Several Higher Differentiation Formulas of Special Functions

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Summary. In this paper, we proved some basic properties of higher differentiation, and higher differentiation formulas of special functions [4].

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The notation and terminology used in this paper are introduced in the following articles: [16], [13], [2], [3], [5], [1], [7], [9], [12], [10], [8], [18], [14], [11], [6], [15], and [17].

For simplicity, we use the following convention: x, r, a, x_0, p are real numbers, n, i, m are elements of $\mathbb{N}$, Z is an open subset of $\mathbb{R}$, and f, f_1, f_2 are partial functions from $\mathbb{R}$ to $\mathbb{R}$.

Next we state a number of propositions:

- (1) For every function f from $\mathbb{R}$ into $\mathbb{R}$ holds $\text{dom}(f|Z) = Z$.
- (2) $(-f_1) - f_2 = f_1 f_2$.
- (3) If $n \geq 1$, then $\text{dom}(\frac{1}{\square^n}) = \mathbb{R} \setminus \{0\}$ and $(\square^n)^{-1}(\{0\}) = \{0\}$.
- (4) $(r \cdot p) \frac{1}{\square^n} = r (p \frac{1}{\square^n})$.
- (5) For all elements n, m of $\mathbb{R}$ holds $n f + m f = (n + m) f$.
- (6) If $f|Z$ is differentiable on Z , then f is differentiable on Z .

- (7) If $n \geq 1$ and f is differentiable n times on Z , then f is differentiable on Z .
- (8) $\square^n$ is differentiable on $\mathbb{R}$.
- (9) If $x \in Z$, then (the function $\sin$)'(Z)(2)(x) = $-\sin x$.
- (10) If $x \in Z$, then (the function $\sin$)'(Z)(3)(x) = $-\cos x$.
- (11) If $x \in Z$, then (the function $\sin$)'(Z)(n)(x) = $\sin(x + \frac{n \cdot \pi}{2})$.
- (12) If $x \in Z$, then (the function $\cos$)'(Z)(2)(x) = $-\cos x$.
- (13) If $x \in Z$, then (the function $\cos$)'(Z)(3)(x) = $\sin x$.
- (14) If $x \in Z$, then (the function $\cos$)'(Z)(n)(x) = $\cos(x + \frac{n \cdot \pi}{2})$.
- (15) If f_1 is differentiable n times on Z and f_2 is differentiable n times on Z , then $(f_1 + f_2)'(Z)(n) = f_1'(Z)(n) + f_2'(Z)(n)$.
- (16) If f_1 is differentiable n times on Z and f_2 is differentiable n times on Z , then $(f_1 - f_2)'(Z)(n) = f_1'(Z)(n) - f_2'(Z)(n)$.
- (17) If f_1 is differentiable n times on Z and f_2 is differentiable n times on Z and $i \leq n$, then $(f_1 + f_2)'(Z)(i) = f_1'(Z)(i) + f_2'(Z)(i)$.
- (18) If f_1 is differentiable n times on Z and f_2 is differentiable n times on Z and $i \leq n$, then $(f_1 - f_2)'(Z)(i) = f_1'(Z)(i) - f_2'(Z)(i)$.
- (19) If f_1 is differentiable n times on Z and f_2 is differentiable n times on Z , then $f_1 + f_2$ is differentiable n times on Z .
- (20) If f_1 is differentiable n times on Z and f_2 is differentiable n times on Z , then $f_1 - f_2$ is differentiable n times on Z .
- (21) If f is differentiable n times on Z , then $(rf)'(Z)(n) = r f'(Z)(n)$.
- (22) If f is differentiable n times on Z , then rf is differentiable n times on Z .
- (23) If f is differentiable on Z , then $f'(Z)(1) = f'_{|Z}$.
- (24) If $n \geq 1$ and f is differentiable n times on Z , then $f'(Z)(1) = f'_{|Z}$.
- (25) If $x \in Z$, then $(r(\text{the function } \sin))'(Z)(n)(x) = r \cdot \sin(x + \frac{n \cdot \pi}{2})$.
- (26) If $x \in Z$, then $(r(\text{the function } \cos))'(Z)(n)(x) = r \cdot \cos(x + \frac{n \cdot \pi}{2})$.
- (27) If $x \in Z$, then $(r(\text{the function } \exp))'(Z)(n)(x) = r \cdot \exp x$.
- (28) $(\square^n)'_{|Z} = (n(\square^{n-1}))_{|Z}$.
- (29) If $x \neq 0$, then $\frac{1}{\square^n}$ is differentiable in x and $(\frac{1}{\square^n})'(x) = -\frac{n \cdot x^{n-1}}{(x^n)^2}$.
- (30) If $n \geq 1$, then $(\square^n)'(Z)(2) = ((n \cdot (n-1))(\square^{n-2}))_{|Z}$.
- (31) If $n \geq 2$, then $(\square^n)'(Z)(3) = ((n \cdot (n-1) \cdot (n-2))(\square^{n-3}))_{|Z}$.
- (32) If $n > m$, then $(\square^n)'(Z)(m) = (((\binom{n}{m}) \cdot m!)(\square^{n-m}))_{|Z}$.
- (33) If f is differentiable n times on Z , then $(-f)'(Z)(n) = -f'(Z)(n)$ and $-f$ is differentiable n times on Z .
- (34) If $x_0 \in Z$, then $(\text{Taylor}(\text{the function } \sin, Z, x_0, x))(n) =$

- $\frac{\sin(x_0 + \frac{n \cdot \pi}{2}) \cdot (x - x_0)^n}{n!}$ and $(\text{Taylor}(\text{the function } \cos, Z, x_0, x))(n) = \frac{\cos(x_0 + \frac{n \cdot \pi}{2}) \cdot (x - x_0)^n}{n!}$.
- (35) If $r > 0$, then $(\text{Maclaurin}(\text{the function } \sin,]-r, r[, x))(n) = \frac{\sin(\frac{n \cdot \pi}{2}) \cdot x^n}{n!}$ and $(\text{Maclaurin}(\text{the function } \cos,]-r, r[, x))(n) = \frac{\cos(\frac{n \cdot \pi}{2}) \cdot x^n}{n!}$.
- (36) If $n > m$ and $x \in Z$, then $(\square^n)'(Z)(m)(x) = \binom{n}{m} \cdot m! \cdot x^{n-m}$.
- (37) If $x \in Z$, then $(\square^m)'(Z)(m)(x) = m!$.
- (38) $\square^n$ is differentiable n times on Z .
- (39) If $x \in Z$ and $n > m$, then $(a(\square^n))'(Z)(m)(x) = a \cdot \binom{n}{m} \cdot m! \cdot x^{n-m}$.
- (40) If $x \in Z$, then $(a(\square^n))'(Z)(n)(x) = a \cdot n!$.
- (41) If $x_0 \in Z$ and $n > m$, then $(\text{Taylor}(\square^n, Z, x_0, x))(m) = \binom{n}{m} \cdot x_0^{n-m} \cdot (x - x_0)^m$ and $(\text{Taylor}(\square^n, Z, x_0, x))(n) = (x - x_0)^n$.
- (42) Let n, m be elements of $\mathbb{N}$ and r, x be real numbers. If $n > m$ and $r > 0$, then $(\text{Maclaurin}(\square^n,]-r, r[, x))(m) = 0$ and $(\text{Maclaurin}(\square^n,]-r, r[, x))(n) = x^n$.
- (43) $\frac{1}{\square^n}$ is differentiable on $]0, r[$.
- (44) If $x_0 \in]0, r[$, then $(\frac{1}{\square^n})'_{|]0, r[}(x_0) = -\frac{n}{(\square^{n+1})(x_0)}$.
- (45) If $x \neq 0$, then $\frac{1}{\square^1}$ is differentiable in x and $(\frac{1}{\square^1})'(x) = -\frac{1}{(x^1)^2}$.
- (46) If $]0, r[\subseteq \text{dom}(\frac{1}{\square^2})$, then $(\frac{1}{\square^2})'_{|]0, r[} = ((-1) \frac{1}{\square^2})_{|]0, r[}$.
- (47) If $x \neq 0$, then $\frac{1}{\square^2}$ is differentiable in x and $(\frac{1}{\square^2})'(x) = -\frac{2 \cdot x^1}{(x^2)^2}$.
- (48) If $]0, r[\subseteq \text{dom}(\frac{1}{\square^3})$, then $(\frac{1}{\square^3})'_{|]0, r[} = ((-2) \frac{1}{\square^3})_{|]0, r[}$.
- (49) If $n \geq 1$, then $(\frac{1}{\square^n})'_{|]0, r[} = ((-n) \frac{1}{\square^{n+1}})_{|]0, r[}$.
- (50) Suppose f_1 is differentiable 2 times on Z and f_2 is differentiable 2 times on Z . Then $(f_1 f_2)'(Z)(2) = f_1'(Z)(2) f_2 + 2((f_1)'_{|Z} (f_2)'_{|Z}) + f_1 f_2'(Z)(2)$.
- (51) If $Z \subseteq \text{dom}(\text{the function } \ln)$ and $Z \subseteq \text{dom}(\frac{1}{\square^1})$, then $(\text{the function } \ln)'_{|Z} = \frac{1}{\square^1}_{|Z}$.
- (52) If $n \geq 1$ and $x_0 \in]0, r[$, then $(\frac{1}{\square^n})'_{|]0, r[}(2)(x_0) = n \cdot (n+1) \cdot (\frac{1}{\square^{n+2}})(x_0)$.
- (53) $((\text{The function } \sin) (\text{the function } \sin))'(Z)(2) = 2(((\text{the function } \cos) (\text{the function } \cos))_{|Z}) + (-2) (((\text{the function } \sin) (\text{the function } \sin))_{|Z})$.
- (54) $((\text{The function } \cos) (\text{the function } \cos))'(Z)(2) = 2(((\text{the function } \sin) (\text{the function } \sin))_{|Z}) + (-2) (((\text{the function } \cos) (\text{the function } \cos))_{|Z})$.
- (55) $((\text{The function } \sin) (\text{the function } \cos))'(Z)(2) = 4 (((-\text{the function } \sin) (\text{the function } \cos))_{|Z})$.
- (56) Suppose $Z \subseteq \text{dom}(\text{the function } \tan)$. Then the function $\tan$ is differentiable on Z and $(\text{the function } \tan)'_{|Z} = (\frac{1}{\text{the function } \cos} \frac{1}{\text{the function } \cos})_{|Z}$.
- (57) Suppose $Z \subseteq \text{dom}(\text{the function } \tan)$. Then $\frac{1}{\text{the function } \cos}$ is differentiable on Z and $(\frac{1}{\text{the function } \cos})'_{|Z} = (\frac{1}{\text{the function } \cos} (\text{the function } \tan))_{|Z}$.

- (58) Suppose $Z \subseteq \text{dom}(\text{the function tan})$. Then $(\text{the function tan})'(Z)(2) = 2(((\text{the function tan}) \frac{1}{\text{the function cos}} \frac{1}{\text{the function cos}})|Z)$.
- (59) Suppose $Z \subseteq \text{dom}(\text{the function cot})$. Then
- (i) the function cot is differentiable on Z , and
 - (ii) $(\text{the function cot})'|_Z = ((-1) (\frac{1}{\text{the function sin}} \frac{1}{\text{the function sin}}))|Z$.
- (60) Suppose $Z \subseteq \text{dom}(\text{the function cot})$. Then
- (i) $\frac{1}{\text{the function sin}}$ is differentiable on Z , and
 - (ii) $(\frac{1}{\text{the function sin}})'|_Z = (-\frac{1}{\text{the function sin}} (\text{the function cot}))|Z$.
- (61) Suppose $Z \subseteq \text{dom}(\text{the function cot})$. Then $(\text{the function cot})'(Z)(2) = 2(((\text{the function cot}) \frac{1}{\text{the function sin}} \frac{1}{\text{the function sin}})|Z)$.
- (62) $((\text{The function exp}) (\text{the function sin}))'(Z)(2) = 2(((\text{the function exp}) (\text{the function cos}))|Z)$.
- (63) $((\text{The function exp}) (\text{the function cos}))'(Z)(2) = 2(((\text{the function exp}) -\text{the function sin})|Z)$.
- (64) Suppose f_1 is differentiable 3 times on Z and f_2 is differentiable 3 times on Z . Then $(f_1 f_2)'(Z)(3) = f_1'(Z)(3) f_2 + (3(f_1'(Z)(2) (f_2)'|_Z) + 3((f_1)'|_Z f_2'(Z)(2))) + f_1 f_2'(Z)(3)$.
- (65) $((\text{The function sin}) (\text{the function sin}))'(Z)(3) = (-8) (((\text{the function cos}) (\text{the function sin}))|Z)$.
- (66) If f is differentiable 2 times on Z , then $(f f)'(Z)(2) = 2(f f'(Z)(2)) + 2(f'_|Z f'_|Z)$.
- (67) Suppose f is differentiable 2 times on Z and for every x_0 such that $x_0 \in Z$ holds $f(x_0) \neq 0$. Then $(\frac{1}{f})'(Z)(2) = \frac{2 f'_|Z f'_|Z - f'(Z)(2) f}{f(f f)}$.
- (68) $((\text{The function exp}) (\text{the function sin}))'(Z)(3) = (2((\text{the function exp}) (-\text{the function sin} + \text{the function cos})))|Z$.

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