PRODUCT OF M -MEASURES

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ABSTRACT. M -measures have been introduced as a useful tool for probability theory on IF-events. We deal with the product of M -measures, a notion which is important for the construction of a joint observable — an analogue of a random vector in the Kolmogorov theory.

It is well-known that, in probability theory, the σ -additivity plays a key role. It can be expressed as a conjunction of continuity and additivity:

$$P(A) + P(B) = P(A \cap B) + P(A \cup B), \quad A, B \subset \Omega.$$

Of course, if instead of (crisp) sets, we consider fuzzy sets (i.e., functions $f: \Omega \rightarrow [0, 1]$), then some operations with fuzzy sets, instead of sets, will be considered. In [1], IF-sets, i.e., pairs

$$A = (\mu_A, \nu_A), \quad \mu_A, \nu_A: \Omega \rightarrow \langle 0, 1 \rangle, \quad \mu_A + \nu_A \leq 1,$$

are considered. If $\mathcal{F}$ is a set of IF-sets, then the probability can be regarded as a mapping

$$m: \mathcal{F} \rightarrow [0, 1].$$

The second important concept of probability theory is the notion of a random variable. It is a measurable mapping

$$\xi: \Omega \rightarrow R_1$$

(if A is a Borel set, then $\xi^{-1}(A) \in \mathcal{S}$, where $\mathcal{S}$ is a given σ -algebra of subsets of Ω). In the IF-probability theory instead of random variables, the so-called observables

$$x: \mathcal{B}(R) \rightarrow \mathcal{F}$$

are considered. For $C \in \mathcal{B}(R)$ we denote

$$x(C) = (x^b(C), 1 - x^\sharp(C)) \in \mathcal{F}.$$

Then the mappings

$$x^b, x^\sharp: \mathcal{B}(R) \rightarrow [0, 1]$$

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have nice properties leading to the notion of an M -measure (see Definition 1).

In the present paper we deal with the existence of products of M -measures. It is motivated by the construction of a joint observable — an analogue of a random vector in the Kolmogorov theory. In Theorem 4 we construct a product of two M -measures defined on σ -algebras, our main result.

DEFINITION 1. Let $\mathcal{R}$ be an algebra of subsets of a set Ω . A mapping $\mu: \mathcal{R} \rightarrow [0, 1]$ is called an M -measure if the following properties are satisfied:

- (i) $\mu(\Omega) = 1, \mu(\emptyset) = 0$;
- (ii) $\mu(A \cup B) = \mu(A) \vee \mu(B), \mu(A \cap B) = \mu(A) \wedge \mu(B)$ for any $A, B \in \mathcal{R}$;
- (iii) $A_n \nearrow A, B_n \searrow B, A_n, B_n, A, B \in \mathcal{R} \implies \mu(A_n) \nearrow \mu(A), \mu(B_n) \searrow \mu(B)$.

THEOREM 2. For every M -measure μ defined on an algebra $\mathcal{R}$ there exists exactly one M -measure $\bar{\mu}$ on $\sigma(\mathcal{R})$ extending μ .

Proof. See [3]. □

DEFINITION 3. Let $\mathcal{C}$ be a family of subsets of Ω . We say that $\mathcal{C}$ is a compact family if for every sequence $(C_n)_n \subset \mathcal{C}$ the following implication holds:

$$\left(\forall n: \bigcap_{i=1}^n C_i \neq \emptyset \right) \implies \bigcap_{i=1}^{\infty} C_i \neq \emptyset.$$

A mapping $\lambda: \mathcal{R} \rightarrow [0, 1]$ is called compact if there exists a compact family $\mathcal{C}$ such that to every $A \in \mathcal{R}$ and every $\varepsilon > 0$

$$B \in \mathcal{R}, C \in \mathcal{C}, B \subset C \subset A \quad \text{with} \quad \lambda(A \setminus B) < \varepsilon.$$

THEOREM 4. Let $(\Omega, \mathcal{S}, \mu), (Y, \mathcal{T}, \nu)$ be two spaces with compact M -measures, where $\mu: \mathcal{S} \rightarrow [0, 1]$ and $\nu: \mathcal{T} \rightarrow [0, 1]$. Then there exists exactly one M -measure $\bar{\kappa}: \sigma(\mathcal{R}) = \mathcal{S} \otimes \mathcal{T} \rightarrow [0, 1]$ such that

$$\bar{\kappa}(A \times B) = \mu(A) \wedge \nu(B) \quad \text{for all} \quad A \in \mathcal{S}, B \in \mathcal{T}.$$

Proof. Let $\mathcal{R}$ be an algebra of all sets of the form

$$\bigcup_{i=1}^n (A_i \times B_i),$$

where

$$n \in \mathbb{N}, A_i \in \mathcal{S}, B_i \in \mathcal{T}, \quad \text{and} \quad (A_i \times B_i) \cap (A_j \times B_j) = \emptyset \quad \text{for} \quad i \neq j.$$

Since the operations $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$ satisfy the distributive law, the expression

$$\bigvee_{i=1}^n (\mu(A_i) \wedge \nu(B_i))$$

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does not depend on the choice of A_i and B_i . Therefore, we can define

$$\kappa \left(\bigcup_{i=1}^n (A_i \times B_i) \right) = \bigvee_{i=1}^n (\mu(A_i) \wedge \nu(B_i)).$$

This way we obtain a mapping $\kappa: \mathcal{R} \rightarrow [0, 1]$. Moreover,

$$\kappa(A \times B) = \mu(A) \wedge \nu(B). \quad (1)$$

We shall prove that κ is an M -measure. Evidently,

$$\begin{aligned} \kappa(X \times Y) &= \mu(X) \wedge \nu(Y) = 1 \wedge 1 = 1, \\ \kappa(\emptyset) &= \mu(\emptyset) \wedge \nu(\emptyset) = 0. \end{aligned}$$

Further,

$$\begin{aligned} \kappa(A \cup B) &= \kappa \left(\bigcup_{i=1}^n (A_i \times B_i) \cup \bigcup_{j=1}^m (C_j \times D_j) \right) = \kappa \left(\bigcup_{i=1}^n \bigcup_{j=1}^m (A_i \times B_i) \cup (C_j \times D_j) \right) \\ &= \kappa \left(\bigcup_{i=1}^n \bigcup_{j=1}^m ((A_i \setminus C_j) \times B_i) \cup ((A_i \cap C_j) \times (B_i \cup D_j)) \cup ((C_j \setminus A_i) \times D_j) \right) \\ &= \bigvee_{i=1}^n \bigvee_{j=1}^m (\mu(A_i \setminus C_j) \wedge \nu(B_i)) \vee (\mu(A_i \cap C_j) \wedge \nu(B_i \cup D_j)) \vee (\mu(C_j \setminus A_i) \wedge \nu(D_j)) \\ &= \bigvee_{i=1}^n \bigvee_{j=1}^m (\mu(A_i) \wedge \nu(B_i)) \vee (\mu(C_j) \wedge \nu(D_j)) \\ &= \kappa(A) \vee \kappa(B). \end{aligned}$$

Similarly,

$$\kappa(A \cap B) = \kappa(A) \wedge \kappa(B) \quad \text{for all } A, B \in \mathcal{R}.$$

Now, we shall prove

$$A_n \in \mathcal{R} \ (n = 1, 2, \dots), \quad A_n \searrow \emptyset \implies \kappa(A_n) \searrow 0. \quad (2)$$

Since μ, ν are compact M -measures, there exist compact families $\mathcal{K}_1 \subset 2^X$, $\mathcal{K}_2 \subset 2^Y$ such that

$$\forall A \in \mathcal{S} \quad \forall \varepsilon > 0 \quad \exists E \in \mathcal{S}, \quad C \in \mathcal{K}_1, \quad E \subset C \subset A, \quad \mu(A \setminus E) < \varepsilon$$

and

$$\forall B \in \mathcal{T} \quad \forall \varepsilon > 0 \quad \exists F \in \mathcal{T}, \quad D \in \mathcal{K}_2, \quad F \subset D \subset B, \quad \nu(B \setminus F) < \varepsilon.$$

Denote

$$\mathcal{K} = \left\{ C; \quad C = \bigcup_{i=1}^n (C_i \times D_i), \quad n \in N, \quad C_i \in \mathcal{K}_1, \quad D_i \in \mathcal{K}_2 \right\}.$$

First, we show that

$$\forall \varepsilon > 0 \quad \forall A \in \mathcal{R} \quad \exists B \in \mathcal{R}, \quad C \in \mathcal{K}, \quad B \subset C \subset A, \quad \kappa(A \setminus B) < \varepsilon. \quad (3)$$

Let $A = \bigcup_{i=1}^n (A_i \times B_i)$, $A_i \in \mathcal{S}$, $B_i \in \mathcal{T}$ ($i = 1, 2, \dots, n$).

Since μ, ν are compact M -measures, then $\exists H_i \in \mathcal{K}_1$, $E_i \in \mathcal{S}$ such that

$$E_i \subset H_i \subset A_i, \quad \mu(A_i \setminus E_i) < \varepsilon \quad \text{and} \quad \exists G_i \in \mathcal{K}_2, \quad F_i \in \mathcal{S}$$

such that

$$F_i \subset G_i \subset B_i, \quad \nu(B_i \setminus F_i) < \varepsilon.$$

Put

$$C = \bigcup_{i=1}^n (H_i \times G_i), \quad B = \bigcup_{j=1}^n (E_j \times F_j),$$

and

$$C \subset B \subset A.$$

Then,

$$\begin{aligned} \kappa(A \setminus B) &= \kappa \left(\bigcup_{i=1}^n (A_i \times B_i) \setminus \bigcup_{j=1}^n (E_j \times F_j) \right) \\ &= \kappa \left(\bigcup_{i=1}^n ((A_i \setminus E_i) \times B_i) \cup \bigcup_{j=1}^n (A_j \times (B_j \setminus F_j)) \right) \\ &= \bigvee_{i=1}^n (\mu(A_i \setminus E_i) \wedge \nu(B_i)) \vee \bigvee_{j=1}^n (\mu(A_j) \wedge \nu(B_j \setminus F_j)) \\ &\leq \bigvee_{i=1}^n \mu(A_i \setminus E_i) \vee \bigvee_{j=1}^n \nu(B_j \setminus F_j) < \varepsilon. \end{aligned}$$

Now return to the sequence (A_n) , $A_n \in \mathcal{R}$, $A_n \searrow \emptyset$.

Using (3) we construct

$$B_n \in \mathcal{R}, \quad C_n \in \mathcal{K}, \quad B_n \subset C_n \subset A_n, \quad \kappa(A_n \setminus B_n) < \varepsilon.$$

Putting

$$D_n = \bigcap_{i=1}^n C_i, \quad C_i \in \mathcal{K}.$$

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We get

$$\bigcap_{n=1}^{\infty} D_n \subset \bigcap_{n=1}^{\infty} A_n = \emptyset.$$

Since $\mathcal{K}$ is a compact family, there exists m such that

$$\bigcap_{i=1}^m B_i \subset D_m = \bigcap_{i=1}^m C_i = \emptyset.$$

We have

$$\begin{aligned} \kappa(A_m) &= \kappa\left(A_m \setminus \bigcap_{i=1}^m B_i\right) \\ &= \kappa\left(\bigcup_{i=1}^m (A_m \setminus B_i)\right) \\ &\leq \kappa\left(\bigcup_{i=1}^m (A_i \setminus B_i)\right) \\ &= \bigvee_{i=1}^m \kappa(A_i \setminus B_i) < \varepsilon. \end{aligned}$$

Further

$$\kappa(A_n) \leq \kappa(A_m) < \varepsilon \quad \text{for any } n \geq m.$$

Therefore,

$$\lim_{n \rightarrow \infty} \kappa(A_n) = 0.$$

Now if

$$B_n \in \mathcal{R}, \quad B_n \nearrow B, \quad B \in \mathcal{R}, \quad \text{then} \quad B \setminus B_n \searrow \emptyset$$

and hence

$$\begin{aligned} \kappa(B) &= \kappa((B \setminus B_n) \cup B_n) \\ &= \kappa(B \setminus B_n) \vee \kappa(B_n) \\ &\leq \kappa(B \setminus B_n) \vee \bigvee_{i=1}^{\infty} \kappa(B_i). \end{aligned}$$

Therefore,

$$\kappa(B) \leq \lim_{n \rightarrow \infty} \kappa(B \setminus B_n) \vee \left(\bigvee_{i=1}^{\infty} \kappa(B_i) \right) = \bigvee_{i=1}^{\infty} \kappa(B_i) \leq \kappa(B)$$

and hence

$$\kappa(B) = \lim_{i \rightarrow \infty} \kappa(B_i).$$

On the other hand, $C_n \searrow C$ implies $C_n \setminus C \searrow \emptyset$,

$$\begin{aligned} \kappa(C_n) &= \kappa((C_n \setminus C) \vee C) = \kappa(C_n \setminus C) \vee \kappa(C), \\ \bigwedge_{n=1}^{\infty} \kappa(C_n) &= \left(\bigwedge_{n=1}^{\infty} \kappa(C_n \setminus C) \right) \vee \kappa(C) = 0 \vee \kappa(C) = \kappa(C). \end{aligned}$$

We have proved that $\kappa: \mathcal{R} \rightarrow [0, 1]$ is an M -measure. By Theorem 1.2 there exists exactly one M -measure $\bar{\kappa}: \sigma(\mathcal{R}) \rightarrow [0, 1]$ such that $\bar{\kappa}|_{\mathcal{R}} = \kappa$.

By (1) we have

$$\bar{\kappa}(A \times B) = \mu(A) \wedge \nu(B) \quad \text{for } A \times B \in \mathcal{R}.$$

Fix B and put

$$\mathcal{L} = \{A \in \mathcal{S}; \bar{\kappa}(A \times B) = \mu(A) \wedge \nu(B)\}.$$

Since $\mathcal{L}$ is monotone and $\mathcal{L} \supset X$, it follows that

$$\mathcal{L} \supset \sigma(X) = \mathcal{S}.$$

Therefore

$$\bar{\kappa}(A \times B) = \mu(A) \wedge \nu(B) \quad \text{for each } A \in \mathcal{S}.$$

Further, for fixed $A \in \mathcal{S}$, consider the family

$$\mathcal{G} = \{B \in \mathcal{T}; \bar{\kappa}(A \times B) = \mu(A) \wedge \nu(B)\}.$$

Clearly, $\mathcal{G} \supset Y$. Since $\mathcal{G}$ is monotone,

$$\mathcal{G} \supset \sigma(Y) = \mathcal{T}$$

and hence,

$$\bar{\kappa}(A \times B) = \mu(A) \wedge \nu(B) \quad \text{whenever } A \in \mathcal{S}, B \in \mathcal{T}.$$

This completes the proof. $\square$

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