ON ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF DISCRETE SYSTEMS

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ABSTRACT. This contribution concerns the asymptotic behavior of solutions of systems of difference equations. It reassumes paper [J. Diblík: *Anti-Lyapunov method for systems of discrete equations*, Nonlinear Anal. **57** (2004), 1043–1057], where conditions are described which ensure that the graph of at least one solution of the studied system stays in a prescribed domain. Those conditions include the existence of a so called connecting function for the sets forming the given domain. The connecting function must have certain properties which enable the use of the retract method for the proof of the main result of the cited work. We show that the conditions can be made simpler, without the need to introduce the connecting function.

1. Introduction

We investigate the asymptotic behavior for $k \rightarrow \infty$ of the solutions of the system

$$\Delta u(k) = F(k, u(k)), \quad (1)$$

where $k \in \mathbb{Z}_a^\infty := \{a, a+1, \dots\}$, $a \in \mathbb{N}$ is fixed, $u(k) = (u_1(k), \dots, u_n(k))$, $\Delta u_i(k) = u_i(k+1) - u_i(k)$, $i = 1, \dots, n$, and $F: \mathbb{Z}_a^\infty \times \mathbb{R}^n \rightarrow \mathbb{R}^n$.

The *solution* of the system (1) is defined as an infinite sequence of number vectors

$$\{u(a), u(a+1), u(a+2), \dots\}$$

such that for any $k \in \mathbb{Z}_a^\infty$, the equality (1) holds.

The *existence and uniqueness* of the solution of the system (1) with a prescribed initial condition

$$u(a) = u^a \in \mathbb{R}^n \quad (2)$$

on $\mathbb{Z}_a^\infty$ is obvious.

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The sequence $\{(k, u(k))\}$, $k \in \mathbb{Z}_a^\infty$, is called the *graph* of the solution $u = u(k)$ for $k \in \mathbb{Z}_a^\infty$ of initial problem (1), (2).

Our aim is to find sufficient conditions with respect to the right-hand side of the system (1) which guarantee *the existence of at least one solution, the graph of which stays in a prescribed domain Ω* . We will base upon the result of [2] but we will show that the conditions can be made simpler than those described there.

2. Description of domain Ω and auxiliary notions

Let us define the set $\Omega \subset \mathbb{Z}_a^\infty \times \mathbb{R}^n$ as

$$\Omega := \bigcup_{k \in \mathbb{Z}_a^\infty} \Omega(k),$$

where for every $k \in \mathbb{Z}_a^\infty$, $\Omega(k)$ is an *open bounded and connected set* in the space $S(k) := \{(k, u) : u \in \mathbb{R}^n\}$. The boundaries, closures and interiors of the sets $\Omega(k)$ are always taken in the corresponding set $S(k)$. The boundary of the set Ω is defined as

$$\partial\Omega := \bigcup_{k \in \mathbb{Z}_a^\infty} \partial\Omega(k).$$

In the paper [2], conditions are found that guarantee the existence of at least one solution $u = u^*(k)$, $k \in \mathbb{Z}_a^\infty$ satisfying

$$(k, u^*(k)) \in \Omega(k),$$

for every $k \in \mathbb{Z}_a^\infty$.

The problem is solved via retract type technique. We will present the main result of [2] in the next section. To be able to do that, define some basic notions that are used there.

Retract and retraction

If $A \subset B$ are any two sets of a topological space and $\pi : B \rightarrow A$ is a continuous mapping of B onto A such that $\pi(p) = p$ for every $p \in A$, then π is said to be a *retraction* of B onto A . If there exists a retraction of B onto A , A is called a *retract* of B .

Connecting function and connecting sets

To construct a “connection” of the sets $\Omega(k)$, $k \in \mathbb{Z}_a^\infty$, we introduce the notion of a connecting function V and the connecting sets.

A continuous function $V: [a, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}$ is called *connecting function* for the sets $\Omega(k)$, $k \in \mathbb{Z}_a^\infty$, if the following conditions hold.

- (1) For every fixed $t \in [a, \infty)$, the set $W(t) := \{(t, u): V(t, u) < 0\}$ is a nonempty open bounded and simply connected set in the space $S(t) := \{(t, u): u \in \mathbb{R}^n\}$.
- (2) The boundary of $W(t)$ in $S(t)$ is equal to the set $\{(t, u): V(t, u) = 0\}$.
- (3) For every $k \in \mathbb{Z}_a^\infty$, $\Omega(k) = W(k)$.

Define the *connecting sets* $\mathcal{V}_{a,\infty}$, $\mathcal{V}_{k,k+1}$, $k \in \mathbb{Z}_a^\infty$, as

$$\begin{aligned}\mathcal{V}_{a,\infty} &:= \{(t, u): a \leq t < \infty, V(t, u) \leq 0\}, \\ \mathcal{V}_{k,k+1} &:= \{(t, u): k \leq t \leq k+1, V(t, u) \leq 0\}\end{aligned}$$

and their u -boundaries as

$$\begin{aligned}\partial_u \mathcal{V}_{a,\infty} &:= \{(t, u): a \leq t < \infty, V(t, u) = 0\}, \\ \partial_u \mathcal{V}_{k,k+1} &:= \{(t, u): k \leq t \leq k+1, V(t, u) = 0\}.\end{aligned}$$

Consequent points

Define the mapping $\mathcal{C}: \mathbb{Z}_a^\infty \times \mathbb{R}^n \rightarrow \mathbb{Z}_a^\infty \times \mathbb{R}^n$ as

$$\mathcal{C}: (k, u) \mapsto (k+1, u + F(k, u)).$$

For any point $M = (k, u) \in \mathbb{Z}_a^\infty \times \mathbb{R}^n$, the point $\mathcal{C}(M)$ is called its *first consequent point*.

For any $n \in \mathbb{N}$ define the n th *consequent point* of the point $M \in \mathbb{Z}_a^\infty \times \mathbb{R}^n$ as

$$\mathcal{C}^n(M) = \mathcal{C}(\mathcal{C}^{n-1}(M)) \quad \text{for } n \geq 1$$

and

$$\mathcal{C}^0(M) = M.$$

Point of egress type

A point $M \in \partial\Omega(k)$, $k \in \mathbb{Z}_a^\infty$, is called a *point of egress type* with respect to the system (1), the set Ω and the connecting function V if $\mathcal{C}(M) \notin \overline{\Omega(k+1)}$ and the line segment connecting the points M and $\mathcal{C}(M)$ has only one point of intersection with the set $\mathcal{V}_{k,k+1}$, namely, the point M itself.

3. Original version of the existence theorem

THEOREM 3.1. *Suppose that $F: \mathbb{Z}_a^\infty \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a continuous mapping and that sets $\Omega(k)$, $k \in \mathbb{Z}_a^\infty$, and a corresponding connecting function $V(t, u)$ are given such that the following properties hold:*

- (1) *(Convexity property) Set $\mathcal{V}_{k,k+1}$ is convex for every $k \in \mathbb{Z}_a^\infty$.*
- (2) *(Strict egress-type property) Every point $M \in \partial\Omega$ is a point of egress type with respect to the system (1), the set Ω and function V .*
- (3) *(Retract property) There exists a retraction π of the set $\partial_u \mathcal{V}_{a,\infty}$ onto the set $\partial\Omega(a)$.*

Then there exists a solution $u = u^(k)$, $k \in \mathbb{Z}_a^\infty$, of (1) satisfying the relation*

$$(k, u^*(k)) \in \Omega(k)$$

for every $k \in \mathbb{Z}_a^\infty$.

When this theorem was presented as one of the main resources of the dissertation thesis of the author of the current paper, there came an objection from one of the members of the committee that the convexity of the connecting sets is not a property of the “input data”. This paper represents an attempt to improve the theorem and to eliminate any assumptions that are not directly connected with the system (1) and the sets $\Omega(k)$.

4. New version of the existence theorem

We will show that the requirement for the existence of the connecting function V and the corresponding connecting sets can be replaced by the assumption that every set $\Omega(k)$, $k \in \mathbb{Z}_a^\infty$, is an *open bounded convex set*. The strict egress-type property has to be reformulated with respect to this new situation.

Point of egress type—new definition

Recall that for a set A , the *convex hull* of A is the least convex set B such that $A \subseteq B$.

A point $M \in \partial\Omega(k)$, $k \in \mathbb{Z}_a^\infty$, is called a *point of egress type* with respect to the system (1) and the set Ω if $\mathcal{C}(M) \notin \overline{\Omega(k+1)}$ and the line segment connecting the points M and $\mathcal{C}(M)$ has only one point of intersection with the convex hull of the set $\overline{\Omega(k)} \cup \overline{\Omega(k+1)}$, namely, the point M itself.

THEOREM 4.1. *Suppose that $F: \mathbb{Z}_a^\infty \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a continuous mapping and that sets $\Omega(k)$, $k \in \mathbb{Z}_a^\infty$, are given such that the following properties hold:*

- (1) *(Convexity property) Set $\Omega(k)$ is convex for every $k \in \mathbb{Z}_a^\infty$.*

- (2) (*Strict egress-type property*) Every point $M \in \partial\Omega$ is a point of egress type with respect to the system (1) and the set Ω .

Then there exists a solution $u = u^*(k)$, $k \in \mathbb{Z}_a^\infty$, of (1) satisfying the relation

$$(k, u^*(k)) \in \Omega(k) \quad (3)$$

for every $k \in \mathbb{Z}_a^\infty$.

Proof. To make the arrangement of the proof clearer, we will divide it into several parts.

Contradiction scheme

Suppose that the initial data $(a, u^*(a)) \in \Omega(a)$, generating a solution $u = u^*(k)$ of the system (1) with property (3) do not exist. In this situation we prove that there exists a retraction of the set $\overline{\Omega(a)}$ (which is topologically equivalent to a closed n -dimensional ball) onto the set $\partial\Omega(a)$. This results in a contradiction, since a well-known fact states that the boundary of an n -dimensional ball cannot be its retract.

The desired retraction will be constructed with the aid of two auxiliary mappings, P and π .

Specification of the leaving value k^*

Let a point $M = (a, u^a) \in \overline{\Omega(a)}$ be arbitrarily chosen. The initial condition $u(a) = u^a$ defines the corresponding solution $u = u(k) = u(k, u^a)$ of the system (1).

If $M \in \Omega(a)$, then from the above supposition we conclude that there exists an integer $s \in \mathbb{N}$ such that

$$\mathcal{C}^i(M) \in \Omega(a + i), \quad i = 0, \dots, s - 1$$

and

$$\mathcal{C}^s(M) \notin \Omega(a + s).$$

In other words, the s th consequent point of the point M does not belong to $\Omega(a + s)$, whereas all the preceding consequent points belong to the corresponding sets $\Omega(a + i)$. Moreover, if $\mathcal{C}^s(M) \in \partial\Omega(a + s)$, then we have, due to the strict egress type property, $\mathcal{C}^{s+1}(M) \notin \overline{\Omega(a + s + 1)}$.

If $M \in \partial\Omega(a)$, then, again due to the strict egress property,

$$\mathcal{C}(M) \notin \overline{\Omega(a + 1)}.$$

Define now an integer characterizing the last moment for which the solution, generated by the initial data $M = (a, u^a) \in \overline{\Omega(a)}$, stays in the set $\overline{\Omega}$. We will

denote this value k^* and will call it the *leaving index*. The value of the leaving index is defined as

$$k^* = \begin{cases} s-1 & \text{if } M \in \Omega(a) \text{ and } \mathcal{C}^s(M) \notin \overline{\Omega(a+s)}, \\ s & \text{if } M \in \Omega(a) \text{ and } \mathcal{C}^s(M) \in \partial\Omega(a+s), \\ 0 & \text{if } M \in \partial\Omega(a). \end{cases}$$

As the value of k^* depends on the chosen initial point M , we could write $k^* = k^*(M)$ but we will mostly omit the argument M , unless it is necessary.

Altogether, we have

$$\mathcal{C}^{k^*}(M) \in \overline{\Omega(a+k^*)} \quad \text{and} \quad \mathcal{C}^{k^*+1}(M) \notin \overline{\Omega(a+k^*+1)}.$$

Auxiliary mapping P

Denote $\mathcal{H}_{k,k+1}$ the convex hull of the set $\overline{\Omega(k)} \cup \overline{\Omega(k+1)}$ in the space $\mathbb{R} \times \mathbb{R}^n$. The boundary of the set $\mathcal{H}_{k,k+1}$ consists of three parts: the sets $\Omega(k)$ and $\Omega(k+1)$ and the remaining part which we will denote $\partial_u \mathcal{H}_{k,k+1}$ and call it its u -boundary. Remark that $\partial\Omega(k) \subset \partial_u \mathcal{H}_{k,k+1}$, too. Define $P(M)$ as the point of intersection of the line segment with the end points $\mathcal{C}^{k^*}(M), \mathcal{C}^{k^*+1}(M)$ with $\partial_u \mathcal{H}_{a+k^*, a+k^*+1}$ (see Figure 1). The mapping P is well defined on $\overline{\Omega(a)}$. If $\mathcal{C}^{k^*}(M) \in \partial\Omega(a+k^*)$, then by the strict egress type property the only intersection is the point $\mathcal{C}^{k^*}(M)$ itself. If $\mathcal{C}^{k^*}(M) \in \Omega(a+k^*)$, then it lies on the boundary of the convex set $\mathcal{H}_{a+k^*, a+k^*+1}$ (not on the u -boundary but on $\Omega(a+k^*)$). In general, for a line segment $\overline{AB}$, where A lies on the boundary of a convex set and B is outside this set, for the intersection of $\overline{AB}$ with the boundary of the set there are three possibilities: it could be one point (the point A itself), two points, A and one more point, or a line segment beginning at A . From the construction of the sets $\mathcal{H}_{k,k+1}$ and from the position of the points $\mathcal{C}^{k^*}(M)$ and $\mathcal{C}^{k^*+1}(M)$ it is clear that here only the second case comes into question and that there is one intersection of the considered line segment with the $\partial_u \mathcal{H}_{a+k^*, a+k^*+1}$.

Now we prove that the mapping P is continuous. Let $\{M_j\}_{j=1}^\infty$ be any sequence with $M_j \in \overline{\Omega(a)}$ such that $M_j \rightarrow M$ for $j \rightarrow \infty$. We will show that $P(M_j) \rightarrow P(M)$. Because of the continuity of the mapping F from the system (1)

$$\mathcal{C}^i(M_j) \rightarrow \mathcal{C}^i(M) \quad \text{for any } i \in \mathbb{N}. \quad (4)$$

We have to consider two cases:

- I) $\mathcal{C}^{k^*}(M) \in \Omega(a+k^*)$,
 - II) $\mathcal{C}^{k^*}(M) \in \partial\Omega(a+k^*)$.
- I) In this case also $\mathcal{C}^{k^*}(M_j) \in \Omega(a+k^*)$ and $\mathcal{C}^{k^*+1}(M_j) \notin \overline{\Omega(a+k^*+1)}$ for j sufficiently large. That means that the leaving index $k^*(M_j)$ is the same as k^* given by M and thus we deal with the same set $\mathcal{H}_{a+k^*, a+k^*+1}$.

 FIGURE 1. Mapping P .

As by (4) $\mathcal{C}^{k^*}(M_j) \rightarrow \mathcal{C}^{k^*}(M)$ and $\mathcal{C}^{k^*+1}(M_j) \rightarrow \mathcal{C}^{k^*+1}(M)$ for $j \rightarrow \infty$, the intersection of the line segment with the end points $\mathcal{C}^{k^*}(M_j)$, $\mathcal{C}^{k^*+1}(M_j)$ with $\partial_u \mathcal{H}_{a+k^*, a+k^*+1}$ tends to $P(M)$.

- II) In this case there can be $\mathcal{C}^{k^*}(M_j) \in \Omega(a+k^*)$ for some members of the sequence $\{M_j\}$ and $\mathcal{C}^{k^*}(M_j) \notin \Omega(a+k^*)$ for the others. In general there could be three subsequences $\{M_{k_j}\}$, $\{M_{l_j}\}$ and $\{M_{m_j}\}$ such that

$$\begin{aligned} \mathcal{C}^{k^*}(M_{k_j}) &\in \Omega(a+k^*), \\ \mathcal{C}^{k^*}(M_{l_j}) &\in \partial\Omega(a+k^*), \\ \mathcal{C}^{k^*}(M_{m_j}) &\notin \overline{\Omega(a+k^*)}. \end{aligned}$$

If $k^* > 0$, all three subsequences come into account, if $k^* = 0$, the subsequence $\{M_{m_j}\}$ can be left out from our considerations.

For the subsequence $\{M_{k_j}\}$ the situation is very similar to that in the case I). Again, the leaving index $k^*(M_{k_j})$ is the same as $k^*(M)$, and we deal with the same set $\mathcal{H}_{a+k^*, a+k^*+1}$. Thus, $P(M_{k_j}) \rightarrow P(M)$ for $j \rightarrow \infty$.

For the subsequence $\{M_{l_j}\}$ there is no problem, too. If $\mathcal{C}^{k^*}(M_{l_j})$ belongs to $\partial_u \mathcal{H}_{a+k^*, a+k^*+1}$, then $P(M_{l_j}) = \mathcal{C}^{k^*}(M_{l_j})$. For the point M itself, $P(M) = \mathcal{C}^{k^*}(M)$. Hence, by (4), $P(M_{l_j}) \rightarrow P(M)$.

As for the subsequence $\{M_{m_j}\}$, the leaving index $k^*(M_{m_j})$ is different from k^* given by M because $\mathcal{C}^{k^*}(M_{m_j})$ is already out of $\overline{\Omega(a+k^*)}$. For j sufficiently large,

$$k^*(M_{m_j}) = k^* - 1$$

because $\mathcal{C}^{k^*-1}(M) \in \Omega(a+k^*-1)$ and thus also $\mathcal{C}^{k^*-1}(M_{m_j}) \in \Omega(a+k^*-1)$. Now, $P(M)$ can be also seen as the point of intersection of the line segment given by the end points $\mathcal{C}^{k^*-1}(M)$, $\mathcal{C}^{k^*}(M)$ with $\partial_u \mathcal{H}_{a+k^*-1, a+k^*}$. The desired

convergence $P(M_{m_j}) \rightarrow P(M)$ can be proved with help of similar considerations as in the case of the subsequence $\{M_{k_j}\}$.

We have shown that $P(M_{k_j}) \rightarrow P(M)$, $P(M_{l_j}) \rightarrow P(M)$ and $P(M_{m_j}) \rightarrow P(M)$ and thus $P(M_j) \rightarrow P(M)$.

Finally, notice that for $M \in \partial\Omega(a)$ is $P(M) = M$. Indeed, if $M \in \partial\Omega(a)$, then due to the strict egress type property from Theorem 4.1, $\mathcal{C}(M) \notin \overline{\Omega(a+1)}$ and the line segment connecting the points M and $\mathcal{C}(M)$ has only one point of intersection with the convex hull of the set $\overline{\Omega(k) \cup \Omega(k+1)}$, namely, the point M itself. But the mentioned intersection is exactly the value of the mapping P at the point M and thus $P(M) = M$.

Auxiliary mapping π

Now we will find a retraction π of the set $\partial_u \mathcal{H}_{a,\infty} := \bigcup_{k=a}^{\infty} \partial_u \mathcal{H}_{k,k+1}$ onto the set $\partial\Omega(a)$ (an analogy to the retraction required in Theorem 3.1, condition (3)). This retraction will be described with help of another auxiliary mapping ρ .

Mapping ρ_A . Let $A \subset \mathbb{R}^m$ be a bounded convex set of dimension m and choose a point $P \in \text{int } A$. As P is an inner point, there exists an $r > 0$ such that $B(P, r) \subset A$, $B(P, r) = \{X \in \mathbb{R}^m : \|X - P\| < r\}$. We can define a homeomorphism $\rho_A : \partial A \rightarrow \partial B(P, r)$ as it is shown in Figure 2. For a point $X \in \partial A$, $\rho_A(X)$ will be the intersection of the line segment PX with the boundary of $B(P, r)$.

FIGURE 2. Mapping ρ_A .

In every $\Omega(k)$, choose an inner point P_k . Choose numbers r_k , $k \in \mathbb{Z}_a^\infty$, such that $B(P_k, r_k) \subset \Omega(k)$. Now, for each $k \in \mathbb{Z}_a^\infty$, denote $\mathcal{B}_{k,k+1}$ the convex hull of

the set $B(P_k, r_k) \cup B(P_{k+1}, r_{k+1})$ and $\mathcal{B}_{a,\infty}$ the union of all the sets $\mathcal{B}_{k,k+1}$. For every $t^* \in [a, \infty)$, the cut of the set $\mathcal{B}_{a,\infty}$ through the hyperplane $t = t^*$ is the open ball $B(P_{t^*}, r_{t^*})$, where $P_{t^*} = (1 - t^* + [t^*])P_k + (t^* - [t^*])P_{k+1}$, $[t^*]$ being the integer part of t^* . The cut of the set $\mathcal{H}_{[t^*], [t^*]+1}$ through the same hyperplane is a convex set, let us denote it $W(t^*)$. There exists the mapping $\rho_{W(t^*)}: \partial W(t^*) \rightarrow \partial B(P_{t^*}, r_{t^*})$. Each set $\partial B(P_{t^*}, r_{t^*})$ can be mapped onto the set $\partial B(P_a, r_a)$ and then the inverse mapping to $\rho_{\Omega(a)}$ can be taken to map $\partial B(P_a, r_a)$ onto $\partial\Omega(a)$. Composing these mappings, we can construct a retraction π of the set $\partial_u \mathcal{H}_{a,\infty}$ onto the set $\partial\Omega(a)$, see Figure 3.

 FIGURE 3. Retraction π .

Construction of the resulting retraction

Define the mapping $R: \Omega(a) \rightarrow \partial\Omega(a)$ as $R = \pi \circ P$. As a composition of two continuous mappings, R is continuous, too. Further, $R(M) = M$ if $M \in \partial\Omega(a)$. Thus R is the desired retraction. As noted above, this fact leads to a contradiction and thus there must exist a solution of the system (1) satisfying (3). $\square$

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IRENA HLAVIČKOVÁ

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