THE SET OF DISCONTINUITIES OF DENSITY-TYPE-APPROXIMATELY CONTINUOUS FUNCTIONS

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ABSTRACT. A characterization of the set of discontinuities for different density-type-approximately continuous functions is given.

1. Introduction

It is well-known that the following two statements for a set $A \subset \mathbb{R}$ are equivalent (see [G]):

- (i) A is an F_σ set,
- (ii) there is a function $f: \mathbb{R} \rightarrow \mathbb{R}$ such that A is the set of discontinuities of f .

An analogous result has been established for approximately continuous functions (see [B, p. 48]):

THEOREM 1.1. *The following two statements for a set $A \subset [a, b]$ are equivalent:*

- (i) A is dense in $[a, b]$ and of type G_δ .
- (ii) *There is an approximately continuous function f such that A is the set of continuities of f .*

The aim of this paper is to verify whether the same result can be established for some density-type topologies. Our main result is as follows:

THEOREM 1.2. *The following two statements for a set $A \subset \mathbb{R}$ (or $A \subset \mathbb{R}^2$) are equivalent:*

- (i) A is an F_σ first category set,

- (ii) *There is a continuous function $f: (\mathbb{R}, \tau) \rightarrow (\mathbb{R}, \mathcal{O})$ (or $f: (\mathbb{R}^2, \tau) \rightarrow (\mathbb{R}, \mathcal{O})$) such that A is the set of discontinuities of f , where τ denotes one of the following topologies: the density topology [W1], simple density topology [WA], complete density topology [WW], ψ -density topology [TW-B], $\langle s \rangle$ -density topology generated by a sequence $\langle s \rangle$ [FH] (respectively density, strong density [W1], product of density topologies or ordinary ψ -density topology on the plane [F]) and $\mathcal{O}$ stands for the Euclidean topology.*

The main idea of the proof comes from [P-WS], where a theorem for the density topology case was proved (as in [B]).

What is interesting the situation in the category case (I -density topology) is different. Some additional assumptions of a set A are needed to receive the existence of I -approximately continuous function $f: (\mathbb{R}, \mathcal{T}_I) \rightarrow (\mathbb{R}, \mathcal{O})$ for which A is equal to the set of all points of ordinary discontinuity (see [W2]).

2. Definitions and basic facts about density type topologies

Each of the topologies mentioned above is defined as a family of those Lebesgue measurable sets A such that each point of A is a point of some kind of density of A :

$$\tau = \{A \in \mathcal{L} : A \subset \Phi(A)\},$$

where $\mathcal{L}$ denotes the σ -algebra of Lebesgue measurable sets on the line, and $\Phi(A)$ denotes the set of some kind of density points of A for $A \in \mathcal{L}$.

We obtain different topologies by using different type of densities.

DEFINITION 2.1 ([W1]). We say that a point $x \in \mathbb{R}$ is a density point of $A \in \mathcal{L}$ if and only if

$$\lim_{h \rightarrow 0^+} \frac{\lambda(A \cap [x - h, x + h])}{2h} = 1,$$

where λ stands for the Lebesgue measure on the line. We denote the density topology by $\mathcal{T}$.

DEFINITION 2.2 ([W1], [TW-B]). Let $\mathcal{C}$ denote the class of continuous increasing functions $\psi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$, where $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$, such that $\lim_{t \rightarrow 0^+} \psi(t) = 0$.

Let $\psi \in \mathcal{C}$. We say that $x \in \mathbb{R}$ is a ψ -density point of $A \in \mathcal{L}$ if and only if

$$\lim_{h \rightarrow 0^+} \frac{\lambda(A' \cap [x - h, x + h])}{2h\psi(2h)} = 0,$$

where A' denotes a complement of the set A .

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We denote the set of ψ -density point of A by $\Phi_\psi(A)$, so ψ -density topology $\mathcal{T}_\psi$ is defined by

$$\mathcal{T}_\psi = \{A \in \mathcal{L} : A \subset \Phi_\psi(A)\}.$$

PROPOSITION 2.3 ([W1], [TW-B]). *For each $\psi \in \mathcal{C} : \mathcal{O} \subsetneq \mathcal{T}_\psi \subsetneq \mathcal{T}$.*

DEFINITION 2.4 ([WA]). We say that $x \in \mathbb{R}$ is a simple density point of the set $A \in \mathcal{L}$ if $\{\chi_{(n \cdot (A-x)) \cap [-1,1]}\}_{n \in \mathbb{N}}$ converges to $\chi_{[-1,1]}$ almost everywhere, where $A - x = \{t - x : t \in A\}$, $n \cdot A = \{nt : t \in A\}$ and χ_A denotes a characteristic function of the set A .

The set of all simple density point of A will be denoted by $\Phi_s(A)$, so the simple density topology $\mathcal{T}_s$ is given by

$$\mathcal{T}_s = \{A \in \mathcal{L} : A \subset \Phi_s(A)\}.$$

DEFINITION 2.5 ([WW]). We say that $x \in \mathbb{R}$ is a complete density point of a set $A \in \mathcal{L}$ if

$$\sum_{n=1}^{\infty} \lambda\left((n(A' - x)) \cap [-1, 1]\right) < +\infty.$$

The set of all complete density points of A will be denoted by $\Phi_c(A)$, so the complete density topology $\mathcal{T}_c$ is defined by

$$\mathcal{T}_c = \{A \in \mathcal{L} : A \subset \Phi_c(A)\}.$$

PROPOSITION 2.6 ([WW], [WA]). $\mathcal{O} \subsetneq \mathcal{T}_c \subsetneq \mathcal{T}_s \subsetneq \mathcal{T}$.

DEFINITION 2.7 ([FH]). Let S be the family of all unbounded and nondecreasing sequences of positive reals. Every sequence $\{s_n\} \in S$ is denoted by $\langle s \rangle$.

Let $\langle s \rangle \in S$. We say that $x \in \mathbb{R}$ is an $\langle s \rangle$ -density point of $A \in \mathcal{L}$ if

$$\lim_{n \rightarrow \infty} \frac{\lambda(A \cap [x - \frac{1}{s_n}, x + \frac{1}{s_n}])}{\frac{2}{s_n}} = 1.$$

The set of all $\langle s \rangle$ -density points of A will be denoted by $\Phi_{\langle s \rangle}(A)$, so $\langle s \rangle$ -density topology $\mathcal{T}_{\langle s \rangle}$ is defined by

$$\mathcal{T}_{\langle s \rangle} = \{A \in \mathcal{L} : A \subset \Phi_{\langle s \rangle}(A)\}.$$

PROPOSITION 2.8 ([FH]). *For every sequence $\langle s \rangle \in S$, the following inclusion $\mathcal{T} \subset \mathcal{T}_{\langle s \rangle}$ holds and $\mathcal{T} = \mathcal{T}_{\langle s \rangle}$ if and only if $\liminf \frac{s_n}{s_{n+1}} > 0$.*

Let $\mathcal{L}_2$ denote the σ -algebra of Lebesgue measurable sets on the plane, and for $h > 0$ and $p = (x_0, y_0) \in \mathbb{R}^2$ put $Q(p, h) = [x_0 - h, x_0 + h] \times [y_0 - h, y_0 + h]$.

DEFINITION 2.9 ([W1]). We say that $p = (x_0, y_0) \in \mathbb{R}^2$ is an ordinary density point of $A \in \mathcal{L}_2$ if

$$\lim_{h \rightarrow 0^+} \frac{\lambda_2(A \cap Q(p, h))}{4h^2} = 1,$$

where λ_2 stands for the Lebesgue measure on the plane.

The set of all ordinary density points of A will be denoted by $\Phi_2(A)$ and the ordinary density topology $\mathcal{T}_2$ is defined by

$$\mathcal{T}_2 = \{A \in \mathcal{L}_2 : A \subset \Phi_2(A)\}.$$

DEFINITION 2.10 ([W1]). We say that a point $p = (x_0, y_0) \in \mathbb{R}^2$ is a strong density point of the set $A \in \mathcal{L}_2$ if

$$\lim_{\substack{h \rightarrow 0^+ \\ k \rightarrow 0^+}} \frac{\lambda_2(A \cap ([x_0 - h, x_0 + h] \times [y_0 - k, y_0 + k]))}{4hk} = 1.$$

The set of all strong density points of A will be denoted by $\Phi^s(A)$, so strong density topology $\mathcal{T}_2^s$ is defined by

$$\mathcal{T}_2^s = \{A \in \mathcal{L}_2 : A \subset \Phi^s(A)\}.$$

PROPOSITION 2.11. $\mathcal{O}_2 \subset \mathcal{T} \times \mathcal{T} \subset \mathcal{T}_2^s \subset \mathcal{T}_2$, where $\mathcal{O}_2$ stands for the Euclidean topology on the plane.

DEFINITION 2.12 ([F]). Let $\psi \in \mathcal{C}$. We say that $p = (x_0, y_0)$ is an ordinary ψ -density point of $A \in \mathcal{L}_2$ if

$$\lim_{h \rightarrow 0^+} \frac{\lambda_2(A' \cap Q(p, h))}{4h^2\psi(4h^2)} = 0.$$

The set of all ordinary ψ -density points of A will be denoted by $\Phi_\psi^0(A)$, so ordinary ψ density topology $\mathcal{T}_\psi^0$ is defined by

$$\mathcal{T}_\psi^0 = \{A \in \mathcal{L}_2 : A \subset \Phi_\psi^0(A)\}.$$

PROPOSITION 2.13 ([F]). For each $\psi \in \mathcal{C} : \mathcal{O}_2 \subsetneq \mathcal{T}_\psi^0 \subsetneq \mathcal{T}_2$.

DEFINITION 2.14. For every topology τ which was mentioned above, we say that a function is τ -approximately continuous if it is continuous as a function $f : (\mathbb{R}, \tau) \rightarrow (\mathbb{R}, \mathcal{O})$ or $f : (\mathbb{R}^2, \tau) \rightarrow (\mathbb{R}, \mathcal{O})$.

PROPOSITION 2.15 ([W1]). Every $\mathcal{T}$ -approximately continuous and every $\mathcal{T}_2$ -approximately continuous function is a Baire one function.

COROLLARY 2.16. Every $\mathcal{T} \times \mathcal{T}$, $\mathcal{T}_2^s$, $\mathcal{T}_\psi$, $\mathcal{T}_\psi^0$ (for any $\psi \in \mathcal{C}$) $\mathcal{T}_s$ or $\mathcal{T}_c$ -approximately continuous function is Baire one function.

PROPOSITION 2.17 ([E, Th. 1.4.7]). If f_n is τ -approximately continuous for any $n \in \mathbb{N}$ and a sequence $\{f_n\}_{n \in \mathbb{N}}$ uniformly converges to f , then f is τ -approximately continuous.

PROPOSITION 2.18 ([L]). *Let $\langle s \rangle \in S$. If f is $\mathcal{T}_{\langle s \rangle}$ -approximately continuous then f is a Baire one function.*

3. Proof of the main theorem

(ii) $\implies$ (i) follows immediately from Proposition 2.18 and Corollary 2.16 for each topology considered here since the set of discontinuities of a Baire one function is a first category F_σ set [L].

(i) $\implies$ (ii)

$\mathcal{T}_\psi$ -case. Let $\psi \in \mathcal{C}$. Let A be an F_σ first category set. There exists a countable set C and a sequence of pairwise disjoint nowhere dense perfect sets P_i such that $A = \bigcup_{i \in \mathbb{N}} P_i \cup C$ (see [S, vol. II, p. 537]).

First, we want to build a $\mathcal{T}_\psi$ -approximately continuous function which is discontinuous at the point 0.

Since $\psi \in \mathcal{C}$, there exists $n_0 \in \mathbb{N}$ such that $\psi\left(\frac{1}{2^{n_0}}\right) \leq 1$. Let

$$a_n = \frac{1}{2^n} - \frac{1}{4^n} \psi\left(\frac{1}{2^n}\right), \quad b_n = \frac{1}{2^n} \text{ for } n \geq n_0$$

and

$$B = \bigcup_{n=n_0}^{\infty} [a_n, b_n].$$

Let $h < \frac{1}{2^{n_0}}$. Then there exists $n \geq n_0$, $n \in \mathbb{N}$ such that $\frac{1}{2^{n+1}} \leq h < \frac{1}{2^n}$, so

$$\begin{aligned} \lambda(B \cap [0, h]) &\leq \lambda\left(B \cap \left[0, \frac{1}{2^n}\right]\right) = \sum_{i=n}^{\infty} \frac{1}{4^i} \psi\left(\frac{1}{2^i}\right) \\ &\leq \psi\left(\frac{1}{2^n}\right) \cdot \sum_{i=n}^{\infty} \frac{1}{4^i} < \psi\left(\frac{1}{2^n}\right) \cdot \frac{1}{4^{n-1}}. \end{aligned}$$

Therefore,

$$\frac{\lambda(B \cap [0, h])}{2h\psi(2h)} < \frac{\psi\left(\frac{1}{2^n}\right) \cdot \frac{1}{4^{n-1}}}{2 \cdot \frac{1}{2^{n+1}} \cdot \psi\left(2 \cdot \frac{1}{2^{n+1}}\right)} = \frac{1}{2^{n-2}}.$$

Since for $h \rightarrow 0^+$ we have n tending to $+\infty$, so $0 \in \Phi_\psi(B')$ and $B' \in \mathcal{T}_\psi$, because each point from $B' \setminus \{0\}$ is an inner one in the Euclidean topology.

Let

$$h_0(x) = \begin{cases} 0 & \text{for } x \in B', \\ 1 & \text{for } x = \frac{b_n + a_n}{2}, n \in \mathbb{N}, \\ \text{linear and continuous} & \text{for } x \in \left[a_n, \frac{b_n + a_n}{2}\right] \text{ and } x \in \left[\frac{b_n + a_n}{2}, b_n\right]. \end{cases}$$

Then h_0 has the desired properties.

We define $h(x) = 0$ if $C = \emptyset$. Otherwise, let $\{r_n\}_{n \in \mathbb{N}}$ be an enumeration of a considered countable set C . We define

$$h_n(x) = \frac{1}{2^n} h_0(x - r_n) \quad \text{and} \quad h(x) = \sum_{n=1}^{\infty} h_n(x) \quad \text{for } x \in \mathbb{R}.$$

Then the series is uniformly convergent on $\mathbb{R}$, so by Proposition 2.17, the function h is $\mathcal{T}_\psi$ -approximately continuous. For a given n_0 , the function $\sum_{n \neq n_0} h_n$ is continuous at the point r_{n_0} . Thus,

$$h = h_{n_0} + \sum_{n \neq n_0} h_n$$

is discontinuous there. Therefore h is discontinuous on C , and continuous on C' .

Let $\mathcal{P}$ be a nowhere dense perfect set.

Let $\{I_n\}_{n \in \mathbb{N}}$ be the family of connected bounded components of the complement of $\mathcal{P}$. Let

$$\sum_{n \in \mathbb{N}} z_n < +\infty$$

such that $z_n > 0$ for $n \in \mathbb{N}$. We denote by J_n an interval centered at the midpoint of I_n and such that

$$\lambda(J_n) = \min \left(z_n \cdot \lambda(I_n) \psi \left(\frac{1}{2} \lambda(I_n) \right), \frac{1}{n} \lambda(I_n) \right), \quad n \in \mathbb{N}.$$

Let $p \in \mathcal{P}$. We want to show that

$$\lim_{h \rightarrow 0^+} \frac{\lambda(\bigcup_{n \in \mathbb{N}} J_n \cap [p, p+h])}{2h\psi(2h)} = 0.$$

We can assume that p is not a left end-point of any I_n (since in the opposite case there exists $h > 0$ such that $\bigcup_{n \in \mathbb{N}} J_n \cap [p, p+h] = \emptyset$).

We fix $n_0 \in \mathbb{N}$. Then there exists $h > 0$ such that

$$[p, p+h] \cap \bigcup_{n=1}^{n_0} J_n = \emptyset.$$

Let $\{n_k\}_{k \in \mathbb{N}}$ be a subsequence of $\{n\}_{n \in \mathbb{N}}$ consisting of all n_k 's such that

$$(p, p+h] \cap J_{n_k} \neq \emptyset.$$

Then $n_k > n_0$ for any $k \in \mathbb{N}$.

It is easily seen that $h > \frac{\lambda(I_{n_k}) - \lambda(J_{n_k})}{2}$, so

$$\lambda(I_{n_k}) < 2h + \lambda(J_{n_k}) \leq 2h + \frac{1}{n_k} \lambda(I_{n_k}),$$

thus

$$(1 - \frac{1}{n_k}) \lambda(I_{n_k}) \leq 2h, \quad k \in \mathbb{N}.$$

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Therefore,

$$\begin{aligned}
\frac{\lambda(\bigcup_{n \in \mathbb{N}} J_n \cap [p, p+h])}{2h\psi(2h)} &= \frac{\lambda(\bigcup_{k \in \mathbb{N}} J_{n_k} \cap [p, p+h])}{2h\psi(2h)} \\
&\leq \sum_{k \in \mathbb{N}} \frac{\lambda(J_{n_k})}{2h\psi(2h)} \\
&\leq \sum_{k \in \mathbb{N}} \frac{z_{n_k} \cdot \lambda(I_{n_k}) \cdot \psi(\frac{1}{2}\lambda(I_{n_k}))}{(1 - \frac{1}{n_k})\lambda(I_{n_k}) \cdot \psi((1 - \frac{1}{n_k})\lambda(I_{n_k}))} \\
&\leq \sum_{k \in \mathbb{N}} \frac{z_{n_k} \psi(\frac{1}{2}\lambda(I_{n_k}))}{\frac{1}{2}\psi(\frac{1}{2}\lambda(I_{n_k}))} \\
&= 2 \cdot \sum_{k \in \mathbb{N}} z_{n_k} \leq 2 \cdot \sum_{n=n_0}^{\infty} z_n.
\end{aligned}$$

Since for $n_0 \rightarrow +\infty$ the rest 0 series $\sum_{n=n_0}^{\infty} z_n$ tends to 0, we have

$$\lim_{h \rightarrow 0^+} \frac{\lambda(\bigcup_{n \in \mathbb{N}} J_n \cap [p, p+h])}{2h\psi(2h)} = 0.$$

The argument similar to that above shows that also

$$\lim_{h \rightarrow 0^+} \frac{\lambda(\bigcup_{n \in \mathbb{N}} J_n \cap [p-h, p])}{2h\psi(2h)} = 0.$$

Consequently,

$$p \in \Phi_{\psi} \left(\left(\bigcup_{n \in \mathbb{N}} J_n \right)' \right).$$

We define g_P to be 0 on the complement of $\bigcup_{n \in \mathbb{N}} J_n$, $g_P(J_n) = [0, 1]$ for $n \in \mathbb{N}$ and g_P continuous on I_n , $n \in \mathbb{N}$. Then g_P is $\mathcal{T}_{\psi}$ -approximately continuous everywhere and discontinuous exactly at all points of $\mathcal{P}$.

Now, for each set $\mathcal{P}_i$, $i \in \mathbb{N}$, we define a function $g_i(x) = \frac{1}{3^i} g_{P_i}(x)$ and let $g(x) = \sum_{i \in \mathbb{N}} g_i(x)$. The function g is $\mathcal{T}_{\psi}$ -approximately continuous everywhere and continuous at all points from the complement of $\bigcup_{i \in \mathbb{N}} \mathcal{P}_i$.

To end the construction, let $f = g + h$. Of course, f is $\mathcal{T}_{\psi}$ -approximately continuous and the set A is exactly the set of discontinuities of f .

$\mathcal{T}_c$ -case. Let $\tilde{\psi}: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be given by

$$\tilde{\psi}(x) = \begin{cases} \frac{1}{n^2} & \text{for } x \in [\frac{2}{n}, \frac{2}{n-1}), \quad n > 1, \\ 1 & \text{for } x \in [1, +\infty). \end{cases}$$

An easy computation shows that if for an arbitrarily chosen set A , $A \subset \mathbb{R}$, if

$$\lim_{n \rightarrow 0^+} \frac{\lambda(A' \cap [-h, h])}{2h\tilde{\psi}(2h)} = 0,$$

then

$$\lim_{n \rightarrow \infty} \frac{\lambda(A' \cap [\frac{1}{n}, \frac{1}{n}])}{\frac{2}{n} \tilde{\psi}(\frac{2}{n})} = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \lambda(n \cdot A' \cap [-1, 1]) < +\infty,$$

so

$$0 \in \phi_c(A).$$

For any non-decreasing function $\tilde{\psi}$ transforming $\mathbb{R}^+$ into $\mathbb{R}^+$ and such that $\lim_{x \rightarrow 0^+} \tilde{\psi}(x) = 0$, we can define a $\tilde{\psi}$ -density topology as for a function from $\mathcal{C}$ and for each function with such properties, we can find a function from the family $\mathcal{C}$ such that the topologies regulated by these two functions coincide [AW].

Therefore, there exists a function $\psi \in \mathcal{C}$ such that $\mathcal{T}_\psi \subset \mathcal{T}_c$, hence the function from $\mathcal{T}_\psi$ -case works for $\mathcal{T}_c$ case.

$\mathcal{T}_s$ -case. According to Proposition 2.6, the function from $\mathcal{T}_c$ -case is also $\mathcal{T}_s$ -approximately continuous on $\mathbb{R}$.

$\mathcal{T}_{(s)}$ -case. Obviously, it follows from the theorem for approximately continuous functions [B], since the approximately continuous function is also $\mathcal{T}_{(s)}$ -approximately continuous on $\mathbb{R}$.

$\mathcal{T}_\psi^0$ -case. Let $\psi \in \mathcal{C}$. Firstly we build a $\mathcal{T}_\psi^0$ -approximately continuous function which is discontinuous at the point $(0, 0)$.

Since $\psi \in \mathcal{C}$, then there exists $n_0 \in \mathcal{N}$ such that $\psi(\frac{1}{2^{n_0}}) \leq 1$. Put

$$B = \bigcup_{n=n_0}^{\infty} \left(\left[\frac{1}{2^n} - \frac{1}{4^n} \psi\left(\frac{1}{4^n}\right), \frac{1}{2^n} \right] \times \left[-\frac{1}{2 \cdot 4^n} \psi\left(\frac{1}{4^n}\right), \frac{1}{2 \cdot 4^n} \psi\left(\frac{1}{4^n}\right) \right] \right).$$

Let $h \in \mathbb{R}$, $h < \frac{1}{2^n}$. Then there exists $n \in \mathbb{N}$, $n \geq n_0$ such that $\frac{1}{2^{n+1}} \leq h < \frac{1}{2^n}$. Monotonicity of ψ yields

$$\begin{aligned} \lambda_2(B \cap Q((0, 0), h)) &\leq \lambda_2\left(B \cap Q\left((0, 0), \frac{1}{2^n}\right)\right) = \left(\sum_{i=n}^{\infty} \left(\frac{1}{4^i} \psi\left(\frac{1}{4^i}\right)\right)\right)^2 \\ &\leq \left(\psi\left(\frac{1}{4^n}\right)\right)^2 \cdot \left(\sum_{i=n}^{\infty} \frac{1}{4^i}\right)^2 < \left(\frac{1}{4^{n-1}}\right)^2 \cdot \psi\left(\frac{1}{4^n}\right), \end{aligned}$$

hence

$$\frac{\lambda_2(B \cap Q((0, 0), h))}{4h^2\psi(4h^2)} < \frac{\left(\frac{1}{4^{n-1}}\right)^2 \cdot \psi\left(\frac{1}{4^n}\right)}{4 \cdot \left(\frac{1}{2^{n+1}}\right)^2 \cdot \psi\left(4 \cdot \left(\frac{1}{2^{n+1}}\right)^2\right)} = \frac{4}{2^n} \cdot \frac{\psi\left(\frac{1}{4^n}\right)}{\psi\left(\frac{1}{4^n}\right)} = \frac{4}{2^n}.$$

Since for $h \rightarrow 0^+$, we have n tending to $+\infty$, so $(0, 0) \in \Phi_\psi^0(B')$ and $B' \in \mathcal{T}_\psi^0$.

Let

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$$h_0(x, y) = \begin{cases} 0 & \text{for } (x, y) \in B', \\ 1 & \text{for } (x, y) = \left(\frac{1}{2^n} - \frac{1}{2 \cdot 4^n} \psi\left(\frac{1}{4^n}\right), 0\right), n \in \mathbb{N}, \\ \text{continuous on } \left[\frac{1}{2^n} - \frac{1}{4^n} \psi\left(\frac{1}{4^n}\right), \frac{1}{2^n}\right] \times \left[-\frac{1}{2 \cdot 4^n} \psi\left(\frac{1}{4^n}\right), \frac{1}{2 \cdot 4^n} \psi\left(\frac{1}{4^n}\right)\right] \\ \text{and such that the image of this interval is equal to } [0, 1], \text{ for } n \in \mathbb{N}. \end{cases}$$

Then h_0 has the desired properties.

Let $\mathcal{P}$ be a nowhere dense perfect set $\mathcal{P} \subset [0, 1] \times [0, 1]$. Let $\{A_l\}_{l \in \mathbb{N}}$ be the family of connected components of $([0, 1] \times [0, 1]) \setminus \mathcal{P}$.

Consider the squares of the following form:

$$S_{(k, m, p)} = \left[\frac{k}{2^p}, \frac{k+1}{2^p}\right] \times \left[\frac{m}{2^p}, \frac{m+1}{2^p}\right], \quad p \in \mathbb{N}, \quad k, m \in \{0, \dots, 2^p - 1\}.$$

Fix $p = 1$. From the squares $S_{(k, m, 1)}$, $k, m \in \{0, 1\}$ we choose those ones which are contained in $\bigcup_{l \in \mathbb{N}} A_l$ and $\text{dist}(S_{(k, m, 1)}, \mathcal{P}) > \frac{1}{4}$.

For $p = 2$, we choose from the squares $S_{(k, m, 2)}$, $k, m \in \{0, \dots, 3\}$ those ones which are contained in $\bigcup_{l \in \mathbb{N}} A_l \setminus \bigcup_{n=1}^{n_1} I_n$ and $\text{dist}(S_{(k, m, 2)}, \mathcal{P}) > \frac{1}{8}$ and we arrange them in a sequence $\{I_n\}_{n=n_1+1}^{n_2}$.

For $p \in \mathbb{N}$ we choose from the squares $S_{(k, m, p)}$, $k, m \in \{0, \dots, 2^p - 1\}$ those which are contained in $\bigcup_{l \in \mathbb{N}} A_l \setminus \bigcup_{n=1}^{n_{p-1}} I_n$ and $\text{dist}(S_{(k, m, p)}, \mathcal{P}) > \frac{1}{2^{p+1}}$ and we arrange them in a sequence $\{I_n\}_{n=n_{p-1}+1}^{n_p}$, and so on. It is easily seen that

$$\bigcup_{l \in \mathbb{N}} A_l = \bigcup_{n \in \mathbb{N}} I_n.$$

Now, in each square $I_n = S_{(k, m, p)}$, $n \in \mathbb{N}$, we put a closed square J_n centered at the midpoint of I_n with sides parallel to the axis and such that

$$\lambda_2(J_1) = \lambda_2(I_1)$$

and

$$\lambda_2(J_n) = \min\left(\frac{1}{n^2} \lambda_2(I_n), \frac{1}{2} \lambda_2(J_{n-1}), \frac{1}{n} \lambda_2(I_n) \psi\left(\frac{\lambda_2(I_n)}{4}\right)\right) \quad \text{for } n > 1.$$

Then

$$\lambda_2\left(\bigcup_{n=n_0}^{\infty} J_n\right) \leq 2\lambda_2(J_{n_0}) \quad \text{for } n_0 > 1.$$

Let $z \in \mathcal{P}$, $r > 0$, and let $n(r) = \min\{n \in \mathbb{N} : Q(z, r) \cap J_n \neq \emptyset\}$. Then there exists exactly one number $p \in \mathbb{N}$ such that $n(r) \in \{n_{p-1} + 1, \dots, n_p\}$.

We denote this number by $p(r)$. Thus, the square $Q(z, r)$ has common points with at least one of the squares $J_{n_{p(r)-1}+1}, \dots, J_{n_{p(r)}}$. Therefore

$$r > \frac{1}{2} \sqrt{\lambda_2(I_{n(r)})} - \frac{1}{2} \frac{1}{n(r)} \cdot \sqrt{\lambda_2(I_{n(r)})} = \frac{1}{2} \sqrt{\lambda_2(I_{n(r)})} \left(1 - \frac{1}{n(r)}\right).$$

Hence for $r \rightarrow 0$, $n(r)$ tends to $+\infty$ and $p(r)$ tends to $+\infty$.

The task is to show that z is a $\mathcal{T}_\psi^0$ density point of the complement of $\bigcup_{n \in \mathbb{N}} J_n$. It follows that for $r > 0$:

$$\lambda_2 \left(\bigcup_{n \in \mathbb{N}} J_n \cap Q(z, r) \right) \leq \lambda_2 \left(\bigcup_{n=n(r)}^{\infty} J_n \cap Q(z, r) \right) \leq \lambda_2 \left(\bigcup_{n=n(r)}^{\infty} J_n \right),$$

so,

$$\begin{aligned} \frac{\lambda_2(\bigcup_{n \in \mathbb{N}} J_n \cap Q(z, r))}{4r^2 \psi(4r^2)} &\leq \frac{2 \cdot \lambda_2(J_{n(r)})}{\lambda_2(I_{n(r)}) \left(1 - \frac{1}{n(r)}\right)^2 \psi \left(\lambda_2(I_{n(r)}) \left(1 - \frac{1}{n(r)}\right)^2 \right)} \\ &\leq \frac{2 \frac{1}{n(r)} \lambda_2(I_{n(r)}) \psi \left(\frac{1}{4} \lambda_2(I_{n(r)}) \right)}{\lambda_2(I_{n(r)}) \left(1 - \frac{1}{2}\right)^2 \psi \left(\left(1 - \frac{1}{2}\right)^2 \lambda_2(I_{n(r)}) \right)} \\ &= \frac{8}{n(r)}. \end{aligned}$$

Hence

$$\lim_{r \rightarrow 0} \frac{\lambda_2(\bigcup_{n \in \mathbb{N}} J_n \cap Q(z, r))}{4r^2 \psi(4r^2)} = 0.$$

Next, we build a function which is $\mathcal{T}_\psi^0$ -approximately continuous and which is discontinuous on $\mathcal{P}$. We define $g_{\mathcal{P}}$ to be 0 on the complement of $\bigcup_{n \in \mathbb{N}} J_n$, $g_{\mathcal{P}}(J_n) = [0, 1]$, $g_{\mathcal{P}}$ continuous on I_n , $n \in \mathbb{N}$.

We have shown that $\mathcal{P} \subset \Phi_\psi^0((\bigcup_{n \in \mathbb{N}} J_n)')$. Of course, $(\bigcup_{n \in \mathbb{N}} J_n)' \setminus \mathcal{P}$ is contained in the interior in the Euclidean topology of the set $(\bigcup_{n \in \mathbb{N}} J_n)'$, so $(\bigcup_{n \in \mathbb{N}} J_n)' \in \mathcal{T}_\psi^0$. Since arbitrarily close to the each point of $\mathcal{P}$, we can find the square J_n , $n \in \mathbb{N}$, the function $g_{\mathcal{P}}$ has the desired properties.

Let A be an F_σ first category set. As in $\mathcal{T}_\psi$ -case, we find a countable set C and a sequence of nowhere dense perfect sets $\mathcal{P}_i$ such that $A = \bigcup_{i \in \mathbb{N}} \mathcal{P}_i \cup C$ and a construction of the function h which is $\mathcal{T}_\psi^0$ -approximately continuous and discontinuous only on C can be also done almost in the same manner.

Let $g_i = \frac{1}{3^i} g_{\mathcal{P}_i}$, $i \in \mathbb{N}$, and $g = \sum_{i \in \mathbb{N}} g_i$. (If $\mathcal{P}_i$ is an unbounded perfect set for some $i \in \mathbb{N}$, we consider sets $\mathcal{P}_{i,n,m} = \mathcal{P}_i \cap ([n, n+1] \times [m, m+1])$ for $n, m \in \mathbb{Z}$.) The function g is $\mathcal{T}_\psi^0$ -approximately continuous function (Proposition 2.17) and it is discontinuous at any point from $\bigcup_{i \in \mathbb{N}} \mathcal{P}_i$. Indeed, if $x \in \bigcup_{i \in \mathbb{N}} \mathcal{P}_i$, then denoting by i_0 the smallest index such that $x \in \mathcal{P}_{i_0}$, we get $\text{osc}(g_{i_0}, x) = \frac{1}{3^{i_0}}$, $\text{osc}(\sum_{i < i_0} g_i, x) = 0$ and $\text{osc}(\sum_{i > i_0} g_i, x) \leq \sum_{i > i_0} \frac{1}{3^i} = \frac{1}{2 \cdot 3^{i_0}}$, therefore $\text{osc}(g, x) > 0$. Of course, when $x \notin \bigcup_{i \in \mathbb{N}} \mathcal{P}_i$, then g is continuous at x .

The function $f = g + h$, where h is defined for the set C as in $\mathcal{T}_\psi$ -case, is the desired one.

$\mathcal{T} \times \mathcal{T}$ -case. We can repeat the construction of the function from the previous case if we can build a $\mathcal{T} \times \mathcal{T}$ -approximately continuous function which is discontinuous at the point $(0, 0)$ and for a nowhere dense perfect set $\mathcal{P}$ we can define

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a family of squares $\{J_n\}_{n \in \mathbb{N}}$ such that if $\{I_n\}_{n \in \mathbb{N}}$ is the sequences of squares defined in $\mathcal{T}_\psi^0$ case, then $J_n \subset I_n$, J_n is centered at the centre of I_n , for $n \in \mathbb{N}$ and each point of $\mathcal{P}$ is an inner point of the complement of $\bigcup_{n \in \mathbb{N}} J_n$ in $\mathcal{T} \times \mathcal{T}$.

We begin by taking an interval set $A = \bigcup_{n \in \mathbb{N}} (a_n, b_n)$, such that

$$a_{n+1} \leq b_{n+1} \leq a_n, \quad n \in \mathbb{N}, \quad a_n \searrow 0 \quad \text{and} \quad 0 \in \Phi(A').$$

Then let

$$B = \bigcup_{n \in \mathbb{N}} (a_n, b_n) \times \left(-\frac{b_n - a_n}{2}, \frac{b_n - a_n}{2} \right).$$

Hence $(0, 0) \in \text{Int}(\mathbb{R}^2 \setminus B)$ since $(\mathbb{R} \setminus A) \times \mathbb{R} \in \mathcal{T} \times \mathcal{T}$ and $(\mathbb{R} \setminus A) \times \mathbb{R} \subset \mathbb{R}^2 \setminus B$.

It follows that the function

$$h_0(x, y) = \begin{cases} 0 & \text{for } (x, y) \in B', \\ 1 & \text{for } (x, y) = \left(\frac{a_n + b_n}{2}, 0 \right), n \in \mathbb{N}, \\ \text{continuous on } [a_n, b_n] \times \left[-\frac{b_n - a_n}{2}, \frac{b_n - a_n}{2} \right] & \text{and such that} \\ \text{the image of this interval is equal to } [0, 1], & \text{for } n \in \mathbb{N} \end{cases}$$

has the desired properties.

Now, let P be a nowhere dense perfect set $P \subset [0, 1] \times [0, 1]$ and $\{I_n\}_{n \in \mathbb{N}}$ be the sequence of squares defined in $\mathcal{T}_\psi^0$ -case. Then, in each square $I_n = S_{(k, m, p)}$, $n \in \mathbb{N}$ we put a square J_n centered at the midpoint of I_n with sides parallel to the axis and such that

$$\lambda_2(J_1) = \lambda_2(I_1) \quad \text{and} \quad \lambda_2(J_n) = \min\left(\frac{1}{n^2} \cdot \lambda_2(I_n), \frac{1}{4} \lambda_2(J_{n-1})\right) \quad \text{for } n > 1.$$

Additionally, we denote

$$I_n = (a_n, b_n) \times (c_n, d_n), \quad n \in \mathbb{N}.$$

Consider $z = (x_0, y_0) \in \mathcal{P}$. Let $A_1 = \bigcup_{n \in N_1} (a_n, b_n)$, where

$$N_1 = \left\{ n \in \mathbb{N} : I_n \subset \{(x, y) : (y - x - y_0 + x_0)(y + x - y_0 - x_0) < 0\} \right\}$$

and $A_2 = \bigcup_{n \in N_2} (c_n, d_n)$, where $N_2 = \mathbb{N} \setminus N_1$. Then $\bigcup_{n \in \mathbb{N}} J_n \subset (A_1 \times \mathbb{R}) \cup (\mathbb{R} \times A_2)$. To show that $z \in \text{Int}_{\mathcal{T} \times \mathcal{T}}(\mathbb{R}^2 \setminus \bigcup_{n \in \mathbb{N}} J_n)$, it suffices to prove that x_0 is a density point of $\mathbb{R} \setminus A_1$ and y_0 is a density point of $\mathbb{R} \setminus A_2$.

Let $r > 0$ and let $n(r) = \min\{n \in N_1 : (x_0, x_0 + r) \cap (a_n, b_n) \neq \emptyset\}$. Then there exists exactly one number $p \in \mathbb{N}$ such that $n(r) \in \{n_{p-1} + 1, \dots, n_p\}$. We denote this number by $p(r)$. Thus, the interval $(x_0, x_0 + r)$ has common points with at least one of the intervals $(a_{n_{p(r)-1}+1}, b_{n_{p(r)-1}+1}), \dots, (a_{n_{p(r)}}, b_{n_{p(r)}})$. Therefore

$$r > \frac{1}{2} \sqrt{\lambda_2(I_{n(r)})} - \frac{1}{2} \cdot \frac{1}{n(r)} \sqrt{\lambda_2(I_{n(r)})} = \frac{1}{2} \cdot \frac{1}{2^{p(r)}} \left(1 - \frac{1}{n(r)} \right),$$

hence for $r \rightarrow 0$, we have $n(r)$ tending to $+\infty$ and $p(r)$ tending to $+\infty$.

We observe that

$$\begin{aligned}\lambda((x_0, x_0 + r) \cap A_1) &= \lambda\left((x_0, x_0 + r) \cap \bigcup_{n \in N_1 \cap [n(r), +\infty)} (a_n, b_n)\right) \\ &\leq \sum_{n \in N_1 \cap [n(r), +\infty)} \lambda(a_n, b_n) \leq 2 \cdot (b_{n(r)} - a_{n(r)}) \\ &\leq 2 \cdot \frac{1}{n(r)} \cdot \frac{1}{2^{p(r)}}.\end{aligned}$$

Consequently,

$$\frac{\lambda((x_0, x_0 + r) \cap A_1)}{r} \leq \frac{2 \cdot \frac{1}{n(r)} \cdot \frac{1}{2^{p(r)}}}{\frac{1}{2 \cdot 2^{p(r)}}} \left(1 - \frac{1}{n(r)}\right) = \frac{4}{n(r) - 1},$$

hence

$$\lim_{r \rightarrow 0^+} \frac{\lambda((x_0, x_0 + r) \cap A_1)}{r} = 0 \quad \text{and similarly,} \quad \lim_{r \rightarrow 0^+} \frac{\lambda((x_0 - r, x_0) \cap A_1)}{r} = 0,$$

which yields that x_0 is a density point of $\mathbb{R} \setminus A_1$.

Similar arguments apply to the set A_2 and the point y_0 .

The construction of the function with desired properties runs as in $\mathcal{T}_\psi^0$ case.

$\mathcal{T}_2$ -case. We get the result immediately by the previous case and Proposition 2.11. $\square$

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