ is bounded on the set $\{s \in \mathbb{C}^+ : |s| > r > 0\}$.

By the **EXS** of $\{S(t)\}_{t\geq 0}$, the resolvent set $\rho(\mathcal{A})$ contains the half-plane $\mathrm{Re}\, s > -\varepsilon$, $\varepsilon > 0$, whence (i) holds. Moreover, $(sI - \mathcal{A})^{-1}$ is then bounded on $\mathbb{C}^+$. Hence $s(sI - \mathcal{A})^{-1}$ is bounded on $\{s \in \mathbb{C}^+ : |s| \leq r\}$. Thus $s(sI - \mathcal{A})^{-1} \in \mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{H}))$, i.e., (**ii**) is met.

Conversely, if (i) and (ii) hold then, by Arendt *et al.* (2011, Cor. 3.7.12, pp. 154–155), $\mathcal{A}$ generates a *bounded* analytic semigroup[1] (Arendt *et al.*, 2011, Def. 3.7.3, p. 150). $0 \in \rho(\mathcal{A})$ means that $\mathcal{A}^{-1} \in \mathbf{L}(\mathrm{H})$, and the resolvents of $\mathcal{A}$ and $\mathcal{A}^{-1}$ are tied by the identity

$$\begin{aligned}(sI - \mathcal{A})^{-1} \\ = -s^{-1}\mathcal{A}^{-1}(s^{-1}I - \mathcal{A}^{-1})^{-1}, \quad s \in \mathbb{C}^+, \quad (13)\end{aligned}$$

whence

$$\begin{aligned}s(sI - \mathcal{A})^{-1} &= -\,\mathcal{A}^{-1}(s^{-1}I - \mathcal{A}^{-1})^{-1} \\ &= -\,s^{-1}(s^{-1}I - \mathcal{A}^{-1})^{-1} + I.\end{aligned}$$

The inversion $s \mapsto 1/s$ is an isometry on $\mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{H}))$, giving $s(sI - \mathcal{A}^{-1})^{-1} \in \mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{H}))$, and, consequently, $s\mathcal{A}^{-1}(sI - \mathcal{A}^{-1})^{-1} \in \mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{H}))$. Applying the inversion $s \mapsto 1/s$ once more, from (13) we get $(sI - \mathcal{A})^{-1} \in \mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{H}))$. By Theorem 4, the semigroup $\{S(t)\}_{t\geq 0}$ is **EXS**. ∎

We can also formulate a sufficient condition for **EXS** semigroup generation.

Definition 6. A densely defined operator $\mathcal{A} : (D(\mathcal{A}) \subset \mathrm{H}) \to \mathrm{H}$ is a *convexoid* if the closure of its *numerical range* $W(\mathcal{A}) := \{\langle \mathcal{A}x, x\rangle_\mathrm{H} : x \in D(\mathcal{A}),\ \|x\|_\mathrm{H} = 1\}$ equals $\mathrm{co}[\sigma(\mathcal{A})]$, i.e., the *convex hull* of its spectrum.

Important facts concerning the numerical range are as follows:

- $W(\alpha I + \beta\mathcal{A}) = \alpha + \beta W(\mathcal{A})$.
- $W(U^*\mathcal{A}U) = W(\mathcal{A})$ for any unitary U.
- $W(\mathcal{A})$ is convex (Toeplitz–Hausdorff theorem).
- $W(\mathcal{A} + \mathcal{B}) \subset W(\mathcal{A}) + W(\mathcal{B})$, $\mathcal{A}, \mathcal{B} \in \mathbf{L}(\mathrm{H})$.
- Let $\mathcal{A}$ be the direct sum operator $\bigoplus_{k=1}^\infty \mathcal{A}_k$ defined on the direct sum Hilbert space $\mathrm{H} = \bigoplus_{k=1}^\infty \mathrm{H}_k$, Then

  $$\overline{W(\mathcal{A})} = \mathrm{clco}\left[\bigcup_{k=1}^{\infty} W(\mathcal{A}_k)\right]$$

  (Bouldin, 1971, Lemma 1, p. 214).
- A hyponormal operator is a convexoid. Recall that a densely defined operator $\mathcal{A}$ is *hyponormal* if $D(\mathcal{A}) \subset D(\mathcal{A}^*)$, and $\|\mathcal{A}f\|_\mathrm{H} \geq \|\mathcal{A}^*f\|_\mathrm{H}$ for $f \in D(\mathcal{A})$. Every hyponormal operator is *paranormal* , i.e., $\|\mathcal{A}f\|_\mathrm{H}^2 \leq \|\mathcal{A}^2 f\|_\mathrm{H}\|f\|_\mathrm{H}$ for $f \in D(\mathcal{A}^2)$[2], however, the paranormal operator may be not a convexoid (Furuta, 2001, p. 114, Example 4).

[1]The analytic extension of a semigroup operator is bounded in a sector $S_{\theta'}$ for each $\theta' \in (0, \theta)$.

[2]Let $\mathcal{A}$ be hyponormal. Then $f \in D(\mathcal{A}^2) \implies \|\mathcal{A}^2 f\|_\mathrm{H} = \langle \mathcal{A}f, \mathcal{A}f\rangle_\mathrm{H}$. By hyponormality, $\mathcal{A}f \in D(\mathcal{A}) \subset D(\mathcal{A}^*)$, so $\|\mathcal{A}^2 f\|_\mathrm{H} = \langle f, \mathcal{A}^*\mathcal{A}f\rangle_\mathrm{H} \leq \|f\|_\mathrm{H}\|\mathcal{A}^*\mathcal{A}f\|_\mathrm{H} \leq \|\mathcal{A}^2 f\|_\mathrm{H}\|f\|_\mathrm{H}$, whence $\mathcal{A}$ is paranormal.

Proposition 2. *If $W(\mathcal{A}) \subset \mathbb{C} \setminus S_\theta$ with $\theta > \pi/2$ and $0 \in \rho(\mathcal{A})$, then $\mathcal{A}$ generates an* **EXS** *analytic semigroup on H.*

Proof. Take $\lambda \notin \overline{W(\mathcal{A})}$. Then, with $x \in D(\mathcal{A})$, $\|x\|_\mathrm{H} = 1$, we have

$$\begin{aligned}0 &< \mathrm{dist}(\lambda, \overline{W(\mathcal{A})}) = \mathrm{dist}(\lambda, \overline{W(\mathcal{A})})\|x\|_\mathrm{H} \\ &\leq |\lambda - \langle \mathcal{A}x, x\rangle_\mathrm{H}| = \big|\lambda\|x\|_\mathrm{H}^2 - \langle \mathcal{A}x, x\rangle_\mathrm{H}\big| \\ &= |\langle(\lambda I - \mathcal{A})x, x\rangle_\mathrm{H}| \leq \|(\lambda I - \mathcal{A})x\|_\mathrm{H}.\end{aligned}$$

Since $\lambda \in \rho(\mathcal{A})$, we have

$$\|(\lambda I - \mathcal{A})^{-1}\|_{\mathbf{L}(\mathrm{H})} \leq \frac{1}{\mathrm{dist}(\lambda, W(\mathcal{A}))} \quad \forall \lambda \notin \overline{W(\mathcal{A})}. \tag{14}$$

The right-hand side of (14) can easily be estimated (draw a figure) by

$$\frac{1}{|\lambda| \sin\left(\theta - \frac{\pi}{2}\right)},$$

provided that $\lambda \in \mathbb{C}^+$. Finally,

$$\|\lambda(\lambda I - \mathcal{A})^{-1}\|_{\mathbf{L}(\mathrm{H})} \leq \frac{1}{\sin\left(\theta - \dfrac{\pi}{2}\right)} < \infty \quad \forall \lambda \in \mathbb{C}^+,$$

and the claim follows from Proposition 1. ∎

Remark 2. In the proof we used the reasoning of Kato (1995, Thm. 3.2, p. 268) and Pazy (1983, pp. 12–13).

The result takes a sharper form for convexoids since then $\overline{W(\mathcal{A})} = \mathrm{co}[\sigma(\mathcal{A})]$. For a discussion of (14) in the last case, see the work of Orland (1964), Furuta (1977, Prop. 2.1) or Gustafson and Rao (1997, p.154).

Proposition 3. *If $\mathcal{A}$ is similar to a convexoid, $\sigma(\mathcal{A}) \subset \mathbb{C} \setminus S_\theta$ with $\theta > \pi/2$ and $0 \in \rho(\mathcal{A})$, then $\mathcal{A}$ generates an* **EXS** *analytic semigroup on H.*

Proof. Let $\mathcal{A}$ be similar to a convexoid $\mathcal{J}$ via a Banach isomorphism T, that is, $T^{-1}\mathcal{A}T = \mathcal{J}$. Moreover,

$$\sigma(\mathcal{A}) = \sigma(\mathcal{J}) \subset \mathrm{co}[\sigma(\mathcal{A})] = \mathrm{co}[\sigma(\mathcal{J})] = \overline{W(\mathcal{J})}.$$

For $\mathcal{J}$ the basic estimate (14) reads as

$$\begin{aligned}\|(\lambda I - \mathcal{J})^{-1}\|_{\mathbf{L}(\mathrm{H})} &\leq \frac{1}{\mathrm{dist}(\lambda, \mathrm{co}[\sigma(\mathcal{J})])} \\ &= \frac{1}{\mathrm{dist}(\lambda, \mathrm{co}[\sigma(\mathcal{A})])},\end{aligned}$$

where $\lambda \notin \mathrm{co}[\sigma(\mathcal{A})]$. Multiplying both the sides on the *condition number* $\mathrm{cond}(T) := \|T\|_{\mathbf{L}(\mathrm{H})}\|T^{-1}\|_{\mathbf{L}(\mathrm{H})}$ of the similarity transform T, we obtain

$$\mathrm{cond}(T)\|(\lambda I - \mathcal{J})^{-1}\|_{\mathbf{L}(\mathrm{H})} \leq \frac{\mathrm{cond}(T)}{\mathrm{dist}(\lambda, \mathrm{co}[\sigma(\mathcal{A})])}.$$

The LHS majorizes $\|T(\lambda I-\mathcal{J})^{-1}T^{-1}\|_{\mathbf{L}(\mathrm{H})}=\|(\lambda I-\mathcal{A})^{-1}\|_{\mathbf{L}(\mathrm{H})}$ and the rest of our proof mimics the final part of that of Proposition 2, which is possible because, by the convexity of $\mathbb{C}\setminus S_\theta$, $\sigma(\mathcal{A})\subset\mathbb{C}\setminus S_\theta$ implies $\mathrm{co}[\sigma(\mathcal{A})]\subset\mathbb{C}\setminus S_\theta$. ∎

Proposition 2 generalizes the result of Janas (1989, Prop. 3.1) proved for hyponormal operators.

Example 1. For $\mathrm{H}=\mathbb{C}^2$,

$$\mathcal{A}=\begin{bmatrix}0 & 1\\ -1 & -1\end{bmatrix}\in\mathbf{L}(\mathbb{C}^2),$$

which is not hyponormal, whence the result of Janas (1989, Prop. 3.1) is not applicable. We have $\sigma(\mathcal{A})=\{\lambda,\overline{\lambda}\}$,

$$\lambda=-\frac{1}{2}+\frac{\sqrt{3}}{2}i,$$

and $\mathrm{co}[\sigma(\mathcal{A})]$ is the segment joining eigenvalues. By the *elliptical range theorem* (Shapiro, 2017), the numerical range is the ellipse

$$W(\mathcal{A})=\{s\in\mathbb{C}:4\left(\mathrm{Re}\,s+\frac{1}{2}\right)^2+\frac{(\mathrm{Im}\,s)^2}{4}\le 1\},$$

depicted in Fig. 3. Since $i\mathbb{R}$ is tangent to $W(\mathcal{A})$ at 0, the assumption of Proposition 2 is not satisfied. However, $\mathcal{A}$ is similar, via a modal matrix

$$T=\begin{bmatrix}1 & 1\\ \lambda & \overline{\lambda}\end{bmatrix}\iff T^{-1}=\frac{1}{2i\,\mathrm{Im}\,\lambda}\begin{bmatrix}-\overline{\lambda} & 1\\ \lambda & -1\end{bmatrix},$$

to its diagonal Jordan form $T^{-1}\mathcal{A}T:=\mathcal{J}=\mathrm{diag}\{\lambda,\overline{\lambda}\}$, which is a normal matrix, whence $\mathcal{A}$ is similar to a convexoid $\mathcal{J}$. $\sigma(\mathcal{A})$ and its convex hull are contained in $\mathbb{C}\setminus S_\theta$ with $\theta>\pi/2$, and Proposition 3 confirms the well-known fact that $\mathcal{A}$ generates an **EXS** analytic semigroup. ♦

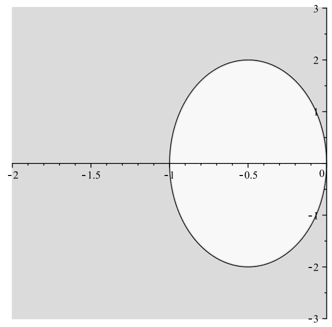

Fig. 3. Numerical range of $\mathcal{A}$.

3.6. Perturbations of analytic semigroup generators.

Theorem 9. *Let $\mathcal{A}$ generate an* **EXS** *analytic semigroup, and assume that $\mathcal{K}\in\mathbf{L}(Y,U)$ is such that (8) holds. Then, the closed-loop system operator $\mathcal{A}_c$ given by (10) generates an analytic* **EXS** *semigroup iff*

$$s\mapsto s\mathcal{A}(sI-\mathcal{A})^{-1}\mathcal{D}\mathcal{K}[I+\widehat{G}(s)\mathcal{K}]^{-1}H\mathcal{A}(sI-\mathcal{A})^{-1}\in H^\infty(\mathbb{C}^+,\mathbf{L}(H)). \tag{15}$$

Proof. In accordance with Proposition 1, we have to prove that $\mathbb{C}^+\cup\{0\}\subset\rho(\mathcal{A}_c)$ and $s(sI-\mathcal{A}_c)^{-1}\in\mathrm{H}^\infty(\mathbb{C}^+,\mathbf{L}(\mathrm{H}))$.

Consider the resolvent equation for $\mathcal{A}_c$,

$$sx-\mathcal{A}(x-\mathcal{D}\mathcal{K}\mathcal{C}x)=z\in\mathrm{H},\qquad s\in\mathbb{C}^+.$$

Applying the resolvent of $\mathcal{A}$ to both the sides, we obtain

$$s(sI-\mathcal{A})^{-1}x-\mathcal{A}(sI-\mathcal{A})^{-1}(x-\mathcal{D}\mathcal{K}\mathcal{C}x)=(sI-\mathcal{A})^{-1}z,$$

which yields

$$x+\mathcal{A}(sI-\mathcal{A})^{-1}\mathcal{D}\mathcal{K}\mathcal{C}x=(sI-\mathcal{A})^{-1}z\ . \tag{16}$$

In order to determine $\mathcal{C}x$, we apply $\mathcal{C}$ to both the sides, which leads to

$$\left[I+\widehat{G}(s)\mathcal{D}\mathcal{K}\right]\mathcal{C}x=H\mathcal{A}(sI-\mathcal{A})^{-1}z\ . \tag{17}$$

Since $\mathcal{A}$ generates an **EXS** analytic semigroup, the RHS of (17) is in $\mathrm{H}^\infty(\mathbb{C}^+,\mathrm{Y})$, which, jointly with (8), yields $\mathcal{C}x\in\mathrm{H}^\infty(\mathbb{C}^+,\mathrm{Y})$,

$$\mathcal{C}x=\left[I+\widehat{G}(s)\mathcal{K}\right]^{-1}H\mathcal{A}(sI-\mathcal{A})^{-1}z. \tag{18}$$

Substituting (18) into (16), we obtain the resolvent of $\mathcal{A}_c$ representation on $s\in\mathbb{C}^+$:

$$\begin{aligned}&(sI-\mathcal{A}_c)^{-1}\\&=(sI-\mathcal{A})^{-1}\\&-\mathcal{A}(sI-\mathcal{A})^{-1}\mathcal{D}\mathcal{K}[I+\widehat{G}(s)\mathcal{K}]^{-1}H\mathcal{A}(sI-\mathcal{A})^{-1},\end{aligned} \tag{19}$$

but, due to Remark 1, (19) is valid on $\overline{\mathbb{C}^+}$. Multiplying both the sides by s, employing (15) and applying Proposition 1, we get the claim. ∎

The assumption that $\mathcal{A}$ generates an **EXS** analytic semigroup can be weakened using the *stabilizability concept* as indicated in Section 6.

Under the parabolic regularity, there holds

$$S(t)x_0\in D(\mathcal{A}^\infty)\qquad\forall t>0,\ \forall x_0\in\mathrm{H}.$$

This enables us to state simpler definitions of the system-theoretic operators. The *observability map* reads now as

$$\begin{aligned}(\mathbf{\Psi}x_0)(t) &= \mathcal{C}S(t)x_0 \\ &= H\mathcal{A}S(t)x_0, \qquad t>0,\ x_0\in \mathrm{H},\end{aligned}$$

$$\begin{aligned}\mathcal{C}(sI-\mathcal{A})^{-1} &= H\mathcal{A}(sI-\mathcal{A})^{-1} \\ &\in \mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{H},\mathrm{Y}))\end{aligned}$$

whilst $\mathcal{C}$ is admissible iff $\mathcal{C}(sI-\mathcal{A})^{-1}x_0 \in \mathrm{H}^2(\mathbb{C}^+,\mathrm{Y})$ for every $x_0 \in \mathrm{H}$.

Moreover, if the latter holds, then (18) means that $\mathcal{C}$ is open-loop admissible if it is closed-loop admissible.

For the *reachability map* one has

$$\mathbf{\Phi}u = \int_0^\infty \mathcal{A}S(t)\mathcal{D}u(t)\,\mathrm{d}t$$

$$\Longrightarrow \mathbf{\Phi}R_t u = \int_0^t \mathcal{A}S(t-\tau)\mathcal{D}u(\tau)\,\mathrm{d}\tau.$$

For the kernel of this convolution one has $\mathcal{A}(sI-\mathcal{A})^{-1}\mathcal{D} \in \mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{H},\mathrm{U}))$ whilst $\mathcal{D}$ is admissible iff $\mathcal{D}^*\mathcal{A}^*(sI-\mathcal{A}^*)^{-1}x_0 \in \mathrm{H}^2(\mathbb{C}^+,\mathrm{Y})$ for every $x_0\in\mathrm{H}$.

The following holds for the *input–output map*:

$$\begin{aligned}(\mathbb{F}u)(t) &= \int_0^t H\mathcal{A}^2 S(t-\tau)\mathcal{D}u(\tau)\,\mathrm{d}\tau, \\ (\widehat{\mathbb{F}u})(s) &= \mathcal{C}\mathcal{A}(sI-\mathcal{A})^{-1}\mathcal{D}\hat{u}(s) = \widehat{G}(s)\hat{u}(s)\end{aligned}$$

and $\mathbb{F} \in \mathbf{L}(\mathrm{L}^2(0,\infty;\mathrm{U}), \mathrm{L}^2(0,\infty;\mathrm{Y}))$, provided that (5) holds.

We can see that (15) establishes a *balance condition* between the growth of the Laplace transforms of the reachability and observability maps and the closed-loop system transfer function.

The assumption (15) of Proposition 9 holds if either $\mathcal{C} = H\mathcal{A}$ or $\mathcal{D}^*\mathcal{A}^*$ is a bounded operator.

4. Example 1: An unloaded RC electric transmission line

Consider the negative feedback system, depicted in Fig. 4, consisting of an unloaded electric $\mathfrak{RC}$ transmission line and a proportional controller (operational amplifier) with gain $\mathcal{K}\in\mathbb{R}$. Here $\mathfrak{R}$, $\mathfrak{C}$ denote respectively the resistance and capacity per unit of length.

The plant dynamics are governed by the equations of a general $\mathfrak{RLCG}$ transmission line with $\mathfrak{G} = 0$ (conductance) and $\mathfrak{L} = 0$ inductivity,

$$\begin{cases} 0 = -V_\theta(\theta,\tau) - \mathfrak{R}I(\theta,\tau), \\ \mathfrak{C}V_t(\theta,\tau) = -I_\theta(\theta,\tau), \\ I(1,\tau) = 0, \\ U(\tau) = V(0,\tau), \\ Y(\tau) = V(1,\tau), \end{cases}$$

$\tau \ge 0$, $\theta\in[0,1]$. Time rescaling $x(\theta,t) = V(\theta,\mathfrak{RC}t)$, $u(t) = U(\mathfrak{RC}t)$, $y(t) = Y(\mathfrak{RC}t)$ yields

$$\begin{cases} x_t(\theta,t) = x_{\theta\theta}(\theta,t), & t\ge 0,\ 0\le\theta\le 1, \\ x_\theta(1,t) = 0, & t\ge 0, \\ u(t) = x(0,t), & t \ge 0, \\ y(t) = x(1,t), & t\ge 0, \end{cases} \tag{20}$$

and we want to know whether this system is well-posed and has desired asymptotic behaviour, possibly **EXS**.

In the Hilbert space $\mathrm{H} = \mathrm{L}^2(0,1)$ with standard scalar product, the dynamics (20) can be written in preliminary abstract form as

$$\begin{cases} \dot{x} = \sigma x, \\ \tau x = u, \\ y = c^\# x, \end{cases} \tag{21}$$

with

$$\begin{aligned}\sigma x = x'', \quad & D(\sigma) = \left\{x\in \mathrm{H}^2(0,1):\ x'(1) = 0\right\}, \\ \tau x = x(0), \quad & D(\tau) = \mathbf{C}[0,1] \supset D(\sigma),\end{aligned}$$

and σ is a closed linear operator, while τ is the so-called *operator of boundary control*.

The observation functional $\mathcal{C} = c^\#$ is given by

$$c^\# x = x(1), \qquad D(c^\#) = \mathbf{C}[0,1]. \tag{22}$$

From the relationships

$$d\in D(\sigma), \quad \sigma d = 0, \qquad \tau d = -1,$$

we determine a factor control vector d,

$$d = -\mathbf{1} \in \mathrm{L}^2(0,1), \qquad \mathbf{1}(\theta) = 1, \quad 0\le\theta\le 1\ , \tag{23}$$

$d\in D(c^\#)$ with $c^\# d = -1$. Thanks to this,

$$\begin{aligned}\tau[x(t)+du(t)] &= \tau x(t) + \tau d u(t) \\ &= \tau x(t) - u(t) = 0,\end{aligned}$$

i.e., $x(t)+du(t)\in\ker\tau$. Next,

$$\begin{aligned}\dot{x}(t) &= \sigma x(t) \\ &= \sigma x(t) + \sigma d u(t) \\ &= \sigma[x(t)+du(t)] = \mathcal{A}[x(t)+du(t)],\end{aligned}$$

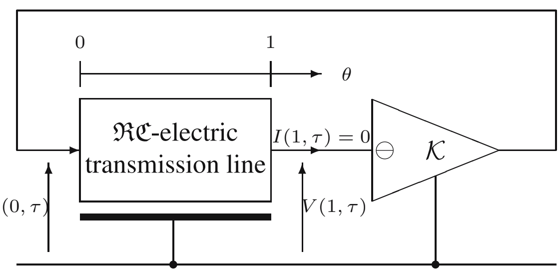

Fig. 4. Proportional feedback control of an $\mathfrak{RC}$ transmission line.

provided that $\mathcal{A} := \sigma|_{\ker\tau}$, here given by

$$\begin{aligned} &\mathcal{A}x = x'', \\ &D(\mathcal{A}) = \{x \in \mathrm{H}^2(0,1) : x'(1) = 0, x(0) = 0\}; \end{aligned} \tag{24}$$

$\mathcal{A} = \mathcal{A}^* < 0$ with the resolvent (for a method of its derivation, see Appendix A)

$$\begin{aligned} &\left((sI - \mathcal{A})^{-1}v\right)(\theta) \\ &= -\int_0^\theta \frac{\sinh\sqrt{s}(\theta-\tau)}{\sqrt{s}} v(\tau)\,\mathrm{d}\tau \\ &\quad + \frac{\sinh\sqrt{s}\theta}{\sqrt{s}\cosh\sqrt{s}} \int_0^1 \cosh\sqrt{s}(1-\tau)v(\tau)\,\mathrm{d}\tau. \end{aligned} \tag{25}$$

The inverse of $\mathcal{A}$

$$\left(\mathcal{A}^{-1}f\right)(\theta) = -\int_0^1 \left\{ \begin{array}{lll} \theta, & \text{if} & \theta < \tau \\ \tau, & \text{if} & \theta > \tau \end{array} \right\} f(\tau)\,\mathrm{d}\tau \tag{26}$$

is a Hilbert–Schmidt operator and, by the discrete version of the spectral theorem, the spectrum of $\mathcal{A}$ consists of countably many eigenvalues $\{\lambda_n\}_{n\in\mathbb{Z}^*}$, $\mathbb{Z}^* := \mathbb{N} \cup \{0\}$, and there exists a system of the corresponding eigenvectors $\{e_n\}_{n\in\mathbb{Z}^*}$ being an *orthonormal basis* of H,

$$\lambda_n = -\left(\frac{\pi}{2} + n\pi\right)^2, \quad e_n(\theta) = \sqrt{2}\sin\left(\pi/2 + n\pi\right)\theta,$$

$0 \le \theta \le 1$, $n \in \mathbb{Z}^*$. $\mathcal{A}$ generates on H an analytic, self-adjoint semigroup $\{S(t)\}_{t\ge 0}$,

$$S(t)x_0 = \sum_{n=0}^\infty e^{\lambda_n t}\langle x_0, e_n\rangle_{\mathrm{H}} e_n \qquad \forall x_0 \in \mathrm{H}, \quad \forall t \ge 0.$$

This semigroup is **EXS** as, by *Parseval's identity*, (2) holds with $M = 1$ and $\alpha = -\lambda_0 = \frac{\pi^2}{4}$.

We have $c^\# \mathcal{A}^{-1}x = \langle x, h\rangle_{\mathrm{H}}$, whence $h(\theta) = -\theta$, $\theta \in [0,1]$.

Similarly

$$\langle \mathcal{A}x, d\rangle_{\mathrm{H}} = -\int_0^1 x''(\theta)\,\mathrm{d}\theta = x'(0),$$

and thus $d^*\mathcal{A}^* = d^*\mathcal{A}$ extends to

$$\begin{aligned} d^\# x &= x'(0), \\ D(d^\#) &= \mathrm{C}^1[0,1] \ni h, \\ d^\# h &= -1 = c^\# d. \end{aligned}$$

Notice that $h' = d$ and

$$\begin{aligned} c^\# e_n &= e_n(1) = (-1)^n\sqrt{2}, \\ d^\# e_n &= e_n'(0) = \sqrt{2}\sqrt{-\lambda_n}, \end{aligned}$$

whence

$$\begin{aligned} &\sum_{n=0}^\infty (-\lambda_n)^{2\kappa} |\langle h, e_n\rangle_{\mathrm{L}^2(0,1)}|^2 \\ &\qquad = \sum_{n=0}^\infty (-\lambda_n)^{2\kappa-2}|c^\# e_n|^2 < \infty \\ &\qquad\qquad \iff \kappa = 3/4 - \varepsilon \\ &\qquad\qquad\qquad \Rightarrow h \in D((-\mathcal{A}))^{3/4-\varepsilon}. \end{aligned} \tag{27}$$

Analogously,

$$\begin{aligned} &\sum_{n=0}^\infty (-\lambda_n)^{2\kappa} |\langle d, e_n\rangle_{\mathrm{L}^2(0,1)}|^2 \\ &\qquad = \sum_{n=0}^\infty (-\lambda_n)^{2\kappa-2}|d^\# e_n|^2 < \infty \\ &\qquad\qquad \iff \kappa = 1/4 - \varepsilon \\ &\qquad\qquad\qquad \Rightarrow d \in D((-\mathcal{A}))^{1/4-\varepsilon}. \end{aligned} \tag{28}$$

Furthermore, for all $f \in \mathrm{L}^2(0,\infty)$, if

$$|\hat{f}(s)| = \left| \int_0^\infty e^{-st}f(t)\,\mathrm{d}t \right| \le \frac{\|f\|_{\mathrm{L}^2(0,\infty)}}{\sqrt{-2\operatorname{Re}\lambda_n}}$$

then

$$\sum_{n=0}^\infty |c^\# e_n|^2 |\hat{f}(-\lambda_n)|^2 \le \sum_{n=0}^\infty \frac{\|f\|^2_{\mathrm{L}^2(0,\infty)}}{-\operatorname{Re}\lambda_n} < \infty,$$

and, by the *spectral criterion of admissibility* (Grabowski, 1995, Prop. 2.1), $c^\#$ is admissible.

For $f(t) = t^{-1/4}e^{-t}$ one has $f \in \mathrm{L}^2(0,\infty)$, $\hat{f}(s) = (s+1)^{-3/4}\Gamma(3/4)$ and

$$\begin{aligned} &\sum_{n=0}^\infty |d^\# e_n|^2 |\hat{f}(-\lambda_n)|^2 \\ &= \frac{4\left[\Gamma\left(\frac{3}{4}\right)\right]^2}{\pi} \sum_{n=0}^\infty \frac{(2n+1)^2}{\left[\frac{4}{\pi^2} + (2n+1)^2\right]^{3/2}} = \infty. \end{aligned}$$

Thus $d^\#$ is not admissible and, by Lemma 2, d is not admissible. This eliminates the use of Theorems 8 and 1 with $p = 2$.

4.1. Perturbation approach.

Using (23) and (25) we find

$$(\mathcal{A}(sI - \mathcal{A})^{-1}d)(\theta) = \frac{\cosh\sqrt{s}\theta}{\cosh\sqrt{s}},$$

and, by (4) and (22),[3]

$$\widehat{G}(s) = sc^\#(sI - \mathcal{A})^{-1}d - c^\# d = \frac{1}{\cosh\sqrt{s}}, \tag{29}$$

[3] One can also use $c^\#(sI - \mathcal{A})^{-1}v = h^*\mathcal{A}(sI - \mathcal{A})^{-1}v = \int_0^1 \frac{\sinh\sqrt{s}\theta}{\sqrt{s}\cosh\sqrt{s}} v(\theta)\,\mathrm{d}\theta$.

where $s \notin \{\lambda_n\}_{n\in\mathbb{Z}^*}$. Since for $s \in \overline{\mathbb{C}^+}$ there holds

$$\begin{aligned}&|\cosh\sqrt{s}|^2\\ &= \sinh^2\sqrt{\frac{|s|+\operatorname{Re}s}{2}} + 1 - \sin^2\sqrt{\frac{|s|-\operatorname{Re}s}{2}} \geq 1,\end{aligned}$$

one has $\widehat{G} \in \mathrm{H}^\infty(\mathbb{C}^+)$ with the norm $\|\widehat{G}\|_{\mathrm{H}^\infty(\mathbb{C}^+)} = 1$ achieved at $s = 0$. The boundedness of $\widehat{G}$ on $j\mathbb{R}$ is confirmed by the *Nyquist curve* depicted in Fig. 5, determining the *spectrum* $\sigma(\mathbb{F}) = \overline{\widehat{G}(\mathbb{C}^+)}$. Notice that $\widehat{G}(0) = 1$,

$$\widehat{G}(\pm 2\pi^2 i) = -\frac{1}{\cosh\pi},$$

and therefore

$$\begin{aligned}&-\frac{1}{\mathcal{K}} \notin \sigma(\mathbb{F}) \cap \mathbb{R}\\ &\quad = \overline{\widehat{G}(\mathbb{C}^+)} \cap \mathbb{R} \iff \mathcal{K} \in (-1, \cosh\pi). \quad (30)\end{aligned}$$

This means that for $\mathcal{K} \in (-1, \cosh\pi)$ we have

$$s \longmapsto [I + \widehat{G}(s)\mathcal{K}]^{-1} = \frac{\cosh\sqrt{s}}{\cosh\sqrt{s} + \mathcal{K}} \in \mathrm{H}^\infty(\mathbb{C}^+).$$

Lemma 5. *For $s \in \mathbb{C}^+$, we have the following estimates*

$$\|h^*\mathcal{A}(sI-\mathcal{A})^{-1}\|^2_{\mathbf{L}(H,\mathbb{C})} \leq \frac{2}{|s|^2 + \left(\frac{\pi}{2}\right)^4} + \frac{1}{\sqrt{2}|s|^{3/2}}, \quad (31)$$

$$\|\mathcal{A}(sI-\mathcal{A})^{-1}d\|^2_H \leq \frac{4}{|s| + \left(\frac{\pi}{2}\right)^2} + \frac{2}{|s|^{1/2}} \ . \quad (32)$$

Proof. Let $s \in \mathbb{C}^+$. Using Parseval's identity, we get

$$\begin{aligned}
&\|h^*\mathcal{A}(sI-\mathcal{A})^{-1}\|^2_{\mathbf{L}(\mathrm{H},\mathbb{C})} = \|\mathcal{A}(\overline{s}I-\mathcal{A})^{-1}h\|^2_{\mathrm{H}}\\
&= \sum_{n=0}^{\infty} \left|\langle \mathcal{A}(\overline{s}I-\mathcal{A})^{-1}h, e_n\rangle_{\mathrm{H}}\right|^2\\
&= \sum_{n=0}^{\infty} \frac{|c^{\#}e_n|^2}{|s-\lambda_n|^2}\\
&= \sum_{n=0}^{\infty} \frac{2}{(\operatorname{Re}s - \lambda_n)^2 + \operatorname{Im}^2 s}\\
&\leq \sum_{n=0}^{\infty} \frac{2}{|s|^2 + \left(\frac{\pi}{2} + \pi n\right)^4}\\
&\leq \frac{2}{|s|^2 + \left(\frac{\pi}{2}\right)^4} + \sum_{n=1}^{\infty} \frac{2}{|s|^2 + \pi^4 n^4}\\
&\leq \frac{2}{|s|^2 + \left(\frac{\pi}{2}\right)^4} + \int_0^\infty \frac{2}{|s|^2 + \pi^4 n^4}\,\mathrm{d}n,
\end{aligned}$$

$$\int_0^\infty \frac{1}{1+y^4}\,\mathrm{d}y = \frac{\pi\sqrt{2}}{4},$$

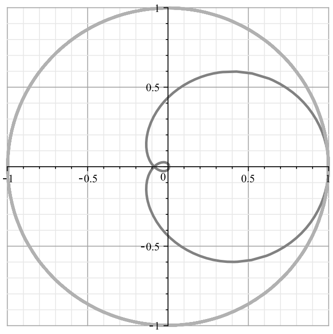

Fig. 5. Nyquist curve $\{\widehat{G}(i\omega)\}_{\omega\in\mathbb{R}}$; $\widehat{G}$ given by (29).

from which we obtain (31). Similarly,

$$\begin{aligned}
&\|\mathcal{A}(sI-\mathcal{A})^{-1}d\|^2_{\mathrm{H}}\\
&= \sum_{n=0}^{\infty} \left|\langle \mathcal{A}(sI-\mathcal{A})^{-1}d, e_n\rangle_{\mathrm{H}}\right|^2 = \sum_{n=0}^{\infty} \frac{|d^{\#}e_n|^2}{|s-\lambda_n|^2}\\
&= \sum_{n=0}^{\infty} \frac{-2\lambda_n}{(\operatorname{Re}s - \lambda_n)^2 + \operatorname{Im}^2 s}\\
&\leq \sum_{n=0}^{\infty} \frac{2\left(\frac{\pi}{2} + \pi n\right)^2}{|s|^2 + \left(\frac{\pi}{2} + \pi n\right)^4}\\
&\leq \sum_{n=0}^{\infty} \frac{\left(\frac{\pi}{2} + \pi n\right)^2}{|s| + \left(\frac{\pi}{2} + \pi n\right)^2} \frac{4}{|s| + \left(\frac{\pi}{2} + \pi n\right)^2}\\
&\leq \sum_{n=0}^{\infty} \frac{4}{|s| + \left(\frac{\pi}{2} + \pi n\right)^2}\\
&\leq \frac{4}{|s| + \left(\frac{\pi}{2}\right)^2} + \sum_{n=1}^{\infty} \frac{4}{|s| + \pi^2 n^2}\\
&\leq \frac{4}{|s| + \left(\frac{\pi}{2}\right)^2} + \int_0^\infty \frac{4\,\mathrm{d}n}{|s| + \pi^2 n^2},
\end{aligned}$$

which yields (32). ∎

Since $\frac{\mathcal{K}}{1+\mathcal{K}\widehat{G}} \in \mathrm{H}^\infty(\mathbb{C}^+)$, from Lemma 5 it follows that

$$\begin{aligned}s \longmapsto &\frac{\mathcal{K}}{1+\mathcal{K}\widehat{G}(s)}\left[s^{1/4}\mathcal{A}(sI-\mathcal{A})^{-1}d\right]\\ &\times \left[s^{3/4}h^*\mathcal{A}(sI-\mathcal{A})^{-1}\right]\end{aligned}$$

is in $\mathrm{H}^\infty(\mathbb{C}^+,\mathbf{L}(\mathrm{H}))$, so (15) is fulfilled.

All assumptions of Theorem 9 hold and, by its assertion, the closed-loop system state operator (10) here equals

$$\mathcal{A}_c x = \mathcal{A}(x - \mathcal{K}dc^{\#}x),$$
$$D(\mathcal{A}_c) = \{x \in D(c^{\#}) : x - \mathcal{K}dc^{\#}x \in D(\mathcal{A})\};$$

equivalently,

$$\begin{aligned}\mathcal{A}_c x &= x'',\\ D(\mathcal{A}_c) &= \{x \in \mathrm{H}^2(0,1) : x'(1) = 0,\\ &\qquad x(0) + \mathcal{K}x(1) = 0\}\end{aligned} \tag{33}$$

generates an analytic **EXS** semigroup on H, provided that $\mathcal{K} \in (-1, \cosh\pi)$.

Remark 3. Let $\mathcal{A}$ be a closed densely defined and boundedly invertible operator, $\mathcal{A}^{-1} \in \mathbf{L}(\mathrm{H})$. Suppose that the *strong stability* of the analytic semigroup $\{e^{t\mathcal{A}^{-1}}\}_{t\geq 0}$ ($\|e^{t\mathcal{A}^{-1}}x\|_{\mathrm{H}} \longrightarrow 0$ as $t \to \infty$ for every $x \in \mathrm{H}$) implies that $\mathcal{A}$ generates an **EXS** analytic semigroup $\{S(t)\}_{t\geq 0}$. One can apply Theorem 9 to find sufficient conditions under which $\mathcal{A}_c$ generates an **EXS** analytic semigroup, too. After DeLaubenfels (1988) and Arendt *et al.* (2011, Thm. 5.5.5b, p. 374), we may conclude that $\{e^{t\mathcal{A}_c^{-1}}\}_{t\geq 0}$ is strongly stable. This method allows getting the robustness of strong stability under a weakened assumption in comparison with the result of Paunonen (2014).

Indeed, in the example under discussion one has that $\mathcal{A}^{-1} = \mathcal{A}^{-*} < 0$ is a *compact* (even a *Hilbert–Schmidt*) operator generating a strongly stable analytic semigroup $e^{t\mathcal{A}^{-1}}$. Hence $\mathcal{A} = \mathcal{A}^* < 0$ generates an **EXS** analytic semigroup. Above we established that $\mathcal{A}_c$ generates an **EXS** analytic semigroup for $\mathcal{K} \in (-1, \cosh\pi)$. By DeLaubenfels (1988) the operator $\mathcal{A}_c^{-1}$ generates a bounded analytic semigroup, and Arendt *et al.* (2011, Thm. 5.5.5b, p. 374) yields strong stability of $e^{t\mathcal{A}_c^{-1}}$. In comparison, the results of Paunonen (2014) require that the sum of fractional powers allowed in (27) and (28) be greater than or equal to 1, which does not hold here.

4.2. Direct approach for $|\mathcal{K}| \neq 1$.

We shall prove that the closed-loop state operator $\mathcal{A}_c$ is similar to a normal operator using the $\mathcal{LM}$-*similarity* method, presented below in several steps.

Step 1. We start from the discrete skew-adjoint operator

$$\mathcal{N}\varphi := \varphi', \quad D(\mathcal{N}) = \{\varphi \in \mathrm{H}^1(0,1) : \varphi(0) = \varphi(1)\}$$

with spectrum $\{2n\pi i\}_{n\in\mathbb{Z}}$.

Step 2. The operator $(\mathcal{N} - aI)^2$, $\operatorname{Im} a \in [0, 2\pi)$ is normal,

$$\begin{aligned}(\mathcal{N} - aI)^2\varphi = \varphi'' - 2a\varphi' + a^2\varphi, D\left((\mathcal{N} - aI)^2\right)\\ = \{\varphi \in \mathrm{H}^2(0,1) : \varphi(0) = \varphi(1),\\ \varphi'(0) = \varphi'(1)\},\end{aligned}$$

with spectrum $\{(2n\pi i - a)^2\}_{n\in\mathbb{Z}}$ located on a parabola (possibly degenerated to a ray). An appropriate selection of a allows us to match the spectra of $(\mathcal{N} - aI)^2$ and $\mathcal{A}_c$.

Step 3. Apply the Banach isomorphism $\mathcal{M} \in \mathbf{L}(\mathrm{H})$,

$$\varphi(\theta) = (\mathcal{M}f)(\theta) := e^{a\theta} f(\theta), \quad \theta \in [0,1],$$

to get the similarity between $(\mathcal{N} - aI)^2$ and the second order differentiation operator

$$\mathcal{M}^{-1}(\mathcal{N} - aI)^2\mathcal{M}f = f'',$$

$$\begin{aligned}&D\left(\mathcal{M}^{-1}(\mathcal{N} - aI)^2\mathcal{M}\right)\\ &\quad = \{f \in \mathrm{H}^2(0,1) : f(0) = e^a f(1),\\ &\qquad f'(0) = e^a f'(1)\}.\end{aligned}$$

Hence, the operator $\mathcal{M}$ matches the formal definition of $(\mathcal{N} - aI)^2$ with the formal definition of $\mathcal{A}_c$; however, their domains are not the same (different boundary conditions).

Step 4. The domains (boundary conditions) can be matched by applying the operator $\mathcal{L} \in \mathbf{L}(\mathrm{H})$,

$$f(\theta) = (\mathcal{L}x)(\theta) = x(\theta) + \gamma x(1 - \theta). \tag{34}$$

which is a Banach isomorphism, provided that $1 - \gamma^2 \neq 0$. Then the inverse operator reads as[4]

$$(\mathcal{L}^{-1}f)(\theta) = \frac{f(\theta) - \gamma f(1 - \theta)}{1 - \gamma^2}.$$

$\mathcal{L}$ makes the formal definition of $\mathcal{M}^{-1}(\mathcal{N} - aI)^2\mathcal{M}$ unchanged because $\mathcal{L}^{-1}(\mathcal{L}x)'' = \mathcal{L}^{-1}\mathcal{L}x'' = x''$, but the boundary conditions

$$\left\{\begin{array}{r} [\gamma e^a - 1]\, x(0) + [\alpha e^a - \gamma]\, x(1) = 0,\\ [1 + \gamma e^a]\, x'(0) + [-\gamma - e^a]\, x'(1) = 0.\end{array}\right.$$

are matched iff

$$1 + \gamma e^a = 0, \quad \gamma + e^a = -1, \quad (e^a - \gamma) = \mathcal{K}(\gamma e^a - 1).$$

Hence

$$\gamma = -e^{-a}, \quad \cosh a = -\mathcal{K}.$$

[4]From (34) we get $f(1-\theta) = x(1-\theta) + \gamma x(\theta)$, whence the values of f and x at θ and $1 - \theta$ are tied by two linear algebraic equations.

Excluding $a = 0 \iff \mathcal{K} = -1$ and $a = \pi i \iff \mathcal{K} = 1$, we get

$$\begin{aligned}(\mathcal{L}x)(\theta) &= x(\theta) - e^{-a}x(1-\theta),\\ \left(\mathcal{L}^{-1}f\right)(\theta) &= -\frac{1}{1-e^{-2a}}\left[x(\theta) + e^{-a}x(1-\theta)\right]\end{aligned}$$

and

$$\cosh a = -\mathcal{K}, \qquad \lambda_n = (2\pi n i - a)^2, \quad n \in \mathbb{Z}.$$

Next, if $\mathcal{K} \le -1$ then

$$a \in \mathbb{R}, \quad a = \operatorname{arccosh}\mathcal{K} = \ln(|\mathcal{K}| + \sqrt{\mathcal{K}^2 - 1}).$$

Since $\cosh(a - \pi i) = -\cosh a = \mathcal{K}$, for $\mathcal{K} \ge 1$ we have

$$a = \operatorname{arccosh}\mathcal{K} + \pi i = \ln(|\mathcal{K}| + \sqrt{\mathcal{K}^2 - 1}) + \pi i$$

and, as $\cosh((\pi - \arccos\mathcal{K})i) = \cos(\pi - \arccos\mathcal{K}) = -\mathcal{K}$, we have

$$|\mathcal{K}| \le 1 \Longrightarrow a = (\pi - \arccos\mathcal{K})i.$$

Moreover, if $\mathcal{K} \in (-1, \cosh\pi)$, then the spectrum of $\mathcal{A}_c$ is in a sector $\mathbb{C} \setminus S_{\frac{\pi}{2}+\varepsilon}$ and λ_n are zeros of $\cosh\sqrt{s} + \mathcal{K}$.

By Proposition 3, $\mathcal{A}_c$ generates an **EXS** analytic semigroup, which confirms the result of the previous section for $\mathcal{K} \ne 1$. The case $\mathcal{K} = -1$ was treated by Ionkin (1977), who proved that then $\mathcal{A}_c$ possesses a system of eigenvectors and generalized eigenvectors which forms a Riesz basis of $\mathrm{H} = \mathrm{L}^2(0,1)$.

4.3. Direct approach for $\mathcal{K} = 1$.

The case $\mathcal{K} = 1 \iff a = i\pi$ will be analysed in the present section using the method of block operators. First of all we notice that

$$\begin{aligned}\left((\mathcal{N} - i\pi)^2 f\right)(\theta) = f''(\theta),\ & D((\mathcal{N} - i\pi)^2)\\ &= \{f \in \mathrm{H}^2(0,1) : f(0) + f(1) = 0,\\ &\quad f'(0) + f'(1) = 0\}\end{aligned}$$

is self-adjoint with the double eigenvalues $-(2k\pi - \pi)^2$, $k \in \mathbb{N}$ as depicted in Fig. 6 and with the ONB of the corresponding eigenvectors

$$\begin{aligned}e_{2k-1}(\theta) &:= \sqrt{2}\cos[(2k\pi - \pi)\theta],\\ e_{2k}(\theta) &:= \sqrt{2}\sin[(2k\pi - \pi)\theta].\end{aligned}$$

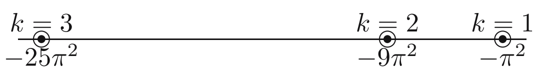

Fig. 6. Spectrum of $(\mathcal{N} - i\pi)^2$.

This operator is no longer similar to the closed-loop operator $\mathcal{A}_c$ for $\mathcal{K} = 1$,

$$\begin{aligned}(\mathcal{A}_c f)(\theta) &= f'',\\ D(\mathcal{A}_c) &= \{f \in \mathrm{H}^2(0,1) : f'(1) = 0,\\ &\qquad f(0) + f(1) = 0\},\end{aligned} \tag{35}$$

with the adjoint operator (its form is consistent with (11))

$$\begin{aligned}(\mathcal{A}_c^* v)(\theta) &= v'',\\ D(\mathcal{A}_c^*) &= \{v \in \mathrm{H}^2(0,1) : v(0) = 0,\\ &\qquad v'(0) + v'(1) = 0\}.\end{aligned} \tag{36}$$

This is because $\mathcal{A}_c$ has generalized eigenvectors.

Lemma 6. *$\mathcal{A}_c$ has a system of eigenvectors and generalized eigenvectors*

$$\begin{aligned}U_{2k-1}(\theta) &:= 4\cos[(2k\pi - \pi)\theta],\\ U_{2k}(\theta) &:= 4(1-\theta)\sin[(2k\pi - \pi)\theta],\end{aligned}$$

which forms a Riesz basis of H.

Moreover, with respect to this basis, $\mathcal{A}_c$ takes a block-diagonal form,

$$\begin{aligned}&\mathcal{A}_c \begin{bmatrix} U_{2k-1} & U_{2k} \end{bmatrix}\\ &\quad = \begin{bmatrix} U_{2k-1} & U_{2k} \end{bmatrix} \begin{bmatrix} \lambda_k & -2\sqrt{-\lambda_k} \\ 0 & \lambda_k \end{bmatrix}, \quad k \in \mathbb{N}.\end{aligned}$$

Similarly, $\mathcal{A}_c^*$ has a set of eigenvectors and generalized eigenvectors forming the biorthogonal Riesz basis,

$$\begin{aligned}V_{2k-1}(\theta) &:= \sin[(2k\pi - \pi)\theta],\\ V_{2k}(\theta) &:= \theta\cos[(2k\pi - \pi)\theta],\end{aligned}$$

with respect to which it has block-diagonal form,

$$\begin{aligned}&\mathcal{A}_c^* \begin{bmatrix} V_{2k-1} & V_{2k} \end{bmatrix}\\ &\quad = \begin{bmatrix} V_{2k-1} & V_{2k} \end{bmatrix} \begin{bmatrix} \lambda_k & 0 \\ -2\sqrt{-\lambda_k} & \lambda_k \end{bmatrix}, \quad k \in \mathbb{N}.\end{aligned}$$

Proof. We shall use the method of Ionkin (1977) with some essential modifications. The *characteristic function* of the eigenproblems for (35) and (36) is $\cosh\sqrt{\lambda} + 1$. All its zeros λ_k are double. Writing $\mu_k := (2k\pi - \pi)$ for simplicity,

$$\begin{aligned}\lambda_k &= -\mu_k^2,\\ -2\sqrt{-\lambda_k} &= -2\mu_k,\\ \cos\mu_k &= -1,\\ \sin\mu_k &= 0,\end{aligned}$$

we can solve the eigenproblem $\mathcal{A}_c f + \mu_k^2 f = 0$ iff

$$\begin{aligned}f''(\theta) + \mu_k^2 f(\theta) &= 0,\\ f(0) + f(1) &= 0,\\ f'(1) &= 0.\end{aligned}$$

A general solution to this problem $f(\theta) = f_1\sin(\mu_k\theta) + f_2\cos(\mu_k\theta)$ is substituted into boundary conditions, which yields $f_1 = 0$, whence $f(\theta) = f_2\cos(\mu_k\theta)$. Next we seek for a *specially normalized generalized eigenvector* determined by $\mathcal{A}_c g + \mu_k^2 g = -2\mu_k f$, $g \in D(\mathcal{A}_c)$. This requires solving the boundary-value problem

$$\begin{aligned} g''(\theta) + \mu_k^2 g(\theta) &= -2\mu_k f(\theta) \\ g(0) + g(1) &= 0, \\ g'(1) &= 0. \end{aligned}$$

A general solution of the first equation,

$$\begin{aligned} g(\theta) =& g_2\sin(\mu_k\theta) + g_1\cos(\mu_k\theta) \\ &- \frac{f_2\left[\cos(\mu_k\theta) + \mu_k\theta\sin(\mu_k\theta)\right]}{\mu_k}, \end{aligned}$$

is again being inserted into boundary conditions, which yields $g_2 = f_2$. Moreover, one should assume $g_1 = f_2/\mu_k$ in order to ensure that a system of generalized eigenvectors will be *quasinormalized*. Recall that quasinormalization is necessary for the existence of a set of eigenvectors and generalized eigenvectors forming a *Riesz basis*. Hence,

$$\begin{aligned} g(\theta) &= f_2\sin(\mu_k\theta) - f_2\theta\sin(\mu_k\theta) \\ &= f_2(1-\theta)\sin(\mu_k\theta). \end{aligned}$$

It is clear that

$$\mathcal{A}_c \begin{bmatrix} f & g \end{bmatrix} = \begin{bmatrix} f & g \end{bmatrix} \begin{bmatrix} -\mu_k^2 & -2\mu_k \\ 0 & -\mu_k^2 \end{bmatrix}.$$

The biorthogonal system with respect to $\{f, g\}$ is sought in the form $\{G, F\}$ in order to have the representation

$$\mathcal{A}_c^* \begin{bmatrix} G & F \end{bmatrix} = \begin{bmatrix} G & F \end{bmatrix} \begin{bmatrix} -\mu_k^2 & 0 \\ -2\mu_k & -\mu_k^2 \end{bmatrix}.$$

Hence F is an eigenvector of $\mathcal{A}_c^*$ and therefore it solves the problem

$$\begin{aligned} F''(\theta) + \mu_k^2 F(\theta) &= 0, \\ F(0) &= 0, \\ F'(0) + F'(1) &= 0. \end{aligned}$$

After substitution of a general solution $F(\theta) = F_1\sin(\mu_k\theta) + F_2\cos(\mu_k\theta)$ of the first equation into the boundary conditions we obtain $F_2 = 0$, $F(\theta) = F_1\sin(\mu_k\theta)$. G solves the boundary-value problem determining the special type of generalized eigenvectors,

$$G''(\theta) + \mu_k^2 G(\theta) = -2\mu_k F(\theta),$$

$$G(0) = 0, \quad G'(0) + G'(1) = 0.$$

Substituting the general solution of the first equation $G(\theta) = G_2\sin(\mu_k\theta) + G_1\cos(\mu_k\theta) + F_1\theta\cos(\mu_k\theta)$ into the boundary conditions, we obtain $G_1 = 0$. Therefore,

$$G(\theta) = G_2\sin(\mu_k\theta) + F_1\theta\cos(\mu_k\theta).$$

Now the biorthogonality conditions

$$\begin{aligned} \langle f, G\rangle_{\mathrm{L}^2(0,1)} &= 1, & \langle f, F\rangle_{\mathrm{L}^2(0,1)} &= 0, \\ \langle g, G\rangle_{\mathrm{L}^2(0,1)} &= 0, & \langle g, F\rangle_{\mathrm{L}^2(0,1)} &= 1 \end{aligned}$$

hold if, e.g., $F_1 = 1$, $f_2 = 4$, $G_2 = 0$, whence

$$\begin{aligned} f(\theta) &= 4\cos(\mu_k\theta), & g(\theta) &= 4(1-\theta)\sin(\mu_k\theta), \\ F(\theta) &= \sin(\mu_k\theta), & G(\theta) &= \theta\cos(\mu_k\theta). \end{aligned}$$

In what follows we shall write $U_{2k-1} := f$, $U_{2k} := g$ and $V_{2k-1} := G$, $V_{2k} := F$. In order to verify that $\{U_k\}_{k\in\mathbb{N}}$ is a Riesz basis on its span, we calculate

$$\begin{aligned} &\sum_{k=1}^{\infty} |\langle x, U_k\rangle_{\mathrm{H}}|^2 \\ &\qquad = \sum_{k=1}^{\infty} |\langle x, U_{2k-1}\rangle_{\mathrm{H}}|^2 + \sum_{k=1}^{\infty} |\langle x, U_{2k}\rangle_{\mathrm{H}}|^2. \end{aligned}$$

Comparing the formulae expressing e_k and U_k, we get (recall that $\mathrm{H} = \mathrm{L}^2(0,1)$)

$$\begin{aligned} \sum_{k=1}^{\infty} |\langle x, U_{2k-1}\rangle_{\mathrm{H}}|^2 &= \sum_{k=1}^{\infty} \left|\left\langle x, \frac{4}{\sqrt{2}} e_{2k-1}\right\rangle_{\mathrm{H}}\right|^2 \\ &\le 8\sum_{k=1}^{\infty} |\langle x, e_k\rangle_{\mathrm{H}}|^2 = 8\|x\|_{\mathrm{H}}^2 \end{aligned}$$

and

$$\begin{aligned} &\sum_{k=1}^{\infty} |\langle x, U_{2k}\rangle_{\mathrm{H}}|^2 \\ &\quad = \sum_{k=1}^{\infty} \left|\left\langle x, \frac{4(1-\cdot)}{\sqrt{2}} e_{2k}\right\rangle_{\mathrm{H}}\right|^2 \\ &\quad = \sum_{k=1}^{\infty} \left|\left\langle (1-\cdot)x, \frac{4}{\sqrt{2}} e_{2k}\right\rangle_{\mathrm{H}}\right|^2 \\ &\quad \le 8\sum_{k=1}^{\infty} |\langle (1-\cdot)x, e_k\rangle_{\mathrm{H}}|^2 \le 8\|x\|_{\mathrm{H}}^2. \end{aligned}$$

Thus we have

$$\sum_{k=1}^{\infty} |\langle x, U_k\rangle_{\mathrm{L}^2(0,1)}|^2 \le 16\|x\|_{\mathrm{L}^2(0,1)}^2. \tag{37}$$

Similarly,

$$\begin{aligned} &\sum_{k=1}^{\infty} |\langle x, V_k\rangle_{\mathrm{H}}|^2 \\ &\qquad = \sum_{k=1}^{\infty} |\langle x, V_{2k-1}\rangle_{\mathrm{H}}|^2 + \sum_{k=1}^{\infty} |\langle x, V_{2k}\rangle_{\mathrm{H}}|^2. \end{aligned}$$

Comparing the formulae expressing e_k and V_k, we get

$$\begin{aligned}\sum_{k=1}^{\infty}&|\langle x,V_{2k}\rangle_{\mathrm{H}}|^2\\ &=\sum_{k=1}^{\infty}\left|\left\langle x,\frac{(\cdot)}{\sqrt{2}}e_{2k-1}\right\rangle_{\mathrm{H}}\right|^2\\ &=\sum_{k=1}^{\infty}\left|\left\langle (\cdot)x,\frac{1}{\sqrt{2}}e_{2k-1}\right\rangle_{\mathrm{H}}\right|^2\\ &\le\frac{1}{2}\sum_{k=1}^{\infty}|\langle(\cdot)x,e_k\rangle_{\mathrm{H}}|^2\le\frac{1}{2}\|x\|_{\mathrm{H}}^2\end{aligned}$$

and

$$\begin{aligned}\sum_{k=1}^{\infty}&|\langle x,V_{2k-1}\rangle_{\mathrm{H}}|^2\\ &=\sum_{k=1}^{\infty}\left|\left\langle x,\frac{1}{\sqrt{2}}e_{2k}\right\rangle_{\mathrm{H}}\right|^2\\ &\le\frac{1}{2}\sum_{k=1}^{\infty}|\langle x,e_k\rangle_{\mathrm{H}}|^2=\frac{1}{2}\|x\|_{\mathrm{H}}^2.\end{aligned}$$

Thus we have

$$\sum_{k=1}^{\infty}|\langle x,V_k\rangle_{\mathrm{L}^2(0,1)}|^2\le\|x\|_{\mathrm{L}^2(0,1)}^2. \tag{38}$$

Since the characteristic function is not identically equal to a constant (*nondegenerate* boundary-value problem), one can apply the result of Marchenko (1977, Ch. 1, Sec. 3, pp. 28–37, especially Thm. 1.3.1, p. 36) to conclude the *completeness* of eigenvectors and generalized eigenvectors $\{U_k\}_{k\in\mathbb{N}}$ as well as $\{V_k\}_{k\in\mathbb{N}}$ on $\mathrm{H}=\mathrm{L}^2(0,1)$; alternatively, one can apply Lang and Locker (1989, Thm. 7.1, p. 557).

By (37), (38) and Bari's criterion (Gohberg and Krein, 1965, Thm. 2.1, pp. 374–375), the system and its biorthogonal system $\{V_k\}_{k\in\mathbb{N}}$ are Riesz bases on H. ■

By Lemma 6, the *projector onto the eigenspace associated with eigenvalue* $\lambda_k=-\mu_k^2$ reads as

$$\begin{aligned}P_kh&=\left[U_{2k-1}V_{2k-1}^*+U_{2k}V_{2k}^*\right]h\\ &=\begin{bmatrix}U_{2k-1} & U_{2k}\end{bmatrix}\begin{bmatrix}V_{2k-1}^*\\ V_{2k}^*\end{bmatrix}h,\end{aligned} \tag{39}$$

$$\exists c>0\,\forall h\in\mathrm{H}:$$
$$\frac{1}{c^2}\|\mathcal{A}_ch\|_{\mathrm{H}}^2\le\sum_{k=1}^{\infty}\|P_kh\|_{\mathrm{H}}^2\le c^2\|\mathcal{A}_ch\|_{\mathrm{H}}^2. \tag{40}$$

It will be verified in Appendix B that P_k coincides with the *Riesz projector*

$$\begin{aligned}P_kv&:=\frac{1}{2\pi i}\int_{C_k}(sI-\mathcal{A}_c)^{-1}v\,\mathrm{d}s\\ &=\operatorname*{Res}_{s=\lambda_k}(sI-\mathcal{A}_c)^{-1}v,\end{aligned} \tag{41}$$

where $v\in\mathrm{H}$ and C_k denotes a small disc containing inside only one eigenvalue λ_k, $k\in\mathbb{N}$.

Here $\lambda_k=-(2k\pi-\pi)^2$ and it is a double eigenvalue for each $k\in\mathbb{N}$.

Definition 7. Subspaces H_n of the Hilbert space H form an *unconditional subspace basis* of H if each vector $x\in\mathrm{H}$ has a unique expansion $x=\sum x_n$ with $x_n\in\mathrm{H}$, and the series unconditionally converges in H. A compact resolvent operator $\mathcal{A}_c$ is called a *discrete spectral operator* if its generalized eigenspaces (ranges of Riesz projectors) form an unconditional subspace basis of H.

The following criterion is known (Dunford and Schwartz, 1971, Corollary, Sec. XVIII.2.33, p. 2257).

Theorem 10. (Dunford–Schwartz) *$\mathcal{A}_c$ is a discrete spectral operator iff the family of sums of finite collections of the Riesz projectors P_k is uniformly bounded, and $P_kh=0$ for all $k\in\mathbb{N}$ implies $h=0$.*

One can see that, by (39) and (40), $\mathcal{A}_c$ is a discrete spectral operator. Observe that $P_k=P_k^2$, $P_kD(\mathcal{A}_c)\subset D(\mathcal{A}_c)$, $\mathcal{A}_cP_k=P_k\mathcal{A}_cP_k$, but $P_k\neq P_k^*$ (P_k is not even a normal operator),

$$\begin{aligned}&\mathcal{A}_cP_kh\\ &=\begin{bmatrix}\mathcal{A}_cU_{2k-1} & \mathcal{A}_cU_{2k}\end{bmatrix}\begin{bmatrix}V_{2k-1}^*\\ V_{2k}^*\end{bmatrix}h\\ &=\begin{bmatrix}U_{2k-1} & U_{2k}\end{bmatrix}\begin{bmatrix}-\mu_k^2 & -2\mu_k\\ 0 & -\mu_k^2\end{bmatrix}\begin{bmatrix}V_{2k-1}^*\\ V_{2k}^*\end{bmatrix}h.\end{aligned}$$

Accordingly, the restriction of $\mathcal{A}_c$ to $R(P_k)$ is represented by the matrix

$$\begin{bmatrix}-\mu_k^2 & -2\mu_k\\ 0 & -\mu_k^2\end{bmatrix},$$

which can be seen by premultiplying $\mathcal{A}_cP_kh$ on x^*.

Lemma 7. *$\mathcal{A}_c$ can be identified with the* block operator*,*

$$\begin{aligned}\mathcal{A}_bh&:=\sum_{k=1}^{\infty}\mathcal{A}_cP_kh,\\ D(\mathcal{A}_b)&=\{h\in H:\\ &\qquad\sum_{k=1}^{\infty}\mathcal{A}_cP_kh\ \textit{strongly converges}\}.\end{aligned} \tag{42}$$

Proof. Indeed, $f_n:=\sum_{k=1}^{n}P_kh$ converges to h as $n\to\infty$ and $f_n\in D(\mathcal{A}_c)$ for every $n\in\mathbb{N}$. If $h\in D(\mathcal{A}_b)$, we have

$$\mathcal{A}_cf_n=\sum_{k=1}^{n}\mathcal{A}_cP_kh\to\mathcal{A}_bh.$$

By the closedness of $\mathcal{A}_c$ ($\mathcal{A}_c$ has a compact inverse), $h\in D(\mathcal{A}_c)$ and $\mathcal{A}_ch=\mathcal{A}_bh$, so we have $\mathcal{A}_b\subset\mathcal{A}_c$.

Conversely, if $h \in D(\mathcal{A}_c)$, then

$$\|\sum_{k=1}^{n} \mathcal{A}_c P_k h - \mathcal{A}_c h\|_{\mathrm{H}} = \|\sum_{k=1}^{n} P_k \mathcal{A}_c h - \mathcal{A}_c h\|_{\mathrm{H}} \longrightarrow 0$$

as $n \to \infty$; by definition of $\mathcal{A}_b$, we have $h \in D(\mathcal{A}_b)$. ■

Remark 4. In (42) $\sum_{k=1}^{\infty} \mathcal{A}_c P_k h$ strongly converges iff $\sum_{k=1}^{\infty} \|\mathcal{A}_c P_k h\|_{\mathrm{H}}^2 < \infty$. Indeed, it follows from (40) that $h \in D(\mathcal{A}_c)$iff $\sum_{k=1}^{\infty} \|\mathcal{A}_c P_k h\|_{\mathrm{H}}^2 = \sum_{k=1}^{\infty} \|P_k \mathcal{A}_c h\|_{\mathrm{H}}^2 < \infty$, and $\mathcal{A}_c = \mathcal{A}_b$.

The facts above justify the notation (here $-i\mathcal{B}$ is a block operator consisting of *classical Jordan cells*)

$$\begin{aligned} \mathcal{A}_c &= \bigoplus_{k=1}^{\infty} \begin{bmatrix} -\mu_k^2 & -2\mu_k \\ 0 & -\mu_k^2 \end{bmatrix} = -\mathcal{B}^2, \\ -i\mathcal{B} &= \bigoplus_{k=1}^{\infty} \begin{bmatrix} \mu_k & 1 \\ 0 & \mu_k \end{bmatrix}, \\ (\lambda I - \mathcal{A}_c)^{-1} &= \bigoplus_{k=1}^{\infty} \begin{bmatrix} \lambda + \mu_k^2 & 2\mu_k \\ 0 & \lambda + \mu_k^2 \end{bmatrix}^{-1}. \end{aligned} \tag{43}$$

Since $\mu_k = 2k\pi - \pi \geq \pi$, the *spectral norm* of the resolvent block is

$$\left\| \begin{bmatrix} \lambda + \mu_k^2 & 2\mu_k \\ 0 & \lambda + \mu_k^2 \end{bmatrix}^{-1} \right\|_s = \frac{\mu_k + \sqrt{\mu_k^2 + |\lambda + \mu_k^2|^2}}{|\lambda + \mu_k^2|^2}.$$

If $\lambda \notin \overline{B(-\mu_k^2, 2\mu_k)}$, the closed ball with centre at $-\mu_k^2$ and radius $2\mu_k$, then

$$\begin{aligned} &\mu_k < |\lambda + \mu_k^2| \\ &\Leftrightarrow \mu_k^2 < |\lambda + \mu_k^2|^2 \\ &\Leftrightarrow \sqrt{\mu_k^2 + |\lambda + \mu_k^2|^2} \leq \sqrt{2}|\lambda + \mu_k^2|, \end{aligned}$$

whence

$$\left\| \begin{bmatrix} \lambda + \mu_k^2 & 2\mu_k \\ 0 & \lambda + \mu_k^2 \end{bmatrix}^{-1} \right\|_s = \frac{1+\sqrt{2}}{|\lambda + \mu_k^2|}, \quad \lambda \in \mathbb{C} \setminus \overline{B(-\mu_k^2, 2\mu_k)}, \quad \mu_k > 0.$$

We have $|\lambda + \mu_k^2|^2 = |\lambda|^2 + 2\mu_k \operatorname{Re}\lambda + \mu_k^2$, and therefore on $\overline{\mathbb{C}^+}$ we have

$$\|\lambda(\lambda I - \mathcal{A}_c)^{-1}\|_{\mathbf{L}(\mathrm{H})} \leq 1 + \sqrt{2}.$$

By Proposition 1, the operator $\mathcal{A}_c$ generates an **EXS** analytic semigroup on H.

The knowledge that (43) holds allows us to apply an approach based on the concept of the numerical range. To be more precise, we have

$$\begin{aligned} \overline{W(\mathcal{A}_c)} &= \operatorname{clco}\left(\bigcup_{k=1}^{\infty} W\left(\begin{bmatrix} -\mu_k^2 & -2\mu_k \\ 0 & -\mu_k^2 \end{bmatrix}\right)\right) \\ &= \operatorname{clco}\left(\bigcup_{k=1}^{\infty} \overline{B(-\mu_k^2, 2\mu_k)}\right), \end{aligned}$$

which is the set depicted in Fig. 7. By Proposition 2, the operator $\mathcal{A}_c$ generates an **EXS** C_0-semigroup.

5. Example 2: A loaded $\mathfrak{RC}$ transmission line for $\mathcal{K} = 1$

Let us modify the example of Section 4 by introducing a *resistance load* $R_0 \in (0, \infty) \setminus \{1\}$ on the output of an $\mathfrak{RC}$-transmission line with $\mathcal{K} = 1$. Then, still in the Hilbert space $\mathrm{H} = \mathrm{L}^2(0,1)$ with a standard scalar product, the preliminary dynamics (21) are the same except for the operator σ, which by *Ohm's law* takes the form

$$\begin{aligned} &\sigma x = x'', \\ &D(\sigma) = \left\{ x \in \mathrm{H}^2(0,1) : \ x'(1) = \frac{1}{R_0} x(1) \right\}. \end{aligned}$$

This, via the relationships $\sigma d = 0$, $\tau d = -1$ and $\mathcal{A} := \sigma|_{\ker \tau}$, leads to the new factor control vector

$$d(\theta) = \frac{1}{1 - R_0}\theta - 1,$$

and the new self-adjoint state operator

$$\begin{aligned} &\mathcal{A}x = x'', \\ &D(\mathcal{A}) = \{x \in \mathrm{H}^2(0,1) : x'(1) = \frac{1}{R_0}x(1), \\ &\qquad x(0) = 0\} \end{aligned} \tag{44}$$

with the resolvent (for a method of its derivation, see Appendix A)

$$\begin{aligned} &\left((sI - \mathcal{A})^{-1} v\right)(\theta) \\ &= -\int_0^\theta \frac{\sinh\sqrt{s}(\theta-\tau)}{\sqrt{s}} v(\tau)\,\mathrm{d}\tau \\ &\quad + \frac{\sinh\sqrt{s}\theta}{\sqrt{s}\cosh\sqrt{s} - \frac{\sinh\sqrt{s}}{R_0}} \int_0^1 \cosh\sqrt{s}(1-\tau) v(\tau)\,\mathrm{d}\tau \\ &\quad - \frac{\sinh\sqrt{s}\theta}{R_0\left(\sqrt{s}\cosh\sqrt{s} - \frac{\sinh\sqrt{s}}{R_0}\right)} \\ &\quad \times \int_0^1 \frac{\sinh\sqrt{s}(1-\tau)}{\sqrt{s}} v(\tau)\,\mathrm{d}\tau. \end{aligned} \tag{45}$$

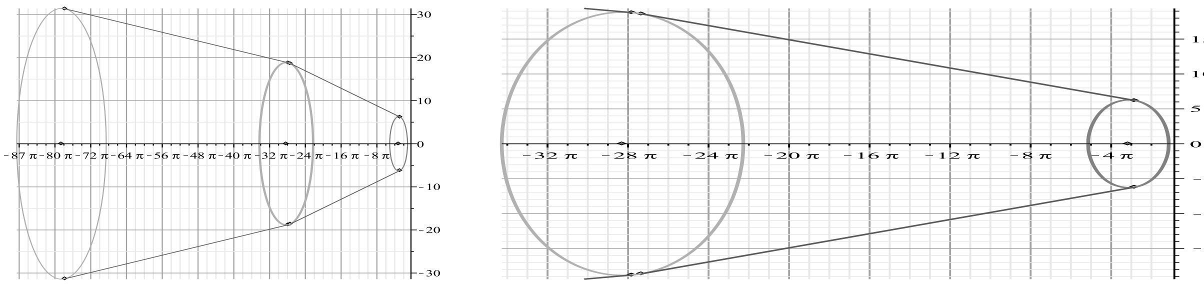

Fig. 7. $\overline{W(\mathcal{A}_c)}$ and its magnification close to 0.

The inverse of $\mathcal{A}$,

$$\left(\mathcal{A}^{-1}f\right)(\theta) = \int_0^1 \left\{ \begin{array}{ll} \left(\frac{\theta}{1-R_0}-1\right)\tau, & \text{if} \quad \tau<\theta \\ \left(\frac{\tau}{1-R_0}-1\right)\theta, & \text{if} \quad \tau>\theta \end{array} \right\} f(\tau)\,\mathrm{d}\tau, \tag{46}$$

is a Hilbert–Schmidt operator and, by the discrete version of the spectral theorem, the spectrum of $\mathcal{A}$ consists of countably many eigenvalues (poles of the resolvent) $\{\lambda_n\}_{n\in\mathbb{N}}$, $\lambda_n = -\tau_n^2$, where τ_n are positive roots to the equation

$$\frac{\tan\mu}{\mu} = R_0,$$

$$R_0>1 \Longrightarrow (n-1)\pi < \tau_n < \frac{\pi}{2} + (n-1)\pi < n\pi,$$
$$R_0<1 \Longrightarrow n\pi < \tau_n < \frac{\pi}{2} + n\pi < 2n\pi,$$
$$\tau_n \approx -\left(\frac{\pi}{2}+n\pi\right), \quad n\in\mathbb{N},$$

and there exists a system of the corresponding eigenvectors $\{e_n\}_{n\in\mathbb{N}}$ being an *orthonormal basis* of H,

$$e_n(\theta) = \sqrt{\frac{4\tau_n}{2\tau_n - \sin 2\tau_n}}\,\sin\tau_n\theta, \quad 0\le\theta\le 1.$$

$\mathcal{A}$ generates on H an analytic, self-adjoint semigroup

$$S(t)x_0 = \sum_{n=1}^{\infty} e^{\lambda_n t}\langle x_0, e_n\rangle_{\mathrm{H}} e_n \quad \forall x_0\in\mathrm{H}, \quad \forall t\ge 0.$$

This semigroup is **EXS** as, by *Parseval's identity*, (2) holds with $M=1$ and $\alpha = -\lambda_1$.

The observation functional $\mathcal{C} = c^{\#}$ is still given by (22). We have $c^{\#}\mathcal{A}^{-1}x = \langle x, h\rangle_{\mathrm{H}}$, whence

$$h(\theta) = \frac{R_0}{1-R_0}\theta, \quad \theta\in[0,1].$$

Similarly, $\langle \mathcal{A}x, d\rangle_{\mathrm{H}} = x'(0)$, and thus $d^*\mathcal{A}^* = d^*\mathcal{A}$ extends to

$$d^{\#}x = x'(0), \quad D(d^{\#}) = \mathbf{C}^1[0,1] \ni h,$$
$$d^{\#}h = \tfrac{R_0}{1-R_0} = c^{\#}d.$$

Notice that

$$d = \frac{1}{R_0}h - \frac{1-R_0}{R_0}h' = \frac{1}{R_0}h - \mathbf{1},$$
$$c^{\#}e_n = \sqrt{\frac{4\tau_n}{2\tau_n-\sin 2\tau_n}}\,\sin\tau_n,$$
$$d^{\#}e_n = \sqrt{\frac{4\tau_n}{2\tau_n-\sin 2\tau_n}}\,\tau_n, \quad n\in\mathbb{N}.$$

From (45) and (22) we obtain

$$c^{\#}(sI-\mathcal{A})^{-1}v = \frac{1}{\sqrt{s}\cosh\sqrt{s} - \dfrac{\sinh\sqrt{s}}{R_0}} \int_0^1 \sinh\sqrt{s}\tau\; v(\tau)\,\mathrm{d}\tau,$$

and (4) enables us to determine the system transfer function

$$\hat{G}(s) = sc^{\#}(sI-\mathcal{A})^{-1}d - c^{\#}d = \frac{1}{\cosh\sqrt{s} - \dfrac{\sinh\sqrt{s}}{R_0\sqrt{s}}}, \quad s\notin\{\lambda_n\}_{n\in\mathbb{N}}, \tag{47}$$

Lemma 8. *$\hat{G}\in H^\infty(\mathbb{C}^+)$ with the norm $\|\widehat{G}\|_{H^\infty(\mathbb{C}^+)} = \frac{R_0}{|1-R_0|}$ achieved at $s=0$.*

Proof. It is enough to prove that

$$\left| R_0\cosh\sqrt{i\omega} - \frac{\sinh\sqrt{i\omega}}{\sqrt{i\omega}} \right|^2 \ge (1-R_0)^2 \quad \forall\omega\in\mathbb{R}. \tag{48}$$

With $i\omega = (1\pm i)\Omega$, $\Omega := \sqrt{|\omega|/2}$, the left-hand side of (48) equals

$$\left[\frac{\sinh\Omega}{2\Omega}\cos\Omega + \cosh\Omega\frac{\sin\Omega}{2\Omega} - R_0\cosh\Omega\cos\Omega\right]^2 + \left[\cosh\Omega\frac{\sin\Omega}{2\Omega} - \frac{\sinh\Omega}{2\Omega}\cos\Omega - R_0\sinh\Omega\sin\Omega\right]^2$$
$$= -\frac{1}{2\Omega^2}\left[2R_0\Omega\sinh\Omega\cosh\Omega - \cosh^2\Omega + 2\Omega R_0\sin\Omega\cos\Omega + \cos^2\Omega - 2R_0^2\cos^2\Omega + 2R_0^2\Omega^2 - 2R_0^2\Omega^2\cosh^2\Omega\right],$$

and we have to prove that

$$\begin{aligned}&\forall R_0>0\ \forall\Omega\in\mathbb{R}:\\ &\quad 2\Omega^2\left[\cos 2\Omega+\cosh 2\Omega-2\right]R_0^2\\ &\quad -2\Omega\left[\sinh 2\Omega+\sin 2\Omega-4\Omega\right]R_0\\ &\qquad +\left[\cosh 2\Omega-\cos 2\Omega-4\Omega^2\right]\geq 0.\end{aligned}$$

Since, for $\Omega\neq 0$, $\cos 2\Omega+\cosh 2\Omega-2>0$, this second order polynomial in R_0 is nonnegative iff its determinant $-4\Omega^2p(\Omega)$ is nonpositive, where

$$\begin{aligned}p(\Omega):=&-2\sinh 2\Omega\sin 2\Omega+8\Omega\sinh 2\Omega\\ &+8\Omega\sin 2\Omega-8\Omega^2\cosh 2\Omega-8\Omega^2\cos 2\Omega\\ &-\cos^2 2\Omega+\cosh^2 2\Omega+4\cos 2\Omega\\ &-4\cosh 2\Omega.\end{aligned}$$

Observe that $p(0)=0$ and, using Taylor's expansion,

$$\begin{aligned}p'(\Omega)=&\,4(\sinh 2\Omega-\sin 2\Omega)\\ &\times(\cosh 2\Omega-\cos 2\Omega-4\Omega^2)\geq 0,\end{aligned}$$

whence $p(\Omega)\geq 0$ and therefore $-4\Omega^2p(\Omega)\leq 0$. ∎

Boundedness of $\widehat{G}$ on $j\mathbb{R}$ is confirmed by the *Nyquist curves* depicted in Fig. 8, determining the *spectrum* $\sigma(\mathbb{F})=\overline{\widehat{G}(\mathbb{C}^+)}$. Notice that

$$1+\widehat{G}(0)=\tfrac{2R_0-1}{R_0-1}$$

and

$$\begin{aligned}-\frac{1}{\mathcal{K}}&=-1\notin\sigma(\mathbb{F})\cap\mathbb{R}\\ &=\overline{\widehat{G}(\mathbb{C}^+)}\cap\mathbb{R}\Leftrightarrow R_0\notin\left(\frac{1}{2},1\right).\end{aligned}\tag{49}$$

Hence (8) holds iff $R_0\in(0,1/2)\cup(1,\infty)$.

Lemma 9. *Let $R_0\neq 1$. For $s\in\mathbb{C}^+$, the following estimates hold true:*

$$\|h^*\mathcal{A}(sI-\mathcal{A})^{-1}\|^2_{\mathbf{L}(H,\mathbb{C})}\leq\frac{3}{|s|^2+\tau_1^4}+\frac{3\sqrt{2}}{4|s|^{3/2}},\tag{50}$$

$$\begin{aligned}\|\mathcal{A}(sI-\mathcal{A})^{-1}d\|_H^2\leq&\frac{2\tau_1^2}{1-\dfrac{\sin 2\tau_1}{2\tau_1}}\frac{1}{|s|^2+\tau_1^4}\\ &+\frac{12}{|s|^{1/2}}.\end{aligned}\tag{51}$$

Proof. Let $s\in\mathbb{C}^+$. Using Parseval's identity, we get

$$\begin{aligned}&\|h^*\mathcal{A}(sI-\mathcal{A})^{-1}\|^2_{\mathbf{L}(\mathrm{H},\mathbb{C})}\\ &\quad=\|\mathcal{A}(\overline{s}I-\mathcal{A})^{-1}h\|^2_{\mathrm{H}}\\ &\quad=\sum_{n=1}^{\infty}\left|\langle\mathcal{A}(\overline{s}I-\mathcal{A})^{-1}h,e_n\rangle_{\mathrm{H}}\right|^2\\ &\quad=\sum_{n=1}^{\infty}\frac{|c^{\#}e_n|^2}{|s-\lambda_n|^2}\\ &\quad=\sum_{n=1}^{\infty}\frac{1-\cos 2\tau_n}{1-\dfrac{\sin 2\tau_n}{2\tau_n}}\frac{1}{(\operatorname{Re}s+\tau_n^2)^2+\operatorname{Im}^2 s}\\ &\quad\leq\sum_{n=1}^{\infty}\frac{3}{|s|^2+\tau_n^4}.\end{aligned}$$

We employed the inequality

$$\frac{1-\cos x}{1-\dfrac{\sin x}{x}}\leq 3\iff q(x):=x-\frac{3\sin x}{2+\cos x}\geq 0$$

for $x\geq 0$, which is true since $q(0)=0$ and

$$q'(x)=\left(\frac{1-\cos x}{2+\cos x}\right)^2\geq 0.$$

Now, if $R_0<1$ then

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{3}{|s|^2+\tau_n^4}&\leq\sum_{n=1}^{\infty}\frac{3}{|s|^2+n^4\pi^4}\\ &\leq\int_0^{\infty}\frac{3\mathrm{d}n}{|s|^2+\pi^4n^4}.\end{aligned}$$

However, if $R_0>1$, then

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{3}{|s|^2+\tau_n^4}\\ &\quad\leq\frac{3}{|s|^2+\tau_1^4}+\sum_{n=2}^{\infty}\frac{3}{|s|^2+(n-1)^4\pi^4}\\ &\quad=\frac{3}{|s|^2+\tau_1^4}+\sum_{n=1}^{\infty}\frac{3}{|s|^2+n^4\pi^4}\\ &\quad\leq\frac{3}{|s|^2+\tau_1^4}+\int_0^{\infty}\frac{3}{|s|^2+\pi^4n^4}\,\mathrm{d}n.\end{aligned}$$

Thus, generally, we have

$$\begin{aligned}&\|h^*\mathcal{A}(sI-\mathcal{A})^{-1}\|^2_{\mathbf{L}(\mathrm{H},\mathbb{C})}\\ &\qquad\leq\frac{3}{|s|^2+\tau_1^4}+\int_0^{\infty}\frac{3\mathrm{d}n}{|s|^2+\pi^4n^4},\end{aligned}$$

whence, substituting

$$y=\frac{\pi n}{\sqrt{|s|}},$$

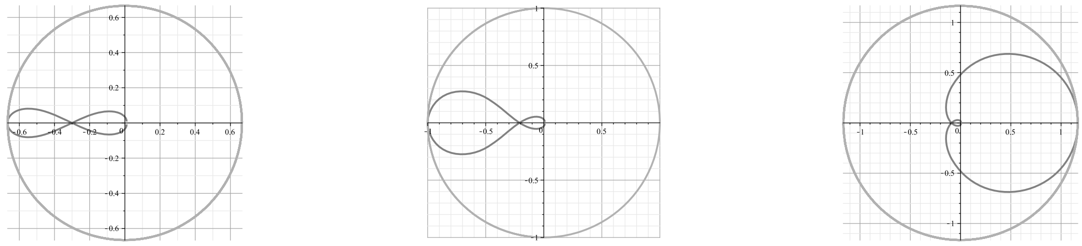

Fig. 8. Nyquist plots of $\widehat{G}(i\omega)$ for $R_0 = 2/5$ (left), $R_0 = 1/2$ (middle) and $R_0 = 7$ (right).

we get (50) because

$$\int_0^\infty \frac{\mathrm{d}y}{1+y^4} = \frac{\pi\sqrt{2}}{4}.$$

Similarly,

$$\begin{aligned}
&\|\mathcal{A}(sI-\mathcal{A})^{-1}d\|_{\mathrm{H}}^2 \\
&= \sum_{n=1}^{\infty} \left|\langle \mathcal{A}(sI-\mathcal{A})^{-1}d, e_n\rangle_{\mathrm{H}}\right|^2 \\
&= \sum_{n=1}^{\infty} \frac{|d^{\#}e_n|^2}{|s-\lambda_n|^2} \\
&\le \sum_{n=1}^{\infty} \frac{2\tau_n^2}{1-\dfrac{\sin 2\tau_n}{2\tau_n}} \frac{1}{|s|^2+\tau_n^4}.
\end{aligned}$$

It is clear that

$$3 \le \frac{2x^2}{1-\dfrac{\sin 2x}{2x}} \le 3x^2 \quad \text{for} \quad x \ge \frac{3}{2},$$

where its left-hand side is larger than 3 even for all $x \ge 0$.

Thus, if $R_0 < 1$, then

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{2\tau_n^2}{1-\dfrac{\sin 2\tau_n}{2\tau_n}} \frac{1}{|s|^2+\tau_n^4} \\
&\le \sum_{n=1}^{\infty} \frac{3\tau_n^2}{|s|^2+\pi^4 n^4} \\
&\le \sum_{n=1}^{\infty} \frac{12n^2\pi^2}{|s|^2+\pi^4 n^4} \\
&\le \sum_{n=1}^{\infty} \frac{24n^2\pi^2}{|s|+n^2\pi^2} \frac{1}{|s|+n^2\pi^2} \\
&\le \sum_{n=1}^{\infty} \frac{24}{|s|+n^2\pi^2},
\end{aligned}$$

while, if $R_0 > 1$, then

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{2\tau_n^2}{1-\dfrac{\sin 2\tau_n}{2\tau_n}} \frac{1}{|s|^2+\tau_n^4} \\
&\le \frac{2\tau_1^2}{1-\dfrac{\sin 2\tau_1}{2\tau_1}} \frac{1}{|s|^2+\tau_1^4} + \sum_{n=2}^{\infty} \frac{3n^2\pi^2}{|s|^2+(n-1)^4\pi^4} \\
&\le \frac{2\tau_1^2}{1-\dfrac{\sin 2\tau_1}{2\tau_1}} \frac{1}{|s|^2+\tau_1^4} + \sum_{n=1}^{\infty} \frac{3(n+1)^2\pi^2}{|s|^2+n^4\pi^4} \\
&\le \frac{2\tau_1^2}{1-\dfrac{\sin 2\tau_1}{2\tau_1}} \frac{1}{|s|^2+\tau_1^4} + \sum_{n=1}^{\infty} \frac{24n^2\pi^2}{|s|+n^2\pi^2} \frac{1}{|s|+n^2\pi^2} \\
&\le \frac{2\tau_1^2}{1-\dfrac{\sin 2\tau_1}{2\tau_1}} \frac{1}{|s|^2+\tau_1^4} + \sum_{n=1}^{\infty} \frac{24}{|s|+n^2\pi^2},
\end{aligned}$$

which for $R_0 \neq 1$ yields (51). ∎

By Lemmas 8 and 9, all assumptions of Theorem 9 hold, provided that $R_0 \in (0, 1/2) \cup (1, \infty)$, and then the closed-loop operator $\mathcal{A}_c$,

$$\begin{aligned}
\mathcal{A}_c x &= \mathcal{A}(x - dc^{\#}x), \\
D(\mathcal{A}_c) &= \{x \in D(c^{\#}) : x - dc^{\#}x \in D(\mathcal{A})\},
\end{aligned}$$

generates an **EXS** analytic semigroup on $\mathrm{H} = \mathrm{L}^2(0,1)$. Its explicit form reads as

$$\begin{aligned}
\mathcal{A}_c x &= x'', \\
D(\mathcal{A}_c) &= \{x \in \mathrm{H}^2(0,1) : x'(1) = \frac{x(1)}{R_0}, \\
&\qquad x(0) + x(1) = 0\}.
\end{aligned}$$

Let us compare the last result and that of Example 1 with an exhaustive study of the spectral properties of the second order differentiation operator with boundary conditions

$$\begin{bmatrix} a_1 & b_1 & a_0 & b_0 \\ c_1 & d_1 & c_0 & d_0 \end{bmatrix} \begin{bmatrix} y'(0) \\ y'(1) \\ y(0) \\ y(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad (52)$$

which was done by Lang and Locker (1989; 1990). The Lang–Locker classification is simplified in Table 1, where A_{ij} denotes the determinant of a submatrix obtained by retaining the i-th and the j-th columns of the matrix in (52). The second column corresponds to the classes I–VIII and X introduced therein. Two last columns say whether or not the operator is a *spectral* one (letter S or NS), while the letter R informs us that boundary conditions are *regular*, see, e.g., the work of Grabowski (1999, Def. 2.9.1) for details.

Boundary conditions appearing in $D(\mathcal{A}_c)$ correspond to

$$\begin{bmatrix} a_1 & b_1 & a_0 & b_0 \\ c_1 & d_1 & c_0 & d_0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & -R_0^{-1} \\ 0 & 0 & 1 & \mathcal{K} \end{bmatrix},$$

giving

$$A_{12} = A_{13} = A_{14} = 0,$$
$$A_{23} = 1, \quad A_{24} = \mathcal{K}, \quad A_{34} = R_0^{-1}.$$

In the case of Example 1, $R_0 = \infty$, so $\mathcal{A}_c$ belongs to the second row of Table 1 if $\mathcal{K} \neq 1$ and to the third row if $\mathcal{K} = 1$. In the case of Example 2, $\mathcal{K} = 1$, so $\mathcal{A}_c$ corresponds to the last row of Table 1. In the last case all eigenvalues of $\mathcal{A}_c$ are single, and there exists a set of the corresponding eigenvectors which is complete in H, but it does not form a Riesz basis, while $\mathcal{A}_c$ is not a spectral operator. Therefore one cannot deduce whether $\mathcal{A}_c$ generates an analytic **EXS** semigroup using the Riesz basis approach. This shows that Theorem 9 yields a stronger result.

6. Discussion and conclusions

Two criteria of $\mathcal{CD} \in \mathbf{L}(\mathrm{U}, \mathrm{Y})$. If $\mathcal{C}$ is closed and $R(\mathcal{D}) \subset D(\mathcal{C})$, then, by the closed-graph theorem, $\mathcal{CD} \in \mathbf{L}(\mathrm{U}, \mathrm{Y})$.

If $\mathrm{Y} = \mathbb{R}^m$ and $\mathrm{U} = \mathbb{R}^r$, then

$$\mathcal{C} = \begin{bmatrix} c_1^\# & c_2^\# & \cdots & c_m^\# \end{bmatrix}^T$$

may even be not closable, but with

$$\mathcal{D} = \begin{bmatrix} d_1 & d_2 & \cdots d_r \end{bmatrix}, \quad d_i \in D(c_j),$$
$$i = 1, 2, \ldots, r; \quad j = 1, 2, \ldots, m$$

one still has $\mathcal{CD} \in \mathbf{L}(\mathrm{U}, \mathrm{Y})$.

Comment on the inversion formula. Let $\mathcal{D} = d \in \mathrm{H}$ and $H = h \in \mathrm{H}$. Then the inversion formula (12) shows that the stated problem has common ingredients with the theory of *rank one perturbations of compact operators*. It follows from the results of Deckard *et al*. (1979) that there exist d and h such that kernels of $\mathcal{A}_c$ and $\mathcal{A}_c^*$ are both trivial and eigenvectors of $\mathcal{A}_c$ span H but the eigenvectors of $\mathcal{A}_c^*$ do not.

Crouzeix's contribution/conjecture. There is a link (take $r(z) = 1/\lambda - z$) between Propositions 2 and 3 and the *functional calculus* based on the numerical range for a closed densely defined operator $\mathcal{A}$ satisfying $\sigma(\mathcal{A}) \subset \overline{W(\mathcal{A})}$. Then we have following:

(a) there holds that

$$\|r(\mathcal{A})\|_{\mathbf{L}(\mathrm{H})} \leq 12 \sup_{z \in W(\mathcal{A})} |r(z)|$$

for any rational function r, bounded on $W(\mathcal{A})$ (Crouzeix, 2008, p. 83);

(b) there exists a Banach similarity isomorphism T with conditional number $\|T\|_{\mathbf{L}(\mathrm{H})}\|T^{-1}\|_{\mathbf{L}(\mathrm{H})} \leq 12$ such that

$$\|r(T^{-1}\mathcal{A}T)\|_{\mathbf{L}(\mathrm{H})} \leq \sup_{z \in W(\mathcal{A})} |r(z)|$$

for any rational function r, bounded on $W(\mathcal{A})$ (Crouzeix, 2008, p. 96). The constant 12 was later improved to $1 + \sqrt{2}$, and it is being conjectured that 2 is its minimal value.

Extension of Theorem 9 by stabilizability. If $\mathcal{A}$ does not generate an **EXS** semigroup out of having a part of spectrum $\sigma(\mathcal{A})$ in $\overline{\mathbb{C}^+}$, one may extract, if possible, the corresponding unstable part from $(sI - \mathcal{A})^{-1}$ using the *Riesz projector*

$$Pf := \frac{1}{2\pi i} \int_{\gamma^+} (s - \mathcal{A})^{-1} f \, \mathrm{d}s,$$

where γ^+ is a positively oriented rectifiable bounded closed curve separating the unstable part of $\mathcal{A}$ from its rest located in the exterior of γ^+. Next, representing $(sI - \mathcal{A}_c)^{-1}$ in the form

$$\begin{aligned} &(sI - \mathcal{A}_c)^{-1} \\ &= (I - P)(sI - \mathcal{A})^{-1} \\ &\quad - \Big\{ \mathcal{A}(sI - \mathcal{A})^{-1}\mathcal{DK}[I + \widehat{G}(s)\mathcal{K}]^{-1}H\mathcal{A}(sI - \mathcal{A})^{-1} \\ &\quad - P(sI - \mathcal{A})^{-1} \Big\}, \end{aligned}$$

we can conclude that $\mathcal{A}_c$ generates an **EXS** analytic semigroup provided that the first component in the RHS and the component in the brackets satisfy the assumption of Proposition 9. This idea is due to T. Kato and was adopted to the control system $\dot{x} = \mathcal{A}x + Bu$, $B \in \mathbf{L}(\mathrm{H})$ by Triggiani (1975).

Table 1. Simplified Lang and Locker classification.

Conditions imposed on A_{ij}	Case	$\Vert \sum P_k \Vert$	Class
$A_{12} \neq 0$	I, II	$< \infty$	R, S
$A_{12} = 0, A_{14} + A_{23} \neq 0, A_{14} + A_{23} \neq \mp(A_{13} + A_{24})$	III, IV	$< \infty$	R, S
$A_{12} = 0, A_{14} + A_{23} \neq 0, A_{14} + A_{23} = \mp(A_{13} + A_{24}), A_{34} \neq 0 \Rightarrow A_{13} = A_{24}$	V, VI, VII	$< \infty$	R, S
$A_{12} = 0, A_{14} + A_{23} = 0, A_{34} \neq 0, A_{13} + A_{24} = 0, A_{13} = A_{24}$	X	$< \infty$	R, S
$0 \neq A_{14} + A_{23} = \mp(A_{13} + A_{24}), A_{12} = 0, A_{34} \neq 0, A_{13} \neq A_{24}$	VIII	∞	R, NS

We have even more: the open-loop state operator may not even generate a C_0-semigroup, but the closed-loop state operator with static feedback $\mathcal{K} \neq 0$ generates an **EXS** analytic semigroup. Indeed, take

$$\begin{gathered} \mathrm{H} = \mathrm{L}^2(0,1), \\ \sigma f = f'', \quad D(\sigma) = \{f \in \mathrm{H}^2(0,1) : f'(0) = 0\}, \\ \tau x = x(0), \quad d = -\mathbf{1}, \quad c^{\#} x = x(1), \\ D(\tau) = D(c^{\#}) = \mathrm{C}[0,1], \end{gathered}$$

whence

$$\begin{aligned} &\mathcal{A} = \sigma|_{\ker \tau} = R^2 \neq \mathcal{A}^*, \\ &Rf = -f', \quad D(R) = \mathrm{W}_0^{1,2}(0,1), \\ &h(\theta) = 1 - \theta, \\ &d^{\#} x = x'(0), \quad D(d^{\#}) = \mathrm{C}[0,1], \\ &d^{\#} h = -1 = c^{\#} d, \end{aligned}$$

where $d^{\#}$ extends $d^* \mathcal{A}^*$. Next, the resolvent

$$\begin{aligned} &((\lambda I - \mathcal{A})^{-1} g)(\theta) \\ &\qquad = \int_0^\theta \frac{\sinh(\sqrt{\lambda}(\theta - \tau))}{\sqrt{\lambda}} g(\tau)\,\mathrm{d}\tau, \quad g \in \mathrm{H} \end{aligned}$$

is an entire operator-valued function and, consequently,

$$\begin{aligned} &h^* \mathcal{A}(\lambda I - \mathcal{A})^{-1} g \\ &\quad = c^{\#}(\lambda I - \mathcal{A})^{-1} g \\ &\quad = -\int_0^1 \frac{\sinh(\sqrt{\lambda}(1 - \tau))}{\sqrt{\lambda}} g(\tau)\,\mathrm{d}\tau, \end{aligned}$$

$$\begin{aligned} (\mathcal{A}(\lambda I - \mathcal{A})^{-1} d)(\theta) &= \cosh(\sqrt{\lambda}\theta), \\ \hat{G}(s) &= \cosh(\sqrt{s}). \end{aligned}$$

The closed-loop state operator has to be identified with

$$\begin{aligned} &\mathcal{A}_c f = f'', \\ &D(\mathcal{A}_c) = \{f \in \mathrm{H}^2(0,2) : f'(0) = 0, \\ &\qquad\qquad f(0) + \mathcal{K} f(1) = 0\}. \end{aligned}$$

The last operator generates an **EXS** analytic semigroup for $\mathcal{K}^{-1} \in (-1, \cosh \pi)$, $\mathcal{K} \neq 1$.

No alternative to Lemma 5 in Example 1. In Example 1 one has $\mathcal{K}(I + \hat{G}\mathcal{K})^{-1} \in \mathrm{H}^\infty(\mathbb{C}^+)$ but $s \longmapsto s\mathcal{K}(I + \hat{G}(s)\mathcal{K})^{-1} \notin \mathrm{H}^\infty(\mathbb{C}^+)$, so only the balance between observation and control ensures that (15) holds.

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Appendix A

Consider the following boundary value problem:

$$z'(\theta) = \mathbf{A}(\lambda)z(\theta) - e_2 v(\theta), \tag{A1}$$

$$\mathbf{M}z(0) + \mathbf{N}z(1) = 0. \tag{A2}$$

$$\mathbf{A}(\lambda) := \begin{bmatrix} 0 & 1 \\ \lambda & 0 \end{bmatrix}$$

$$\Longrightarrow e^{\theta \mathbf{A}(\lambda)} = \begin{bmatrix} \cosh\sqrt{\lambda}\theta & \dfrac{\sinh\sqrt{\lambda}\theta}{\sqrt{\lambda}} \\ \sqrt{\lambda}\sinh\sqrt{\lambda}\theta & \cosh\sqrt{\lambda}\theta \end{bmatrix}.$$

A general solution of (A1) is

$$z(\theta) = e^{\theta \mathbf{A}(\lambda)} z(0) - \int_0^\theta e^{(\theta-\tau)\mathbf{A}(\lambda)} e_2 v(\tau)\,\mathrm{d}\tau.$$

Substituting it into (A2), we get

$$(\mathbf{M} + \mathbf{N} e^{\mathbf{A}(\lambda)}) z(0) = \mathbf{N} p,$$

$$p = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} := \begin{bmatrix} \left\langle \dfrac{\sinh\sqrt{\lambda}(1-(\cdot))}{\sqrt{\lambda}}, v \right\rangle_{\mathrm{H}} \\ \left\langle \cosh\sqrt{\lambda}(1-(\cdot)), v \right\rangle_{\mathrm{H}} \end{bmatrix}.$$

If $\det\left(\mathbf{M} + \mathbf{N} e^{\mathbf{A}(\lambda)}\right) \neq 0$ for some $\lambda \in \mathbb{C}$, then

$$\begin{aligned} z(\theta) =& e^{\theta \mathbf{A}(\lambda)} (\mathbf{M} + \mathbf{N} e^{\mathbf{A}(\lambda)})^{-1} \mathbf{N} p \\ & - \int_0^\theta \begin{bmatrix} \dfrac{\sinh\sqrt{\lambda}(\theta-\tau))}{\sqrt{\lambda}} v(\tau) \\ \cosh\sqrt{\lambda}(\theta-\tau) v(\tau) \end{bmatrix} \mathrm{d}\tau, \end{aligned}$$

whence the resolvent of $\mathcal{A}$ is

$$\left((sI - \mathcal{A}_c)^{-1} v\right)(\theta) = e_1^T z(\theta). \tag{A3}$$

Appendix B

To justify that the projector P_n (39) coincides with the *Riesz projector*, we start from finding the resolvent of $\mathcal{A}_c$ for $\mathcal{K} = 1$, which reduces to solving the two-boundary value problem

$$\begin{cases} \lambda f(\theta) - f''(\theta) = v(\theta), \quad v \in \mathrm{L}^2(0,1), \\ f(0) + f(1) = 0, \\ f'(1) = 0. \end{cases} \tag{B1}$$

Introducing

$$z(\theta) := \begin{bmatrix} f(\theta) \\ f'(\theta) \end{bmatrix},$$

we reduce (B1) to (A1), (A2) with $M = e_1 e_1^T$, $N = I$ and $\det\left(\mathbf{M} + \mathbf{N} e^{\mathbf{A}(\lambda)}\right) = \cosh\sqrt{\lambda} + 1 \neq 0$, that is, λ is not an eigenvalue of $\mathcal{A}_c$. Here

$$\begin{aligned} & e_1^T e^{\theta \mathbf{A}(\lambda)} (\mathbf{M} + \mathbf{N} e^{\mathbf{A}(\lambda)})^{-1} \mathbf{N} p \\ &= \begin{bmatrix} \cosh\sqrt{\lambda}\theta & \dfrac{\sinh\sqrt{\lambda}\theta}{\sqrt{\lambda}} \end{bmatrix} \\ & \times \begin{bmatrix} \dfrac{1}{1+\cosh\sqrt{\lambda}} \left(p_1 \cosh\sqrt{\lambda} - p_2 \dfrac{\sinh\sqrt{\lambda}}{\sqrt{\lambda}} \right) \\ -\dfrac{\sqrt{\lambda}\sinh\sqrt{\lambda}}{1+\cosh\sqrt{\lambda}} p_1 + p_2 \end{bmatrix} \\ & - \frac{\cosh\sqrt{\lambda}\theta}{1+\cosh\sqrt{\lambda}} \left\langle \frac{\sinh\sqrt{\lambda}(\cdot)}{\sqrt{\lambda}}, v \right\rangle_{\mathrm{L}^2(0,1)} \\ & - \frac{\sinh\sqrt{\lambda}\theta \sinh\sqrt{\lambda}}{1+\cosh\sqrt{\lambda}} p_1 + \frac{\sinh\sqrt{\lambda}\theta}{\sqrt{\lambda}} p_2, \end{aligned} \tag{B2}$$

and therefore, by (A3), with $v \in \mathrm{H}$,

$$\begin{aligned} & \left((\lambda I - \mathcal{A}_c)^{-1} v\right)(\theta) \\ &= \frac{\sinh\sqrt{\lambda}\theta}{\sqrt{\lambda}} \int_0^1 \cosh\sqrt{\lambda}(1-y) v(\tau)\,\mathrm{d}y \\ & - \int_0^\theta \frac{\sinh\sqrt{\lambda}(\theta-y))}{\sqrt{\lambda}} v(\tau)\,\mathrm{d}y \\ & - \frac{\cosh\sqrt{\lambda}\theta}{\cosh\sqrt{\lambda}+1} \int_0^1 \frac{\sinh\sqrt{\lambda}y}{\sqrt{\lambda}} v(y)\,\mathrm{d}y \\ & - \frac{\sinh\sqrt{\lambda}\theta \sinh\sqrt{\lambda}}{\cosh\sqrt{\lambda}+1} \\ & \times \int_0^1 \frac{\sinh\sqrt{\lambda}(1-y)}{\sqrt{\lambda}} v(y)\,\mathrm{d}y, \end{aligned}$$

where the last line consists of *meromorphic functions* $L_1(\lambda)/M(\lambda)$, $L_2(\lambda)/M(\lambda)$ of λ,

$$\begin{aligned} M(\lambda) &:= \cosh\sqrt{\lambda} + 1, \\ L_1(\lambda) &:= -\cosh\sqrt{\lambda}\theta \int_0^1 \frac{\sinh\sqrt{\lambda}y}{\sqrt{\lambda}} v(y)\,\mathrm{d}y \\ L_2(\lambda) &:= -\sinh\sqrt{\lambda}\theta \sinh\sqrt{\lambda} \\ & \quad \times \int_0^1 \frac{\sinh\sqrt{\lambda}(1-y)}{\sqrt{\lambda}} v(y)\,\mathrm{d}y, \end{aligned}$$

and the third line is an entire function of λ (here θ and y are fixed).

To determine an explicit form of P_k we need the following result.

Lemma B1. *The second order residue of $f(s)/g(s)$, where f, g are entire, $g(s_0) = 0$, $g'(s_0) = 0$ and $g''(s_0) \neq 0$ is expressed as*

$$\begin{aligned} \operatorname*{Res}_{s=s_0} \frac{f(s)}{g(s)} &:= \lim_{s\to s_0} \frac{\mathrm{d}}{\mathrm{d}s} \left[\frac{f(s)(s-s_0)^2}{g(s)} \right] \\ &= \frac{2f'(s_0)}{g''(s_0)} - \frac{2f(s_0)g'''(s_0)}{3[g''(s_0)]^2}. \end{aligned} \tag{B3}$$

Proof. Indeed,

$$\begin{aligned}\left[\frac{f(s)(s-s_0)^2}{g(s)}\right]' &= \frac{f'(s)(s-s_0)^2}{g(s)} \\ &\quad + f(s)\frac{2(s-s_0)g(s)-g'(s)(s-s_0)^2}{[g(s)]^2},\end{aligned}$$

and the *Taylor series* of g around s_0 reads as

$$g(s) = \frac{g''(s_0)}{2}(s-s_0)^2 + \frac{g'''(s_0)}{6}(s-s_0)^3 + \cdots,$$

whence, as $s \to s_0$, we have

$$\begin{aligned}&\frac{(s-s_0)^2}{g(s)} \to \frac{2}{g''(s_0)}, \\ &\frac{2(s-s_0)g(s)-g'(s)(s-s_0)^2}{[g(s)]^2} \\ &= \frac{\frac{1}{6}g'''(s_0)(s-s_0)^4+\cdots}{\left[\frac{g''(s_0)}{2}\right]^2(s-s_0)^4+\cdots} \longrightarrow \frac{2g'''(s_0)}{3[g''(s_0)]^2}.\end{aligned}$$

Those facts with regularity of f and f' at s_0 give the assertion (B3). ■

Applying the rule (B3) to (41), we get

$$\begin{aligned}P_k v = \sum_{m=1}^{m=2}\int_0^1 &\left[\frac{2L'_m(\lambda_k)}{M''(\lambda_k)}\right. \\ &\left. -\frac{2L_m(\lambda_k)M'''(\lambda_k)}{3[M''(\lambda_k)]^2}\right] v(y)\mathrm{d}y.\end{aligned}$$

Using computer aided symbolic calculations offered by MAPLE®, we obtain

$$\begin{aligned}(P_k v)(\theta) &= 4(1-\theta)\sin(\mu_k\theta)\int_0^1 \sin(\mu_k y)v(y)\,\mathrm{d}y \\ &\quad + 4\cos(\mu_k\theta)\int_0^1 \cos(\mu_k y)yv(y)\,\mathrm{d}y \\ &= U_{2k-1}(\theta)V^*_{2k-1}v + U_{2k}(\theta)V^*_{2k}v, \\ &\quad k\in\mathbb{N},\ v\in \mathrm{L}^2(0,1),\end{aligned}$$

as expected.

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