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CONSECUTIVE RATIOS IN SECOND-ORDER LINEAR RECURRENCE SEQUENCES

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ABSTRACT. Let $(a_n)_{n=0}^{\infty}$ be a second-order linear recurrence sequence with constant coefficient. We study the limit points and asymptotic distribution of the sequence of consecutive ratios a_{n+1}/a_n.

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1. Introduction

The Fibonacci sequence $(F_n)_{n=0}^{\infty}$ is defined by $F_0 = 1$, $F_1 = 1$, and

$$F_n = F_{n-1} + F_{n-2}, \qquad n > 1. \tag{1}$$

The limit of the ratios of consecutive terms of $(F_n)_{n=0}^{\infty}$ is well known to be

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \phi,$$

where $\phi = \frac{1+\sqrt{5}}{2} = 1.618\ldots$ is the golden mean [22, p.240].

In general, a sequence of complex numbers $(a_n)_{n=0}^{\infty}$ is a linear recurrence sequence of order k with constant coefficients if it satisfies:

$$a_n = c_{k-1}a_{n-1} + \cdots + c_0 a_{n-k}, \qquad n \geq k,\ (c_0, \ldots, c_{k-1} \in \mathbf{C}). \tag{2}$$

Möbius transformation, Cauchy distribution, circular Cauchy distribution, unique ergodicity.

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The polynomial

$$p(\lambda) = \lambda^k - c_{k-1}\lambda^{k-1} - \cdots - c_0$$

is the *characteristic polynomial* of $(a_n)_{n=0}^{\infty}$. Let $\lambda_1, \ldots, \lambda_h$ be the roots of p, with respective multiplicities $k_1, \ldots, k_h$. The general term a_n may be written explicitly in the form

$$a_n = \sum_{i=1}^{h} \sum_{j=1}^{k_i} c_{i,j} n^{j-1} \lambda_i^n, \qquad n \in \mathbf{N}, \tag{3}$$

where the coefficients $c_{i,j}$ are complex numbers, uniquely determined by $a_0, a_1, , \ldots, a_{k-1}$ (see [14, Theorem 3.6]). Usually, there is one term on the right-hand side of (3) that dominates all others. In fact, order the roots λ_i so that $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_h|$; between roots λ_i and λ_j of the same modulus, λ_i precedes λ_j if $k_i > k_j$. To avoid trivialities, we assume that $c_{i,k_i} \neq 0$ for each i. If $|\lambda_1| > |\lambda_2|$, or $|\lambda_1| = |\lambda_2|$ and $k_1 > k_2$, then the term $c_{1,k_1} n^{k_1-1} \lambda_1^n$ is much larger in absolute value than all other terms for large n. In this case, for large n we have

$$\frac{a_{n+1}}{a_n} \approx \frac{c_{1,k_1}(n+1)^{k_1-1}\lambda_1^{n+1}}{c_{1,k_1} n^{k_1-1} \lambda_1^n},$$

and therefore $a_{n+1}/a_n \xrightarrow[n\to\infty]{} \lambda_1$. (Here and later, if finitely many a_n-s vanish, we consider the ratios a_{n+1}/a_n only for sufficiently large n.) In particular, if $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_h|$, then, by omitting 0 terms from the right-hand side of (3), we see that the consecutive ratios a_{n+1}/a_n converge to one of the λ_i as $n \to \infty$. (This is a special case of a result of Poincaré [27].) On the other hand, if there exist two distinct roots of p with the same modulus, then it is always possible to find initial conditions so that $\lim_{n\to\infty} a_{n+1}/a_n$ does not exist [7].

The special case where $(a_n)_{n=0}^{\infty}$ is a sequence of integers was studied in [15–18, 20, 21]. Suppose the roots of its characteristic polynomial are distinct and satisfy $|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_h|$. We may write a_n in the form

$$a_n = c_1\lambda_1^n + c_2\lambda_2^n + \cdots + c_h\lambda_h^n, \qquad n \geq 0,$$

where $c_1, c_2, \ldots, c_h$ are algebraic numbers and assume that $c_1 \neq 0$. The sequence of ratios $(a_{n+1}/a_n)_{n=0}^{\infty}$ converges to λ_1. In [15, 17, 18], the rate of convergence of the ratios a_{n+1}/a_n to this limit was studied. If $(a_n)_{n=0}^{\infty}$ is of order 2 and $|\lambda_1| = |\lambda_2|$, it was shown that $|\lambda_1|$ is a partial limit of the sequence of ratios, and the distances between the terms of the sequence and this partial limit were discussed [16, 18, 21].

Following [7], we call $\lim_{n\to\infty} a_{n+1}/a_n$, if it exists, the *Kepler limit* of $(a_n)_{n=0}^{\infty}$. The *Kepler set* of $(a_n)_{n=0}^{\infty}$ is the set of limit points of the sequence $(a_{n+1}/a_n)_{n=0}^{\infty}$. In this paper, by a linear recurrence sequence we mean a linear recurrence sequence with constant coefficients. To avoid trivialities, we will always assume that the sequence is not identically 0.

Bagdasar, Hedderwick, and Popa [2] discussed the Kepler set of second-order linear recurrence sequences. According to the discussion above, unless the roots λ_1 and λ_2 of the characteristic polynomial p are of equal moduli, the Kepler set reduces to a single point. For $|\lambda_1| = |\lambda_2|$, they noted that the Kepler set may be a finite set or a circle in the complex plane. We will notice that the Kepler set may be also a line. Our first goal will be to understand the exact dependence of the Kepler set on the parameters in the representation $a_n = c_1\lambda_1^n + c_2\lambda_2^n$. We will also characterize the lines and circles that arise as Kepler sets.

For general linear recurrence sequences, in addition to the topological information given by the Kepler set, one may ask about the distribution of the sequence $(a_{n+1}/a_n)_{n=0}^{\infty}$ in the complex plane. For example, suppose that the Kepler set is a circle. Does the sequence spend roughly the same time in equal arcs of this circle? Taking any example, one is readily convinced that this is not the case (see, for example Figure 1 (B) in the sequel). Our main goal in this paper is to understand how the sequence of consecutive ratios is distributed for any linear recurrence sequence of order 2. Moreover, we will characterize the family of distributions on the complex plane arising this way.

The distribution modulo 1 of the sequence of consecutive ratios was studied in several papers [8, 19] when $(a_n)_{n=0}^{\infty}$ is a real-valued recurrence sequence. Kiss and Tichy [19] studied the distribution for order-2 real-valued sequences $(a_n)_{n=0}^{\infty}$. Under suitable conditions, they determined the asymptotic distribution function of the sequence (a_{n+1}/a_n) and gave an estimate of the error term. The asymptotic distribution function of the sequence $(a_{n+1}/a_n)_{n=0}^{\infty}$, for linear recurrence sequence of any order $k \geq 2$, was discussed by Goldstern, Tichy, and Turnwald [8].

The sequence of consecutive ratios is easily seen to be the orbit if some point in the complex plane under a certain Möbius transformation. We will discuss the ergodic-theoretical properties of this transformation and derive some properties of the sequence of ratios.

In Section 2, we state our main results. Section 3 introduces an extended family of Cauchy distributions in the complex plane and studies their behaviour under Möbius transformations. In Section 4, we discuss the intuition behind the main results. Section 5 presents the proofs.

2. Main results

Let $(a_n)_{n=0}^{\infty}$ be a second-order linear recurrence sequence

$$a_n = ca_{n-1} + da_{n-2}, \qquad n \geq 2, \tag{4}$$

with some initial values a_0, a_1, where c, d are fixed complex numbers (with some restrictions listed below, designed to avoid trivialities). We have

$$\frac{a_{n+2}}{a_{n+1}} = c + \frac{d}{a_{n+1}/a_n}. \tag{5}$$

Denote $r_n = a_{n+1}/a_n$ for $n \geq 0$. By (5), the sequence $(r_n)_{n=1}^{\infty}$ satisfies the recurrence

$$r_{n+1} = c + d/r_n.$$

Consider the Möbius transformation on the extended complex plane $\mathbf{C}_{\infty}$., defined by

$$S(z) = c + d/z, \qquad z \in \mathbf{C}_{\infty}.$$

In terms of S, the Kepler set of $(a_n)_{n=0}^{\infty}$ is the set of limit points of the sequence $\left(S^n(a_1/a_0)\right)_{n=0}^{\infty}$. (If $a_n = 0$ for some n, we take $a_{n+1}/a_n = \infty$. The assumptions below will guarantee that we cannot have $a_n = a_{n+1} = 0$.) In particular, the Kepler set is S-invariant.

Rewrite a_n explicitly

$$a_n = c_1\lambda_1^n + c_2\lambda_2^n, \qquad n = 0, 1, 2, \ldots, \tag{6}$$

where λ_1 and λ_2 are the roots of the characteristic polynomial p and c_1, c_2 are complex numbers. As explained above, unless $|\lambda_1| = |\lambda_2|$, the sequence has a Kepler limit. Thus, we will assume throughout that $|\lambda_1| = |\lambda_2|$ (but $\lambda_1 \neq \lambda_2$) and that $c_1, c_2 \neq 0$. Moreover, if λ_2/λ_1 is a root of unity, then the sequence of consecutive ratios is periodic; we will exclude this trivial case.

Lemma 2.1. *Let λ_1 and λ_2 be the roots of the characteristic polynomial of $(a_n)_{n=0}^{\infty}$ given in (4). If $|\lambda_1| = |\lambda_1|$ and λ_2/λ_1 is not a root of unity, then $d < -c^2/4$. Moreover, $a_n = 0$ for at most one value of n.*

Denote by $O(C, R)$ the circle of radius R, centered at C. The following theorem identifies the infinite Kepler sets of order 2 linear recurrence sequences.

Theorem 2.2.

1. *Let $(a_n)_{n=0}^{\infty}$ be a second-order linear recurrence sequence, given by $a_n = c_1\lambda_1^n + c_2\lambda_2^n$ for some non-zero complex numbers c_1, c_2, λ_1, λ_2 with $|\lambda_1| = |\lambda_2| > 0$, where λ_2/λ_1 is not a root of unity. Then:*
   (i) *If $|c_1| = |c_2|$, then the Kepler set of $(a_n)_{n=0}^{\infty}$ is the line passing through the origin and the point $(\lambda_1 + \lambda_2)/2$.*

(ii) *If* $|c_1| \neq |c_2|$, *then the Kepler set of* $(a_n)_{n=0}^{\infty}$ *is the circle* $O(C, R)$ *with*

$$C = \frac{|c_1|^2\lambda_1 - |c_2|^2\lambda_2}{|c_1|^2 - |c_2|^2},$$

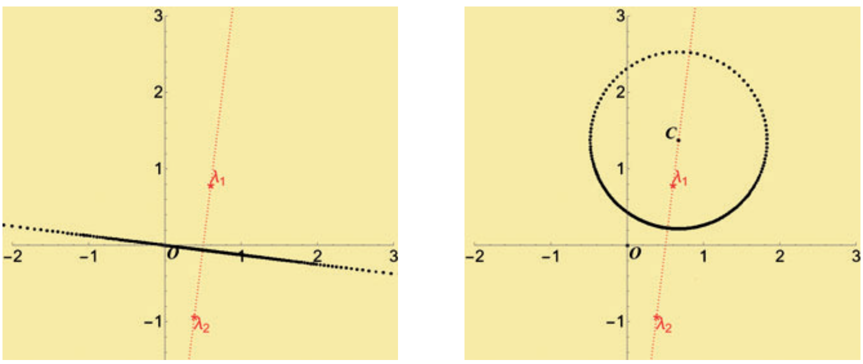
and

$$R = \frac{|c_1||c_2||\lambda_1 - \lambda_2|}{||c_1|^2 - |c_2|^2|}.$$

The circle does not include the origin either on its circumference or inside it.

2. *Conversely, if* K *is a line passing through the origin, or a circle not containing the origin either on its circumference or inside, then there exists a second--order linear recurrence sequence whose Kepler set is* K.

Example. Let $\lambda_1 = (3+4i)/5$ and $\lambda_2 = (5-12i)/13$. For $c_1 = c_2 = 1$ the Kepler set is the line passing through origin and the point $(\lambda_1+\lambda_2)/2 = 32/65 - 4/65i$, and for $c_1 = 2, c_2 = 1$, it is the circle $O\big(131/195+268/195i, 28/(3\sqrt{65})\big)$ (see Figure 1).



(A) (B)

Figure 1. (A) The Kepler set of $a_n = \lambda_1^n + \lambda_2^n$, where $\lambda_1 = (3+4i)/5$, $\lambda_2 = (5-12i)/13$. (B) The Kepler set of $a_n = 2\lambda_1^n + \lambda_2^n$.

Remark 1. The role of the coefficients c_1 and c_2 in determining the Kepler set of $(a_n)_{n=0}^{\infty}$ is only via $|c_2/c_1|$. Thus, in part 1. (i) of the theorem, the line depends only on λ_1 and λ_2 (as long as $|c_1| = |c_2|$). In part 1. (ii) of the theorem, denoting $r = |c_2/c_1|$, we may rewrite C and R in a much simpler form:

$$C = \frac{\lambda_1 - r^2\lambda_2}{1 - r^2}, \qquad R = \frac{r|\lambda_1 - \lambda_2|}{|1 - r^2|}. \tag{7}$$

Remark 2. If λ_2/λ_1 is a primitive root of unity of order m, then the sequence $(a_{n+1}/a_n)_{n=0}^{\infty}$ is of period m, and in particular the Kepler set is of size m. It is contained in the line (if $|c_1| = |c_2|$) or circle (if $|c_1| \neq |c_2|$) specified in Theorem 2.2. (See Figure 2 for two examples of Kepler sets of size 45.)

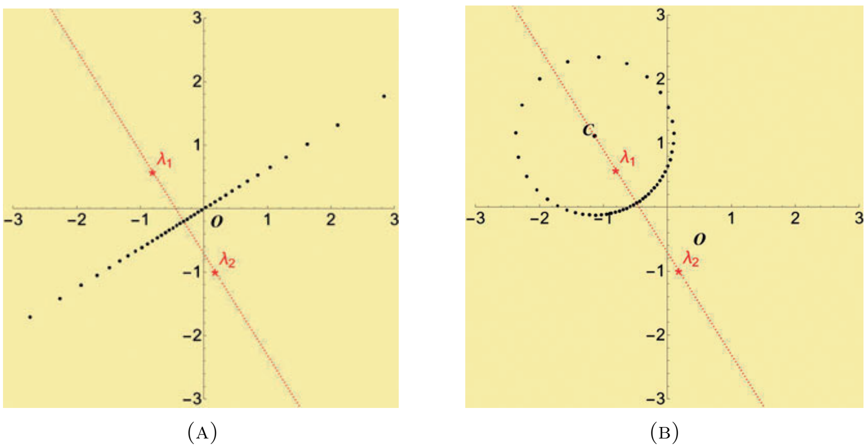


Figure 2. (A.) The Kepler set of $a_n = \lambda_1^n + \lambda_2^n$, where $\lambda_1 = \exp(4\pi i/5)$, $\lambda_2 = \exp(14\pi i/9)$. (B) The Kepler set of $a_n = 2\lambda_1^n + \lambda_2^n$. Both Kepler sets comprise $9 \cdot 5 = 45$ points.

Theorem 2.2 specifies the "topology" of the sequence $(a_{n+1}/a_n)_{n=0}^{\infty}$. How is the sequence distributed in the Kepler set? To this end, recall the notion of the distribution of a (deterministic) sequence in a topological space. Let X be a locally compact Hausdorff space, $\mathcal{B}$ its Borel σ-field, and $(x_n)_{n=0}^{\infty}$ a sequence in X. The sequence is *distributed according to* some probability measure ν on $(X, \mathcal{B})$ if

$$\lim_{N\to\infty} \sum_{n=0}^{N-1} f(x_n) = \int_X f \mathrm{d}\nu$$

for all continuous function $f : X \to \mathbf{C}$ (see [23, p. 178]). Intuitively, it means that, for a "well-behaved" set $A \in \mathcal{B}$, the asymptotic density of the set $\{n : x_n \in A\}$ is $\nu(A)$.

Definition 2.3.
Let $(a_n)_{n=0}^{\infty}$ be a sequence in $\mathbf{C}$. The *Kepler measure* of $(a_n)_{n=0}^{\infty}$ is the probability measure according to which the sequence $(a_{n+1}/a_n)_{n=0}^{\infty}$ is distributed, if any.

We define two families of distributions that will be relevant in the sequel. A complex-valued random variable Y is *Cauchy distributed* with median $\mu \in \mathbf{C}$, *scale* $\sigma \in \mathbf{R}_+$, and *direction* $\alpha \in [0, 2\pi]$, and we denote $Y \sim C(\mu, \sigma, \alpha)$ if:

(i) Y is supported on the line L passing through μ and making an angle α with the positive real axis. (Here, if $\alpha = 0$ then L is parallel to the real axis or coincides with it.)

(ii) The density of Y (with respect to arc-length) is

$$f_Y(z) = \frac{1}{\sigma\pi} \cdot \frac{1}{1 + \left|\frac{z-\mu}{\sigma}\right|^2}, \qquad z \in L. \tag{8}$$

A complex-valued random variable Z, supported on a circle $O(C, R)$, is *circular Cauchy distributed* with *location* C, *scale* R, and *eccentricity* ψ if its density function (with respect to arc-length) is given by

$$f_Z(z) = \frac{R}{2\pi} \cdot \frac{|1 - |\psi|^2|}{|z - (C + R\psi)|^2}, \qquad z \in O(C, R), \ \ (\psi \in \mathbf{C}, |\psi| \neq \mathbf{1}). \tag{9}$$

We write $Z \sim C^\star(C, R, \psi)$ in this case.

Theorem 2.4. *Let (a_n) be a second-order linear recurrence sequence, defined by $a_n = c_1\lambda_1^n + c_2\lambda_2^n$ for some non-zero complex numbers c_1, c_2, λ_1, λ_2 with $|\lambda_1| = |\lambda_2| > 0$, where λ_2/λ_1 is not a root of unity.*

1. *If $|c_1| = |c_2|$, then the Kepler measure of $(a_n)_{n=0}^\infty$ is $C\big(\mu, \sigma, \arg(\mu)\big)$, where*

$$\mu = \frac{\lambda_1 + \lambda_2}{2}, \qquad \sigma = \left|\frac{\lambda_2 - \lambda_1}{2}\right|.$$

2. *If $|c_1| \neq |c_2|$, then the Kepler measure of $(a_n)_{n=0}^\infty$ is $C^\star(C, R, \psi)$, where R and C are as in Theorem 2.2 and*

$$\psi = \frac{|c_2|}{|c_1|} \cdot \frac{||c_1|^2 - |c_2|^2|}{|c_1|^2 - |c_2|^2} \cdot \frac{\lambda_2 - \lambda_1}{|\lambda_2 - \lambda_1|}.$$

In the following theorem we identify the collection of all Kepler measures of second-order linear recurrence sequences with any fixed (infinite) Kepler set. It is analogous to Theorem 2.2. 2.

Theorem 2.5.

1. *The Cauchy measure $C(\mu, \sigma, \alpha)$ is the Kepler measure of some second-order linear recurrence sequence if and only if $\alpha = \arg(\mu)$ and*

$$\frac{1}{\pi} \arctan\left(\sigma/|\mu|\right) \notin \mathbf{Q}. \tag{10}$$

2. *The circular Cauchy distribution $C^{\star}(C, R, \psi)$ is the Kepler measure of some second-order linear recurrence sequence if and only if:*
   (a) *R¡—C—,*

   (b) *ψ lies on the circle $O(C', R')$, where we set*
   $$C' = -C/R \quad \textit{and} \quad R' = \sqrt{(|C|/R)^2 - 1},$$
   (c)
   $$\frac{1}{\pi} \arg\left(\frac{C + R/\bar{\psi}}{C + R\psi}\right) \notin \mathbf{Q}. \tag{11}$$

The Kepler measure ν is invariant under the transformation S, namely for every measurable subset A of the Kepler set we have $\nu(S^{-1}A) = \nu(A)$. Thus, we may study the ergodic-theoretical properties of the system. We recall several basic definitions and results from ergodic theory. (See [31] for more details.)

Let $(X, \mathcal{B}, \nu)$ be a probability space and $T : X \to X$ be measure-preserving. The pointwise ergodic theorem states that

$$\frac{1}{n}\sum_{k=0}^{n-1} f(T^k x) \xrightarrow[n\to\infty]{} \overline{f}(x), \qquad f \in L_1(\nu), \tag{12}$$

for almost all $x \in X$ with respect to ν, for some T-invariant function $\overline{f} \in L_1(\nu)$, namely a function satisfying $\overline{f} \circ T = \overline{f}$. T is *ergodic* if, for every T-invariant set $E \subseteq \mathcal{B}$, either $\nu(E) = 0$ or $\nu(E) = 1$. If T is ergodic, the ergodic theorem takes the simpler form

$$\frac{1}{n}\sum_{k=0}^{n-1} f(T^k x) \xrightarrow[n\to\infty]{\text{a.e.}} \int f \mathrm{d}\nu, \qquad f \in L_1(\nu). \tag{13}$$

Now let X be a compact metric space and $\mathcal{B}$ its Borel σ-field. Let T be a continuous transformation from X to itself. It is well known that there exist T-invariant probability measures on $(X, \mathcal{B})$ (see [13, Theorem 4.1.1]). Let ν be such a measure and suppose T is ergodic. A point $x_0 \in X$ is *generic* if

$$\frac{1}{n}\sum_{k=0}^{n-1} f(T^k x_0) \xrightarrow[n\to\infty]{} \int f \mathrm{d}\nu \tag{14}$$

for every continuous function $f : X \to \mathbf{C}$. A system (X, T) is *uniquely ergodic* if T admits a unique invariant probability measure. If (X, T) is uniquely ergodic, then every point $x \in X$ is generic.

Proposition 2.6. *In the setup of Theorems 2.2.* 1. (ii) *and 2.4.* 2.*, the transformation S is uniquely ergodic, and the unique S-invariant probability measure* *is the Kepler measure $\nu = C^\star(C,R,\psi)$. In particular, for every continuous function f from $O(C,R)$ to $\mathbf{C}$, we have*

$$\frac{1}{n}\sum_{k=0}^{n-1} f\left(S^k(z)\right) \xrightarrow[n\to\infty]{} \int_{O(C,R)} f\,\mathrm{d}\nu \tag{15}$$

uniformly on $O(C,R)$.

In some cases, (15) takes an especially attractive form. Recall that a function h from an open region $\Omega \subseteq \mathbf{R}^2$ into $\mathbf{R}$ is *harmonic* if it is C^2 and

$$\frac{\partial^2 h(x,y)}{\partial x^2} + \frac{\partial^2 h(x,y)}{\partial y^2} = 0, \qquad (x,y)\in\Omega.$$

A function $h : \Omega \to \mathbf{C}$ is *harmonic* if both its real part and its imaginary part are harmonic in Ω. (See [30, §11] for more details on harmonic functions.) Denote

$$D(C,R) = \{z \in \mathbf{C} :\ |z - C| \le R\},$$

and

$$D^{\mathrm{o}}(C,R) = \{z \in \mathbf{C} :\ |z - C| < R\}.$$

Theorem 2.7. *In the setup of Proposition 2.6, if $h : D(C,R) \to \mathbf{C}$ is continuous and is harmonic in $D^{\mathrm{o}}(C,R)$, then*

$$\frac{1}{N}\sum_{n=0}^{N-1} h\left(\frac{a_{n+1}}{a_n}\right) \xrightarrow[N\to\infty]{} \begin{cases} h(\lambda_1), & |c_2| < |c_1|, \\ h(\lambda_2), & |c_2| > |c_2|. \end{cases} \tag{16}$$

Consider the claim of the theorem for the case where f is the identity function. Suppose, say, that $|c_2| < |c_1|$. The theorem implies

$$\frac{1}{N}\sum_{n=0}^{N-1} \frac{c_1\lambda_1^{n+1} + c_2\lambda_2^{n+1}}{c_1\lambda_1^n + c_2\lambda_2^n} = \frac{1}{N}\sum_{n=0}^{N-1}\frac{a_{n+1}}{a_n} \xrightarrow[N\to\infty]{} \lambda_1. \tag{17}$$

When $|c_2/c_1|$ is close to 0, all terms in the sum on the left-hand side of (17) are very close to λ_1, so that one should expect the sequence to be close to λ_1. The surprising thing is that the limit is exactly λ_1. Moreover, even when $|c_2/c_1|$ becomes near (but less than) 1, so that the weight of the term $c_2\lambda_2^n$ is almost as large as that of $c_1\lambda_1^n$, the limit stays λ_1. One may say that, on average, the term $c_2\lambda_2^n$ has no effect.

3. Extended Cauchy and circular Cauchy distributions

In this section, we will first recall: (i) the basics of Möbius transformations and their dynamics, and (ii) the Cauchy distribution and its analogue on the unit circle S^1. Next, we will discuss at length the distributions defined by the density functions in (8) and (9).

3.1. Möbius Transformations

A Möbius transformation M is a mapping of the form

$$M(z) = \frac{az+b}{cz+d}, \qquad (a, b, c, d \in \mathbf{C}, ad - bc \neq 0), \tag{18}$$

of the extended complex plane $\mathbf{C}_\infty = \mathbf{C} \cup \{\infty\}$ into itself. The image of a line or a circle under a Möbius transformation is again a line or circle:

- When $|c| = |d|$, the image of S^1 under M is the line passing through $(a/c + b/d)/2$, and making an angle of $\arg\big(i(a/c - b/d)\big)$ with the positive real axis (see the proof of Theorem 3.1. (i)).
- When $|c| \neq |d|$, the image of S^1 under M is $O(C, R)$, where C and R are given by

$$C = \frac{a\bar{c} - b\bar{d}}{|c|^2 - |d|^2}, \qquad R = \left| \frac{ad - bc}{|c|^2 - |d|^2} \right|, \tag{19}$$

(see [4, § 41–44]). The set of all Möbius transformations is a group under composition. (For more delails on Möbius transformations, we refer to [4].)

3.2. The Dynamics of Möbius transformations

Let M_1 and M_2 be two Möbius transformations. The transformations are *conjugate* if $M_2 = M \circ M_1 \circ M^{-1}$ for some Möbius transformation M. Any Möbius transformation $M \neq I$ has precisely two fixed points in the extended complex plane $\mathbf{C}_\infty$, counting multiplicities [3, Theorem 2.6. 2]. If M has a single fixed point, say ζ, then $M^n(z) \xrightarrow[n\to\infty]{} \zeta$ for all $z \in \mathbf{C}_\infty$. If M has two distinct fixed points ζ_1 and ζ_2, then M is conjugate to a Möbius transformation M_a of the form

$$M_a(z) = az, \qquad z \in \mathbf{C}_\infty, \tag{20}$$

for some $a \neq 1$. Therefore, the sequence $\big(M^n(z)\big)_{n=0}^{\infty}$ either

1. converges to one of the fixed points of M, say ζ_1, for all $z \neq \zeta_2$ (corresponding to $|a| \neq 1$ in (20)), or
2. moves cyclically through a finite set of points for all z (if a is a root of unity), or
3. forms a dense subset of some line or circle (if $|a| = 1$, but a is not a root of unity).

(We refer to [3] for more details.)

3.3. Extensions and analogues of the Cauchy distribution

The *standard Cauchy distribution* is the probability distribution on the real line, defined by the density function

$$f(x) = \frac{1}{\pi(1+x^2)}, \qquad x \in \mathbf{R}. \tag{21}$$

Allowing translations of Cauchy random variables, we get the *generalized Cauchy distribution*, with a density function of the form (see [29])

$$f(x) = \frac{1}{\pi(1+(x-\mu)^2)}, \qquad x \in \mathbf{R},\ (\mu \in \mathbf{R}).$$

A further generalization is obtained by allowing a scale change as well. For real μ, σ, with $\sigma > 0$, a random variable Y is *Cauchy distributed*, with *median* μ and *scale* σ, if $(Y-\mu)/\sigma$ has a standard Cauchy distribution [32]. McCullagh [26] combined the two parameters into a single complex-valued parameter $\theta = \mu + i\sigma$ in the upper half-plane. The density function is given by

$$f_Y(x) = \frac{\sigma}{\pi|x-\theta|^2}, \qquad x \in \mathbf{R}. \tag{22}$$

In fact, McCullagh found it convenient to let σ be negative also. (The numerator on the right-hand side of (22) is then replaced by $|\sigma|$.) Note that θ and $\bar{\theta}$ give rise to the same distribution. Also, in the degenerate case $\sigma = 0$, the distribution reduces to a point mass at μ.

In Section 2, we have introduced a further generalization, allowing the distribution to be supported on any line L in $\mathbf{C}$. The median μ is now any complex number, σ is still the scale, and we add a third parameter α indicating the direction of L with respect to the positive real axis. (In principle, α ranges over $[0,\pi)$, but it will be more convenient for us to let it range over $[0,2\pi)$.) One readily checks that (8) defines the density function of this distribution.

Let Y be a Cauchy distributed random variable, with parameter $\theta = \mu + i\sigma$. Since the transformation

$$x \to \frac{ix+1}{-ix+1}, \qquad x \in \mathbf{R},$$

maps $\mathbf{R}$ into S^1, the complex-valued random variable

$$Z = \frac{iY+1}{-iY+1}$$

is supported on S^1. The density of Z is given by

$$f_Z(z) = \frac{|1-|\psi|^2|}{2\pi|z-\psi|^2}, \qquad z \in S^1, \tag{23}$$

where $\psi = (i\theta+1)/(-i\theta+1)$ (see [26]).

The distribution defined by this density function is the *circular Cauchy* distribution, and we write $Z \sim C^\star(\psi)$. (The parameter ψ was referred to as eccentricity in Section 2.)

The circular Cauchy distribution (a.k.a. the *wrapped Cauchy distribution*) is a lesser known distribution, although defined already by Lévy [24]. The family of circular Cauchy distributions is closed under Möbius transformations [26]. Moreover, circular Cauchy distributions enjoy the following properties [12, 26]:

1) $C^\star(0)$ is the uniform measure on $S^{1\cdot}$

2) For every ψ, the distributions $C^\star(\psi)$ and $C^\star(1/\overline{\psi})$ coincide. Thus, it suffices to consider ψ-s in the unit disc.

3) As $|\psi|$ increases from 0 to 1, the distribution deviates more and more from the uniform distribution and becomes concentrated near $\psi/|\psi|$. As $\psi \to \psi_0$ for some $\psi_0 \in S^1$, the distribution converges to a point mass at ψ_0.

4) If M is the Möbius transformation defined by $M(z) = \beta_0 z$ with $\beta_0 \in S^1$, then $M\big(C^\star(\psi)\big) = C^\star(\beta_0\psi)$.

5) If M is the Möbius transformation defined by $M(z) = (z + \beta_1)/(\bar{\beta}_1 z + 1)$ with $\beta_1 \in \mathbf{C}$, then $M\big(C^\star(\psi)\big) = C^\star\big((\psi + \beta_1)/(\bar{\beta}_1\psi + 1)\big)$.

6) If $M(z)$ is the Möbius transformation defined by $M(z) = \beta_0\cdot(z+\beta_1)/(\bar{\beta}_1 z+1)$ with $\beta_0 \in S^1$ and $\beta_1 \in \mathbf{C}$, then $M\big(C^\star(0)\big) = C^\star(\beta_0\beta_1)$. (This property follows directly from the two preceding ones, but it will be convenient to have it handy.)

7) If Z_1 and Z_2 are independent and $Z_1 \sim C^\star(\psi_1)$, $Z_2 \sim C^\star(\psi_2)$ with $|\psi_1|$, $|\psi_2| \leq 1$, then $Z_1 Z_2 \sim C^\star(\psi_1\psi_2)$.

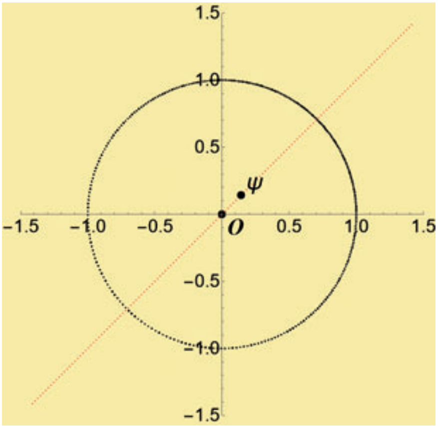
The following example helps understanding the property 3) better.

Example. In Figure 3, we have "depicted" $C^\star(\psi)$ for four values of the parameter $\psi_l = 0.2l \cdot \exp(i\pi/4)$, $1 \leq l \leq 4$. We have started with the 300 points $\exp(2\pi ik/300)$, $0 \leq k \leq 299$. The (discrete) uniform distribution over these 300 points approximates the uniform distribution over S^1, which is $C^\star(0)$. By the property 5), the measure $C^\star(0)$ is taken under the Möbius transformation

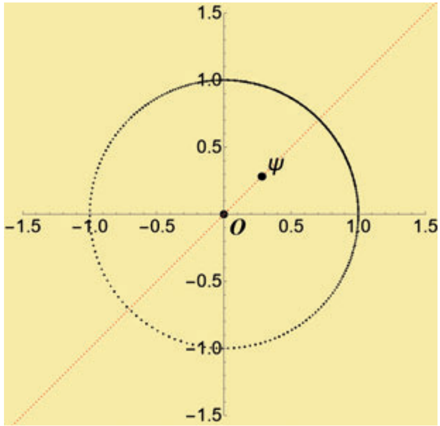
$$M_\psi(z) = (z + \psi)/(\bar{\psi}z + 1), \qquad z \in \mathbf{C}_\infty, \quad \text{to } C^\star(\psi).$$

Thus, for each of the four values above of ψ, we have drawn the images of those 300 points under M_ψ. The uniform measure over these points is an approximation of $C^\star(\psi)$.

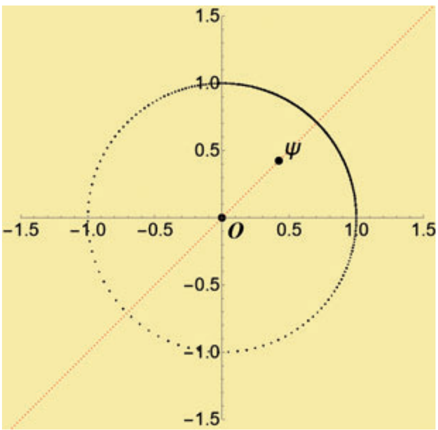
For more information on the circular Cauchy distribution, see [12, 24–26, 34].



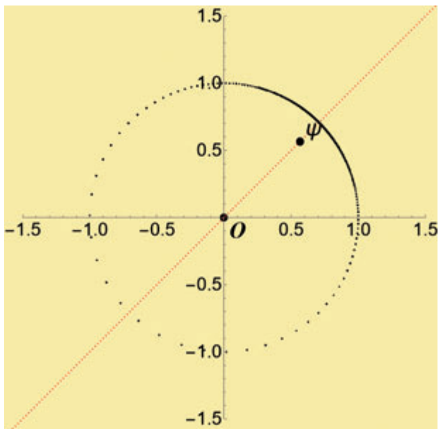
(A) $\psi = 0.2\exp(i\pi/4)$.



(B) $\psi = 0.4\exp(i\pi/4)$.



(C) $\psi = 0.6\exp(i\pi/4)$.



(D) $\psi = 0.8\exp(i\pi/4)$.

FIGURE 3. The distribution $C^\star(\psi)$ for several values of ψ. Note how $|\psi|$ and $\arg(\psi)$ are reflected by the figures.

In Section 2, we have defined a generalization of the circular Cauchy distribution, allowing the distribution to be supported on any circle in $\mathbf{C}$. We added two new parameters; *location* – the center of the circle, and *scale* – the radius of the circle. The eccentricity means the same as in the case of an S^1-supported circular Cauchy distribution. One readily checks that (9) defines the density function of this distribution.

Our main result in this section specifies how Möbius transformations act on the distribution $C^\star(0)$. We will explain below that it allows finding how they act on any $C(\mu,\sigma,\alpha)$ and $C^\star(C,R,\psi)$.

Theorem 3.1. *Let M be any Möbius transformation:*

$$M(z)=\frac{az+b}{cz+d},\qquad z\in\mathbf{C}_\infty,\quad (a,b,c,d\in\mathbf{C},\ ad-bc\neq 0).$$

(i) *If $|c|=|d|$, then*

$$M\big(C^\star(0)\big)=C(\mu,\sigma,\alpha),$$

where

$$\mu=\frac{1}{2}\left(\frac{a}{c}+\frac{b}{d}\right),\qquad \sigma=\frac{1}{2}\left|\frac{a}{c}-\frac{b}{d}\right|,\qquad \alpha=\arg\left(i\left(\frac{a}{c}-\frac{b}{d}\right)\right).$$

(ii) *If $|c|\neq|d|$, then*

$$M\big(C^\star(0)\big)=C^\star(C,R,\psi),$$

where C and R are as in (19) *and*

$$\psi=\frac{\big||c|^2-|d|^2\big|}{|c|^2-|d|^2}\cdot\frac{bc-ad}{|bc-ad|}\cdot\frac{\bar c}{d}.$$

Proof.

(i) In this case $d\neq 0$, so we may rewrite M in the form $M=M_1\circ M_2$, where

$$M_1(z)=\frac{\frac{a}{c}z+\frac{b}{d}}{z+1},\qquad M_2(z)=\frac{c}{d}z,\qquad z\in\mathbf{C}_\infty.$$

By the property 4) above

$$M\big(C^\star(0)\big)=M_1\circ M_2\big(C^\star(0)\big)=M_1\big(C^\star(0)\big). \tag{24}$$

The Möbius transformation

$$T_2(z)=\frac{iz+1}{-iz+1},\qquad z\in\mathbf{C}_\infty,$$

maps $\mathbf{R}\cup\{\infty\}$ onto S^1. Therefore, the image of S^1 under M_1 is same as the image of $\mathbf{R}\cup\{\infty\}$ under $M_1\circ T_2$:

$$\begin{aligned} M_1\circ T_2(x)&=\frac{\frac{a}{c}\left(\frac{ix+1}{-ix+1}\right)+\frac{b}{d}}{\frac{ix+1}{-ix++1}+1}\\ &=\frac{1}{2}\left(\frac{a}{c}+\frac{b}{d}\right)+i\frac{1}{2}\left(\frac{a}{c}-\frac{b}{d}\right)x\\ &=\mu+\sigma_1 x,\qquad x\in\mathbf{R}\cup\{\infty\}, \end{aligned}\tag{25}$$

where $\mu=\frac{1}{2}\left(\frac{a}{c}+\frac{b}{d}\right)$ and $\sigma_1=i\left(\frac{a}{c}-\frac{b}{d}\right)$. By (25), the image of $\mathbf{R}$ under $M_1\circ T_2$ is the line L passing through the point μ and making an angle of $\alpha=\arg(\sigma_1)$

with the positive real axis. By [26] and [5, p.83], under the transformation T_2, the image of the standard Cauchy distribution $C(0,1,0)$ is $C^\star(0)$. Hence we may rewrite (24) in the form

$$M\big(C^\star(0)\big) = M_1 \circ T_2\big(C(0,1,0)\big). \tag{26}$$

By (21) and (25), the density function f_1 of the distribution $M_1 \circ T_2\big(C(0,1,0)\big)$ on the line L is

$$\begin{aligned} f_1(z) &= f\left((M_1 \circ T_2)^{-1}(z)\right) \left|\frac{\mathrm{d}}{\mathrm{d}z}(M_1 \circ T_2)^{-1}(z)\right| \\ &= \frac{1}{\pi} \cdot \frac{1}{1 + |(z-\mu)/\sigma_1|^2} \left|\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{z-\mu}{\sigma_1}\right)\right| \\ &= \frac{1}{\pi|\sigma_1|} \cdot \frac{1}{1 + |(z-\mu)/\sigma_1|^2}, \qquad z \in L. \end{aligned} \tag{27}$$

Hence, by (8), (26), and (27), we have $M\big(C^\star(0)\big) = C(\mu, \sigma, \alpha)$, where $\sigma = |\sigma_1|$.

(ii) Define a Möbius transformation M_3 by

$$M_3(z) = \frac{z - C}{R}, \qquad z \in \mathbf{C}_\infty. \tag{28}$$

By (19), the Möbius transformation $M_3 \circ M$ maps S^1 onto itself. By (19) and (28), we have for $z \in \mathbf{C}$,

$$\begin{aligned} M_3 \circ M(z) &= \left(\frac{az+b}{cz+d} - \frac{a\bar{c} - b\bar{d}}{|c|^2 - |d|^2}\right) \cdot \frac{||c|^2 - |d|^2|}{|bc - ad|} \\ &= \frac{||c|^2 - |d|^2|}{|c|^2 - |d|^2} \cdot \frac{1}{|bc - ad|} \cdot \frac{(az+b)(|c|^2 - |d|^2) - (cz+d)(a\bar{c} - b\bar{d})}{cz+d} \\ &= \frac{||c|^2 - |d|^2|}{|c|^2 - |d|^2} \cdot \frac{bc - ad}{|bc - ad|} \cdot \frac{\bar{d}z + \bar{c}}{cz+d} \\ &= \frac{||c|^2 - |d|^2|}{|c|^2 - |d|^2} \cdot \frac{bc - ad}{|bc - ad|} \cdot \frac{\bar{d}}{d} \cdot \frac{z + \bar{c}/\bar{d}}{(c/d)z + 1}. \end{aligned} \tag{29}$$

By the property 6) above and (29),

$$M_3 \circ M\big(C^\star(0)\big) = C^\star(\psi), \tag{30}$$

where $\psi = \frac{||c|^2-|d|^2|}{|c|^2-|d|^2} \cdot \frac{bc-ad}{|bc-ad|} \cdot \frac{\bar{c}}{d}$. By (30),

$$M\big(C^\star(0)\big) = M_3^{-1} C^\star(\psi). \tag{31}$$

The density function f_2 of the distribution $C^\star(\psi)$ may be written in the form

$$f_2\big(\exp(i\theta)\big) = \frac{|1-|\psi|^2|}{2\pi|\exp(i\theta)-\psi|^2}, \qquad \theta \in [0,2\pi). \tag{32}$$

By (32), the density function f_3 of the distribution $M_3^{-1}\big(C^\star(\psi)\big)$ on $O(C,R)$ is

$$\begin{aligned} f_3\big(C+R\exp(i\theta)\big) &= f_2\Big(M_3\big(C+R\exp(i\theta)\big)\Big)\left|\frac{1}{R}\cdot\frac{\mathrm{d}}{\mathrm{d}\theta}\Big(M_3\big(C+R\exp(i\theta)\big)\Big)\right| \\ &= \frac{|1-|\psi|^2|}{2\pi\left|\exp(i\theta)-\psi\right|^2}\cdot\left|\frac{1}{R}\cdot\frac{\mathrm{d}}{\mathrm{d}\theta}\big(\exp(i\theta)\big)\right| \\ &= \frac{|1-|\psi|^2|}{2\pi\left|\exp(i\theta)-\psi\right|^2}\cdot\frac{1}{R}, \qquad \theta\in[0,2\pi). \end{aligned} \tag{33}$$

Hence, by (9), (31), and (33), we have

$$M\big(C^\star(0)\big) = M_3^{-1}\big(C^\star(\psi)\big) = C^\star(C,R,\psi). \qquad \square$$

Note that we have stated in Theorem 3.1 only to what measure the uniform distribution $C^\star(0)$ on S^1 is mapped under any Möbius transformation. This allows us finding the image of any Cauchy measure and circular Cauchy measure. Indeed, let L be a line in $\mathbf{C}$, endowed with measure $C(\mu,\sigma,\alpha)$, and M any Möbius transformation. By Theorem 3.1. (i), letting $M_{\mu,\sigma,\alpha}(z)=(az+b)/(z+1)$, where $a=\mu+\exp\big(i(\alpha-\pi/2)\big)\sigma$ and $b=\mu-\exp\big(i(\alpha-\pi/2)\big)\sigma$, we have

$$M_{\mu,\sigma,\alpha}\big(C^\star(0)\big) = C(\mu,\sigma,\alpha).$$

Therefore, $M\big(C(\mu,\sigma,\alpha)\big) = M\circ M_{\mu,\sigma,\alpha}\big(C^\star(0)\big)$. Similarly, let $O(C,R)$ be any circle, endowed with the measure $C^\star(C,R,\psi)$. It follows from the property 6) and Theorem 3.1. (ii) that

$$M\big(C^\star(C,R,\psi)\big) = M\circ M_3^{-1}M_\psi\big(C^\star(0)\big),$$

where $M_3(z)=(z-C)/R$ and $M_\psi(z)=(z+\psi)/(\bar{\psi}z+1)$.

We mention in passing that the circular Cauchy (as well as its generalization to higher-dimensional spheres [5, 6, 9, 11]) distribution is usually used to model angular data. Examples of such data are migration of turtles [28], orientation of ants towards a black target [1, 28], and wind direction data [1, 12]. The data lies in $[0,2\pi]$, but it makes little sense to treat it as real data in this interval. It is more natural to view it as data on the unit circle in the plane. Thus, viewing the data as corresponding to complex numbers on the unit circle is a matter of convenience, but the data is not really complex-valued. In our case, though, the circular Cauchy distribution really describes the behaviour of complex-valued data.

4. Discussion and Intuition

Suppose we fix λ_1 and λ_2, with $|\lambda_1| = |\lambda_2|$ and λ_2/λ_1 not a root of unity. What can be said about the Kepler sets of all sequences $a_n = c_1\lambda_1^n + c_2\lambda_2^n$, as c_1, c_2 vary over $\mathbf{C}$? By Remark 1, only the absolute ratio $r = |c_2/c_1|$ plays a role in determining the Kepler set. Without loss of generality, we may therefore restrict our attention to (a_n) of the form

$$a_n = \lambda_1^n + r\lambda_2^n, \qquad n \geq 0, \ \bigl(r \in (0, \infty)\bigr).$$

Let us describe how the Kepler set changes as r varies from 0 to ∞. For $r \approx 0$, by (7), the circle $O(C, R)$ has its center close to λ_1 and its radius close to 0. Indeed,

$$\frac{a_{n+1}}{a_n} = \frac{\lambda_1^{n+1} + r\lambda_2^{n+1}}{\lambda_1^n + r\lambda_2^n} \approx \frac{\lambda_1^{n+1}}{\lambda_1^n} = \lambda_1,$$

which shows that the consecutive ratios become close to λ_1.

Rewrite the formula for C in (7) in the form:

$$\lambda_1 = (1 - r^2)\, C + r^2\lambda_2 \tag{34}$$

Thus, for $0 < r < 1$, it follows from (34) that λ_1 is a convex combination of C and λ_2. Put differently, C lies on the line passing through λ_1 and λ_2, so that λ_1 is between λ_2 and C. It will follow from the proof of Theorem 2.2 that the Kepler set of $(a_n)_{n=0}^{\infty}$ is the image of S^1 under the Möbius transformation $T(z) = \frac{r\lambda_2 z + \lambda_1}{rz+1}$. The Möbius transformation defined by

$$M_3(z) = \frac{1}{R}(z - C), \qquad z \in \mathbf{C}_\infty,$$

maps $O(C, R)$ to S^1. By (19),

$$\begin{aligned} M_3\bigl(T(z)\bigr) &= \frac{|1 - r^2|}{r|\lambda_2 - \lambda_1|} \cdot \left(\frac{\lambda_1 + r\lambda_2 z}{1 + rz} - \frac{\lambda_1 - r^2\lambda_2}{1 - r^2}\right) \\ &= \frac{|1 - r^2|}{1 - r^2} \cdot \frac{\lambda_2 - \lambda_1}{|\lambda_2 - \lambda_1|} \cdot \frac{z + r}{rz + 1} \\ &= \alpha \cdot \frac{z + r}{rz + 1}, \qquad z \in S^1, \end{aligned}$$

where $\alpha = \frac{|1-r^2|}{1-r^2} \cdot \frac{\lambda_2 - \lambda_1}{|\lambda_2 - \lambda_1|} \in S^1$. Since M_3 maps $D^{\circ}(C, R)$ onto $D^{\circ}(0, 1)$, and the complement of $D^{\circ}(C, R)$ onto the complement of $D^{\circ}(0, 1)$, we have

$$M_3\bigl(T(0)\bigr) = \alpha r \quad \text{and} \quad \bigl|M_3\bigl(T(0)\bigr)\bigr| = r < 1.$$

Hence, $T(0) = \lambda_1$ is inside $O(C, R)$ and $T(\infty) = \lambda_2$ is outside.

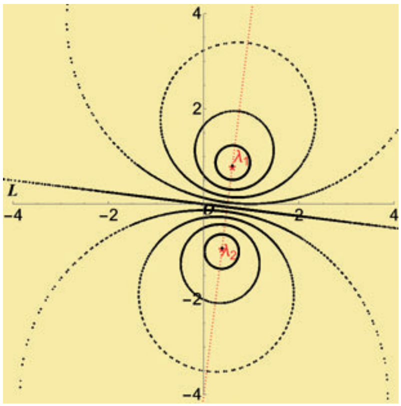


FIGURE 4. The Kepler sets of the sequences $a_n = \lambda_1^n + r\lambda_2^n$, where $\lambda_1 = (3+4i)/5$, $\lambda_2 = (5-12i)/13$, with various values of r. For $r = 1$, the Kepler set is the line L. For $r = 0.2, 0.4, 0.6, 0.8$ we get four circles above L, and for $r = 1/0.2, 1/0.4, 1/0.6, 1/0.8$ — their mirror images with respect to L.

As r increases from 0 to 1, the center gets further away from λ_1 and the radius increases. As $r \to 1^-$, both the center and the radius tend to ∞. By Theorem 2.4, in the process, $|\psi|$ increases from 0 to 1, while $\arg(\psi) = \arg(\lambda_2 - \lambda_1)$ remains fixed. Thus, the distribution we get on $O(C,R)$ changes gradually from a uniform measure (in the limit as $r \to 0$) to become more and more concentrated near the intersection of $O(C,R)$ with the interval $[\lambda_1, \lambda_2]$. The density on $O(C,R)$ is largest at the point closest to λ_2, and becomes smaller as we get further from λ_2. (See Figure 4.)

The circles we obtain, considered as circles in the Riemann sphere, are pairwise disjoint and converge to a great circle as $r \to 1^-$. This great circle is the Kepler set for $r = 1$, which is the line mean perpendicular to the interval $[\lambda_1, \lambda_2]$ (see Figure 4).

For $r > 1$, the situation is similar. The Kepler set of $(a_n)_{n=0}^{\infty}$ is the same as that of $\frac{1}{r}a_n = \lambda_2^n + \frac{1}{r}\lambda_1^n$, so that we get the same picture as for $r < 1$, where λ_1, λ_2 switch roles.

REMARK 3. Since the Kepler sets are invariant under the transformation S, the transformation S partitions $\mathbf{C}_\infty$ into disjoint S-invariant circles and an S--invariant line. Thus, S is a *semi-simple* transformation according to the terminology in [33, p. 121].

We explain intuitively why only $|c_2/c_1|$ is important. Shifting the sequence, we may consider the sequence $a_{n+k} = c_1\lambda_1^k\lambda_1^n + c_2\lambda_2^k\lambda_2^n$ with arbitrary k instead of a_n. Thus, the coefficients c_1, c_2 give rise to the same Kepler set as do the coefficients $c_1\lambda_1^k$, $c_2\lambda_2^k$ for any k. Taking a sequence $(k_i)_{i=0}^{\infty}$ for which $(c_2/c_1)(\lambda_2/\lambda_1)^{k_i} \xrightarrow[i\to\infty]{} |c_2/c_1|$, we see that c_1, c_2 may be replaced by $|c_1|, |c_2|$.

5. Proofs

Proof of Lemma 2.1. The characteristic polynomial of $(a_n)_{n=0}^{\infty}$ is given by $f(x) = x^2 - cx - d$, and its roots are $\lambda_{1,2} = (c \pm \sqrt{c^2+4d})/2$. If $c = 0$ or $c^2 + 4d = 0$, then λ_2/λ_1 is a root of unity. Also, λ_1 and λ_2 cannot be both real, as otherwise the equality $|\lambda_1| = |\lambda_2|$ would imply that $c = 0$.

Since $|\lambda_1| = |\lambda_2|$, the line segment $[\lambda_1, \lambda_2]$ is perpendicular to $[0, c/2]$. It follows that $\sqrt{c^2+4d}/c \in i\mathbf{R}$, so that $(c^2+4d)/c^2 \in \{x : x < 0\}$. Hence $d < -c^2/4$.

Write $a_n = c_1\lambda_1^n + c_2\lambda_2^n$ for suitable non-zero c_1, c_2, depending on the initial values a_0, a_1. Suppose that $a_n = 0$ for some n. Then

$$(\lambda_2/\lambda_1)^n = (-c_1/c_2). \tag{35}$$

If λ_2/λ_1 is not a root of unity, then (35) can have at most one solution $n \in \mathbf{Z}$. $\square$

Proof of Theorem 2.2. 1. We have

$$\frac{a_{n+1}}{a_n} = \lambda_1 \cdot \frac{c_2/c_1 \cdot (\lambda_2/\lambda_1)^{n+1} + 1}{c_2/c_1 \cdot (\lambda_2/\lambda_1)^n + 1}. \tag{36}$$

Since λ_2/λ_1 is not a root of unity, the sequence $\left((\lambda_2/\lambda_1)^n\right)_{n=0}^{\infty}$ is dense in S^1 (see [31, Proposition 1.3.4]). Therefore, the Kepler set of $(a_n)_{n=0}^{\infty}$ is

$$K = \left\{\lambda_1 \cdot \frac{(c_2/c_1)(\lambda_2/\lambda_1)z + 1}{(c_2/c_1)z + 1} : z \in S^1\right\}. \tag{37}$$

We now continue separately in the two cases $|c_1| = |c_2|$ and $|c_1| \neq |c_2|$.

If $|c_1| = |c_2|$, as z varies over S^1, so does $(c_2/c_1)z$. Hence we may write K in a simpler form

$$K = \left\{\frac{\lambda_2 z + \lambda_1}{z+1} : z \in S^1\right\}. \tag{38}$$

It follows that $K = T_1(S^1)$, where T_1 is the Möbius transformation given by

$$T_1(z) = \frac{\lambda_2 z + \lambda_1}{z+1}, \qquad z \in \mathbf{C}_\infty. \tag{39}$$

Let T_2 be the Möbius transformation defined by

$$T_2(z) = \frac{iz+1}{-iz+1}, \qquad z \in \mathbf{C}_\infty, \tag{40}$$

and notice that it takes $\mathbf{R} \cup \{\infty\}$ onto S^1. Thus, $K = T_1 \circ T_2(\mathbf{R} \cup \{\infty\})$. Now

$$T_1 \circ T_2(x) = \frac{\lambda_1 + \lambda_2}{2}\left(i \cdot \frac{\lambda_2 - \lambda_1}{\lambda_1 + \lambda_2} x + 1\right), \qquad x \in \mathbf{R} \cup \{\infty\}. \tag{41}$$

Since $i \cdot \frac{\lambda_2 - \lambda_1}{\lambda_1 + \lambda_2}$ is real, the image of $\mathbf{R} \cup \{\infty\}$ under $T_1 \circ T_2$ is the line passing through the origin and the point $(\lambda_1 + \lambda_2)/2$.

Now consider the case $|c_1| \neq |c_2|$. Let $c_2/c_1 = r\exp(i\phi)$ for some $r > 0$ and $\phi \in [0, 2\pi)$, with $r \neq 1$. Replacing z in (37) by $\exp(i\phi)z$, we see that

$$K = \left\{\frac{r\lambda_2 z + \lambda_1}{z+1} : z \in S^1\right\}.$$

Denoting by T the Möbius transformation given by

$$T(z) = \frac{r\lambda_2 z + \lambda_1}{rz+1}, \qquad z \in \mathbf{C}_\infty, \tag{42}$$

we obtain $K = T(S^1)$. By [4, §41–44] (see the result mentioned in Subsection 3.1), $T(S^1) = O(C, R)$, where the center C and radius R are given by

$$C = \frac{\lambda_1 - r^2\lambda_2}{1 - r^2} = \frac{|c_1|^2\lambda_1 - |c_2|^2\lambda_2}{|c_1|^2 - |c_2|^2}, \tag{43}$$

and

$$R = \frac{|r\lambda_1 - r\lambda_2|}{|1 - r^2|} = \frac{|c_1||c_2||\lambda_1 - \lambda_2|}{||c_1|^2 - |c_2|^2|}. \tag{44}$$

A routine calculation shows now that

$$R^2 + |\lambda_1|^2 = |C|^2. \tag{45}$$

Thus, $R < |C|$, so that that the origin is outside $O(C, R)$.

2. Let L be a line passing through the origin and making an angle θ with the positive real axis. Choose $\lambda_1 = \exp(i(\theta - \varepsilon))$ and $\lambda_2 = \exp(i(\theta + \varepsilon))$ for some $\varepsilon \in (0, \pi/2)$ which is not a rational multiple of π. By part 1. (i), the line L is the Kepler set of the sequence $(a_n)_{n=0}^\infty$, given by $a_n = \lambda_1^n + \lambda_2^n$ for each n.

Let $O(C, R)$ be a circle such that $R < |C|$. We need to construct a second--order linear recurrence sequence $(a_n)_{n=0}^\infty$ whose Kepler set is $O(C, R)$. It is easily seen that the circles $O(0, \sqrt{|C|^2 - R^2})$ and $O(C, R)$ intersect orthogonally. (Refer to Figure 5 in what follows.) Draw through C a line intersecting $O(0, \sqrt{|C|^2 - R^2})$ at two points. Let λ_1 be the intersection point closer to C, and λ_2 – the one farther away. Perturbing this line if necessary, we may assume that λ_2/λ_1 is not a root of unity. We can write λ_1 as a convex combination of λ_2 and C,

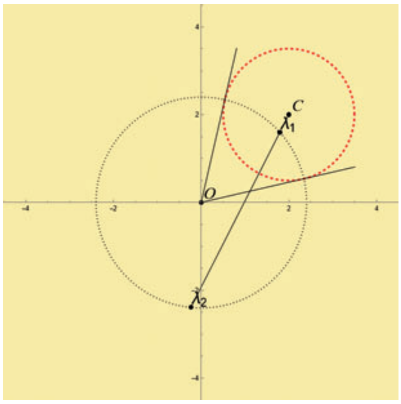


FIGURE 5. Construction of a second-order recurrence sequence whose Kepler set is the dashed red circle $O(2+2i, 3/2)$. The roots need to lie on the dotted black circle $O(0, \sqrt{|2+2i|^2 - (3/2)^2}) = O(0, \sqrt{23}/2)$.

say $\lambda_1 = (1-r^2)C + r^2\lambda_2$ for some $r \in (0,1)$. Since $|\lambda_1|^2 = |\lambda_2|^2 = |C|^2 - R^2$, we have $R = r\frac{|\lambda_1-\lambda_2|}{1-r^2}$. It follows from Theorem 2.2. 1. (ii) and (42), (43), (44) that $O(C,R)$ is the Kepler set of the sequence $(a_n)_{n=0}^{\infty}$, given by $a_n = \lambda_1^n + r\lambda_2^n$ for each n. $\square$

In the next proof, we will make use of

PROPOSITION 5.1 ([23, p.178]). *Let $(x_n)_{n=0}^{\infty}$ be a sequence in a compact Hausdorff space X, distributed according to some probability measure ν on X. Let Y be another compact Hausdorff space and $f : X \to Y$ a continuous function. Then the sequence $(f(x_n))_{n=0}^{\infty}$ is distributed in Y according to $f(\nu)$.*

The proof is immediate.

Proof of Theorem 2.4. 1. Refer again to (36):

$$\frac{a_{n+1}}{a_n} = \lambda_1 \cdot \frac{c_2/c_1 \cdot (\lambda_2/\lambda_1)^{n+1} + 1}{c_2/c_1 \cdot (\lambda_2/\lambda_1)^n + 1}. \tag{36}$$

The sequence $(c_2/c_1 \cdot (\lambda_2/\lambda_1)^n)_{n=0}^{\infty}$ is distributed according to the uniform measure $C^\star(0)$ on S^1 (see [23, Example 2.1]). By Proposition 5.1, the Kepler measure of the sequence $(a_n)_{n=0}^{\infty}$ is $T_1\big(C^\star(0)\big)$, where $T_1(z) = \frac{\lambda_2 z + \lambda_1}{z+1}$. From Theorem 3.1 we obtain $T_1\big(C^\star(0)\big) = C(\mu, \sigma, \alpha)$, with $\mu = \frac{\lambda_1+\lambda_2}{2}$, $\sigma = \left|\frac{\lambda_2-\lambda_1}{2}\right|$, and $\alpha = \arg\big(i(\lambda_1-\lambda_2)\big) = \arg(\lambda_1+\lambda_1) = \arg(\mu)$.

2. Rewrite (36) in the form

$$\frac{a_{n+1}}{a_n} = \frac{r \cdot \exp(i\theta) \cdot \lambda_2 \left(\lambda_2/\lambda_1\right)^n + \lambda_1}{r \cdot \exp(i\theta) \cdot \left(\lambda_2/\lambda_1\right)^n + 1},$$

where $c_2/c_1 = r\exp(i\theta)$ for some $r > 0$ and $\theta \in (0, 2\pi)$. Thus the sequence $\big(\exp(i\theta) \cdot (\lambda_2/\lambda_1)^n\big)_{n=0}^{\infty}$ is distributed according to $C^\star(0)$ on S^1. By Proposition 5.1, Theorem 3.1, (42), (43), and (44), the Kepler measure of the sequence $(a_n)_{n=0}^{\infty}$ is $T\big(C^\star(0)\big) = C^\star(C, R, \psi)$, where

$$T(z) = \frac{r\lambda_2 z + \lambda_1}{rz + 1}, \quad \text{and} \quad \psi = r \cdot \frac{|1 - r^2|}{1 - r^2} \cdot \frac{\lambda_2 - \lambda_1}{|\lambda_2 - \lambda_1|}. \qquad \square$$

Proof of Theorem 2.5. 1. Only if: Let $(a_n)_{n=0}^{\infty}$ be a second-order linear recurrence sequence with Kepler measure $C(\mu, \sigma, \alpha)$. Let λ_1 and λ_2 be the roots of the characteristic polynomial of $(a_n)_{n=0}^{\infty}$. It follows from the discussion in Sections 1 and 2 that $|\lambda_1| = |\lambda_2|$ and λ_2/λ_1 is not a root of unity. By Theorem 2.2. 1. (i), we have $\mu = (\lambda_1 + \lambda_2)/2$, $\sigma = |(\lambda_1 - \lambda_2)/2|$ and $\alpha = \arg(\mu)$. The triangle $\triangle O\lambda_1\lambda_2$ (where O denote the origin) is an isosceles triangle. The line passing through O and μ bisects the side $[\lambda_1, \lambda_2]$ of the triangle. Hence, this line is also a height of the triangle. It follows that $\lambda_1 = \mu + i(\mu/|\mu|)\sigma$ and $\lambda_2 = \mu - i(\mu/|\mu|)\sigma$. We have

$$\frac{\lambda_2}{\lambda_1} = \frac{1 - i\sigma/|\mu|}{1 + i\sigma/|\mu|} = \exp(-2i\arctan(\sigma/|\mu|)).$$

Since λ_2/λ_1 is not a root of unity, $\arctan(\sigma/|\mu|)$ is not a rational multiple of π.
If: Choose $\lambda_1 = \mu + i(\mu/|\mu|)\sigma$ and $\lambda_2 = \mu - i(\mu/|\mu|)\sigma$. We have

$$\frac{1}{\pi}\arg\left(\lambda_2/\lambda_1\right) = \frac{-2}{\pi}\arctan\left(\sigma/|\mu|\right) \notin \mathbf{Q}.$$

It follows from Proof of Theorem 2.4. 1. that the Kepler measure of the sequence $(a_n)_{n=0}^{\infty}$, given by $a_n = \lambda_1^n + \lambda_2^n$ for $n \geq 0$, is $C(\mu, \sigma, \alpha)$.
2. Since $C^\star(C, R, \psi) = C^\star(C, R, 1/\bar{\psi})$ for all $\psi \in \mathbf{C}$ with $|\psi| \neq 1$, it suffices to prove the theorem for $|\psi| < 1$.
Only if: Let $(a_n)_{n=0}^{\infty}$ be a second-order linear recurrence sequence with Kepler measure $C^\star(C, R, \psi)$. Let λ_1 and λ_2 be the roots of the characteristic polynomial of $(a_n)_{n=0}^{\infty}$. As in the first part of the proof, $|\lambda_1| = |\lambda_2|$ and λ_2/λ_1 is not a root of unity. Write a_n in the form

$$a_n = c_1\lambda_1^n + c_2\lambda_2^n, \qquad n = 0, 1, 2, \dots,$$

for suitable non-zero c_1 and c_2 with $|c_2| \neq |c_1|$. Suppose, say, that, $|c_2| < |c_1|$. Put $r = |c_2/c_1| < 1$. By Theorem 2.4. 2,

$$\psi = r\frac{\lambda_2 - \lambda_1}{|\lambda_2 - \lambda_1|}. \tag{46}$$

By (7) and (46),

$$\begin{aligned}\left|\frac{\psi - C'}{R'}\right| &= \frac{1}{R'}\left|\frac{r(\lambda_2-\lambda_1)}{|\lambda_2-\lambda_1|} + \frac{C}{R}\right| \\ &= \frac{1}{RR'}\left|\frac{rR(\lambda_2-\lambda_1)}{|\lambda_2-\lambda_1|} + C\right| \\ &= \frac{1}{|\lambda_1|}\left|\frac{r^2}{1-r^2}(\lambda_2-\lambda_1) + \frac{\lambda_1 - r^2\lambda_2}{1-r^2}\right| \\ &= \frac{1}{|\lambda_1}\left|\frac{(1-r^2)\lambda_1}{(1-r^2)}\right| = 1. \qquad (47)\end{aligned}$$

Hence ψ lies on the circle $O(C', R')$. By (7), Theorem 2.4.2, and (46), we have $C + R\psi = \lambda_1$ and $C + R/\bar{\psi} = \lambda_2$. Hence $\frac{1}{\pi}\arg(\lambda_2/\lambda_1) = \frac{1}{\pi}\arg\left(\frac{C+R/\bar{\psi}}{C+R\psi}\right) \notin \mathbf{Q}$.

If: Let $\lambda_1 = C + R\psi$ and $\lambda_2 = C + R/\bar{\psi}$. Since ψ lies on the circle $O(C', R')$, we have

$$\begin{aligned}C\bar{\psi} + \bar{C}\psi &= R(-C'\bar{\psi} - \bar{C}'\psi) \\ &= R\left(|C'-\psi|^2 - |C'|^2 - |\psi|^2\right) \\ &= R\left(R'^2 - |C'|^2 - |\psi|^2\right) \\ &= R(-1-|\psi|^2). \qquad (48)\end{aligned}$$

By (48),

$$\begin{aligned}\left|\frac{\lambda_2}{\lambda_1}\right|^2 &= \frac{|C + R/\bar{\psi}|^2}{|C + R\psi|^2} \\ &= \frac{|C|^2 + R^2/|\psi|^2 + R(C\bar{\psi} + \bar{C}\psi)/|\psi|^2}{|C|^2 - R^2} \\ &= \frac{|C|^2 + R^2/|\psi|^2 + R(-R - R|\psi|^2)/|\psi|^2}{|C|^2 - R^2} \\ &= 1. \qquad (49)\end{aligned}$$

Hence $|\lambda_1| = |\lambda_2|$, and from (11) it follows that λ_2/λ_1 is not a root of unity. Define a linear recurrence sequence $(a_n)_{n=0}^{\infty}$ by

$$a_n = \lambda_1^n + |\psi|\lambda_2^n, \qquad n = 0, 1, 2, \ldots$$

By Theorem 2.4. 2. the Kepler measure of $(a_n)_{n=0}^{\infty}$ is $C^\star(C, R, \psi)$. $\square$

P r o o f o f P r o p o s i t i o n 2.6. Define the Möbius transformations R and T by

$$R(z) = \frac{\lambda_2}{\lambda_1}z, \qquad T(z) = \frac{r\lambda_2 z + \lambda_1}{z+1}, \quad \text{for } z \in \mathbf{C}_\infty.$$

The uniform measure on S^1 is the unique R-invariant probability measure [31, Proposition 6.4.4]. By (5) and (36), the diagram

$$\begin{tikzcd} S^1 \arrow[r, "R"] \arrow[d, "T"] & S^1 \arrow[d, "T"] \\ O(C,R) \arrow[r, "S"] & O(C,R) \end{tikzcd} \tag{50}$$

is commutative. Therefore, $\nu = C^\star(C, R, \psi)$ is the unique S-invariant measure [13, p.145], where ψ is as in Theorem 2.4. By (14), for every continuous function $f : O(C, R) \to \mathbf{C}$,

$$\frac{1}{N}\sum_{n=0}^{N-1} f\big(S^n(x)\big) \xrightarrow[n\to\infty]{} \int_{O(C,R)} f \mathrm{d}\nu \tag{51}$$

uniformly on $O(C, R)$. □

P r o o f o f T h e o r e m 2.7. It is enough to prove the theorem for real-valued h. Let f_3 be as in the proof of Theorem 3.1. By (33) and (51), for $z \in O(C, R)$

$$\begin{aligned} \frac{1}{N}\sum_{n=0}^{N-1} h\big(S^n(z)\big) &\xrightarrow[n\to\infty]{} \int_{O(C,R)} h \mathrm{d} f_3 \\ &= \int_0^{2\pi} h\big(C + R\exp(i\theta)\big) f_3\big(C + R\exp(i\theta)\big) R \cdot \mathrm{d}\theta \\ &= \int_0^{2\pi} h\big(C + R\exp(i\theta)\big) \frac{|1-|\psi|^2|}{2\pi\,|\exp(i\theta)-\psi|^2}\frac{1}{R} \cdot R\mathrm{d}\theta \\ &= \int_0^{2\pi} h\big(C + R\exp(i\theta)\big) \frac{|1-|\psi|^2|}{2\pi\,|\exp(i\theta)-\psi|^2}\mathrm{d}\theta. \end{aligned} \tag{52}$$

By [30, Theorem 11.9] and (52), for every $z \in O(C, R)$

$$\frac{1}{N}\sum_{n=0}^{N-1} h\big(S^n(z)\big) \xrightarrow[n\to\infty]{} h(C + R\psi) = \begin{cases} h(\lambda_1), & |c_2| < |c_1|, \\ h(\lambda_2), & |c_1| < |c_2|. \end{cases} \tag{53}$$

Putting $z = a_1/a_0$ in (53), we obtain

$$\frac{1}{N}\sum_{n=0}^{N-1} h\left(\frac{a_{n+1}}{a_n}\right) \xrightarrow[n\to\infty]{} \begin{cases} h(\lambda_1), & |c_2| < |c_1|, \\ h(\lambda_2), & |c_1| < |c_2|. \end{cases} \tag{54}$$

□

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