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Log m-Convex Functions

Teodoro Lara [1,a] and Edgar Rosales [1,b]

Abstract. In this research we lay the concept of log m-convex functions defined on real intervals containing the origin, some algebraic properties are exhibit, in the same token discrete Jensen type inequalities and integral inequalities are set and shown.
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1. Introduction

Logarithmically convex (log-convex) functions are of interest in many areas of mathematics and science. They have been found to play an important role in the theory of special functions and mathematical statistics [10, 12]. In this paper we introduce the concept of log m-convex function as combination of the well known definitions of m-convex and log-convex functions, some examples are shown. At the same time, we exhibit basic algebraic properties of this new type of functions same as integral inequalities of Hermite-Hadamard type, we recall them [4, 11, 12] and references therein.

Definition 1. ([1, 2, 12]) A function $f : I \to (0, +\infty)$, I an interval, is said to be log-convex or multiplicatively convex if $\log f$ is convex, or, equivalently, if for all $x, y \in I$ and $t \in [0, 1]$ one has the inequality:

$$f(tx + (1-t)y) \leq [f(x)]^t [f(y)]^{1-t}. \tag{1.1}$$

When we work on m-convexity, almost always, it is necessary to know the concept of m-convex set.

Definition 2. ([9])A subset D of a real linear space X is said to be m-convex if, for all $x, y \in D$ and for all t in the interval $[0, 1]$, the point $tx + m(1-t)y$ also belongs to D.

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[1] *Departamento de Física y Matemáticas, Universidad de los Andes. Trujillo. Venezuela*
[a] *e-mail: tlara@ula.ve*
[b] *e-mail: edgarr@ula.ve* .

In [9, Theorem 2.6] it was proved that a subset $D \subseteq X$ containing 0 is m-convex if and only if $conv\left(\{0, x, \frac{m}{m+1}(x+y)\}\right) \setminus \{0\} \subseteq D$ for all $x, y \in D$, where $conv$ denote the convex hull. From this result, is not hard to prove that if I is an interval, such that $0 \in I$, then I is an m-convex set. So, it has sense de following

Definition 3. ([4, 7]) A function $f : I \to \mathbb{R}$, I an interval and $0 \in I$, is called m-convex, $0 \le m \le 1$, if for any $x, y \in I$ and $t \in [0, 1]$ we have

$$f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y).$$

Our definition, and key part of this work, runs as follow.

Definition 4. A function $f : I \to (0, +\infty)$, I an interval and $0 \in I$, is called log m-convex, $0 \le m \le 1$, if for all $x, y \in I$ and $t \in [0, 1]$ one has

$$f(tx + m(1-t)y) \le [f(x)]^t [f(y)]^{m(1-t)}. \tag{1.2}$$

If $m = 1$, we recapture the concept of log-convex function defined on I, given in Definition 1.

Just as it is stated in Definition 1, some authors indifferently use the terms log-convex or multiplicatively convex to define the functions which verify (1.1). Nevertheless, in our context, we consider the multiplicative convexity as follows.

Definition 5. A function $f : I \to (0, +\infty)$, I an interval and $0 \in I$, is called multiplicatively m-convex, $0 \le m \le 1$, if for all $x, y \in I$ and $t \in [0, 1]$ one has

$$f\left(x^t y^{m(1-t)}\right) \le [f(x)]^t [f(y)]^{m(1-t)}. \tag{1.3}$$

Remark 6. *It is not difficult to verify that the* log m*-convexity of* f *is equivalent to the* m*-convexity of* $\log f$.

Remark 7. *It is important to point out that (1.2) is equivalent to*

$$f(mtx + (1-t)y) \le [f(x)]^{mt} [f(y)]^{(1-t)}, \tag{1.4}$$

with x, y *and* t *as before.*

From now on I always will be an interval containing zero unless other thing is stated.

Remark 8. *The following properties of a* log m*-convex function* f *are a direct consequence of (1.2).*

(1) $f(my) \le [f(y)]^m$, *if* $t = 0$ *in (1.2).*
(2) $f(tx) \le [f(x)]^t$, *if* $m = 0$ *in (1.2).*
(3) *By taking* $x = y = 0$ *in (1.2), and* $m \ne 1$, *we have* $f(0) \le [f(0)]^{t+m(1-t)}$. *So,* $\log f(0) \le (t + m(1-t)) \log f(0)$, *or* $\log f(0)[1 - (t + m(1-t))] \le 0$. *Since* $1 - (t + m(1-t)) \ge 0$, *it follows that* $\log f(0) \le 0$. *Therefore, if* f *is log* m*-convex, necessarily* $f(0) \le 1$. *This fact, in turn, implies (by taking* $y = 0$ *in (1.2)) that* $f(tx) \le [f(x)]^t$ *holds for any* $m \in [0, 1)$. *This property is the equivalent to be starshaped of the* m*-convex functions ([8]).*

Example 1. The function $f : I \to (0, +\infty)$, given as $f(x) = e^x$, is log m-convex, but for $m \ne 1$, f is not an m-convex function (since $f(0) \not\le 0$, [6]). This fact shows a difference between log m-convexity and the usual log-convexity, since log-convexity implies convexity ([12]), while the converse is not always true as this function indicates.

2. Basic Properties and Examples

It is well known that the sum of log-convex functions is also log-convex ([12]). However, the sum of log m-convex functions is not necessarily log m-convex, as the next example shows.

Example 2. If $f(x) = e^x$, then $(f + f)(x) = 2e^x$, and $(f + f)(0) = 2$. Thus, by Remark 8, $f + f$ is not a log m-convex function.

Proposition 9. *If* $f : I \to (0, +\infty)$ *is a log* m*-convex function, and* $0 \le \alpha \le 1$, *then* αf *is log* m*-convex function as well.*

Proof. Let $x, y \in I$, and $t, \alpha \in [0,1]$, and by the log m-convexity of f,

$$(\alpha f)(tx + m(1-t)y) \leq \alpha \left([f(x)]^t [f(y)]^{m(1-t)} \right).$$

On the other hand $0 \leq \alpha \leq 1$ and $t + m(1-t) \leq 1$, hence $\alpha^{t+m(1-t)} \geq \alpha$. Indeed,

$$\begin{aligned}(\alpha f)(tx + m(1-t)y) &\leq \alpha^{t+m(1-t)} \left([f(x)]^t [f(y)]^{m(1-t)} \right) \\ &\leq [(\alpha f)(x)]^t [(\alpha f)(y)]^{m(1-t)}.\end{aligned}$$

Remark 10. *If $\alpha > 1$ in Proposition 9, the result is not true. Again, we consider the function $f(x) = e^x$; then for $\alpha > 1$, $(\alpha f)(x) = \alpha e^x > e^x$ and consequently $(\alpha f)(0) > 1$, so αf can not be a log m-convex function by Remark 8.*

Proposition 11. *If $f, g : I \to (0, +\infty)$ are two log m-convex functions, so is the product function fg.*

Proof. Let $x, y \in I$, and $t \in [0,1]$. Then,

$$\begin{aligned}(fg)(tx + m(1-t)y) &= f(tx + m(1-t)y)\, g(tx + m(1-t)y) \\ &\leq [f(x)]^t [f(y)]^{m(1-t)} [g(x)]^t [g(y)]^{m(1-t)} \\ &= [(fg)(x)]^t [(fg)(y)]^{m(1-t)}.\end{aligned}$$

Example 3. Once again, the exponential function, together with Remark 8, allow us to notice certain aspects regarding the basic operations of log m-convex functions. This time to check that the composition of log m-convex functions is not necessarily log m-convex. In fact, if $f(x) = e^x$ then $(f \circ f)(0) = e > 1$; hence $f \circ f$ is not log m-convex. Nonetheless, we have the following result.

Proposition 12. *Let $f : I \to (0, +\infty)$ be a log m-convex function, and let $g : J \to (0, +\infty)$ be a nondecreasing multiplicatively m-convex function, such that $range(f) \subseteq J$, then $g \circ f : I \to (0, +\infty)$ is log m-convex too.*

Proof. Let $x, y \in I$, and $t \in [0,1]$, then

$$\begin{aligned}g\left[f(tx + m(1-t)y)\right] &\leq g\left([f(x)]^t [f(y)]^{m(1-t)} \right) \\ &\leq [g(f(x))]^t [g(f(y))]^{m(1-t)}.\end{aligned}$$

Proposition 13. *Let $f : I \to (0, +\infty)$ be a function. Then*

(1) If f is log m-convex, so is $(f)^\alpha$ for all $\alpha > 0$.
(2) If $(f)^\alpha$ is an m-convex function for all $\alpha > 0$, then f is a log m-convex.

Proof. (1) Follows directly from the m-convexity of $\log f$ and [4, Proposition 4].

(2) It is not difficult to show that if a real function g is m-convex, so is $g - 1$. In fact, for all $x, y \in I$ and $t \in [0,1]$,

$$\begin{aligned}g(tx + m(1-t)y) - 1 &\leq tg(x) + m(1-t)g(y) - 1 \\ &= t(g(x) - 1) + m(1-t)(g(y) - 1) - (1-t)(1-m) \\ &\leq t(g(x) - 1) + m(1-t)(g(y) - 1).\end{aligned}$$

Thus, if for all $\alpha > 0$, f^α is m-convex, so is the function $(f)^{1/n} - 1$ for all $n \in \mathbb{N}$. We have then the sequence $\{n[(f)^{1/n} - 1]\}$ of m-convex functions, which converges pointwise to the function $\log f$; and from [5, Proposition 2.2], $\log f$ is m-convex function.

By picking $m = 1$ in Proposition 13, we have the following.

Corollary 14. *Let $f : I \to (0, +\infty)$. Then, $(f)^\alpha$ is convex for all $\alpha > 0$ if and only if f is log-convex.*

We close this section with a couple of discrete Jensen type inequalities for log m-convex functions.

Theorem 15. *Let $t_1,\ldots,t_n>0$ and $T_n=\sum_{i=1}^{n} t_i$. If $f:[0,+\infty)\to(0,+\infty)$ is a log m-convex function, with $m\in(0,1]$, then*

$$f\left(\frac{1}{T_n}\sum_{i=1}^{n} t_i x_i\right)\le\prod_{i=1}^{n}\left[f\left(\frac{x_i}{m^{n-i}}\right)\right]^{\frac{m^{n-i}t_i}{T_n}},\ \textit{for all } x_1,\ldots,x_n\in[0,+\infty).$$

Proof. Because the function $\log f:[0,+\infty)\to\mathbb{R}$ is m-convex (Remark 6), it follows from [5, Theorem 3.1], that for all $x_1,\ldots,x_n\in[0,+\infty)$,

$$\begin{aligned}\log f\left(\frac{1}{T_n}\sum_{i=1}^{n} t_i x_i\right)&\le\frac{1}{T_n}\sum_{i=1}^{n} m^{n-i}t_i\log f\left(\frac{x_i}{m^{n-i}}\right)\\&=\frac{1}{T_n}\sum_{i=1}^{n}\log\left[f\left(\frac{x_i}{m^{n-i}}\right)\right]^{m^{n-i}t_i}\\&=\log\left(\prod_{i=1}^{n}\left[f\left(\frac{x_i}{m^{n-i}}\right)\right]^{\frac{m^{n-i}t_i}{T_n}}\right).\end{aligned}$$

It only remains to apply the exponential function.

Theorem 16. *If $f:I\to(0,+\infty)$ is a log m-convex function, then for all $t_1,\ldots,t_n\ge 0$ with $\sum_{i=1}^{n} t_i\in(0,1]$ and all $x_1,\ldots,x_n\in I$, we have*

$$f\left(\sum_{i=1}^{n} m^{1-\delta_{i1}}t_i x_i\right)\le\prod_{i=1}^{n}[f(x_i)]^{m^{1-\delta_{i1}}t_i},\tag{2.1}$$

where δ_{ij} is the well-known Delta of Kronecker function.

Proof. For $m=1$, the result follows by applying the classical discrete Jensen's inequality to the convex function $\log f$. For $m\ne 1$, the proof goes by induction on n. So, if $n=1$, we have $f(t_1x_1)\le[f(x_1)]^{t_1}$ which is true by Remark 8.

Let us assume that the result holds for n.

Now, let $x_1,\ldots,x_{n+1}\in I$, and $t_1,\ldots,t_{n+1}\ge 0$ with $\sum_{i=1}^{n+1} t_i\in(0,1]$. First of all, if $t_{n+1}=1$, then $t_1,\ldots,t_n=0$ and hence,

$$\begin{aligned}f\left(\sum_{i=1}^{n+1} m^{1-\delta_{i1}}t_i x_i\right)&=f(mx_{n+1})\\&\le[f(x_{n+1})]^m\ \text{(by Remark 8)}\\&=\prod_{i=1}^{n+1}[f(x_i)]^{m^{1-\delta_{i1}}t_i}.\end{aligned}$$

We assume that $t_{n+1}\ne 1$, and we put

$$f\left(\sum_{i=1}^{n+1} m^{1-\delta_{i1}}t_i x_i\right)=f\left((1-t_{n+1})\sum_{i=1}^{n} m^{1-\delta_{i1}}t_i' x_i+mt_{n+1}x_{n+1}\right),$$

where $t_i'=\dfrac{t_i}{1-t_{n+1}}$.

Because I is m-convex and $\sum_{i=1}^{n} t_i' \in (0,1]$, it follows, from [9, Lemma 3.3], that $\sum_{i=1}^{n} m^{1-\delta_{i1}} t_i' x_i \in I$. Thus, by log m-convexity of f, and in accordance with Remark 7, we have

$$f\left((1-t_{n+1})\sum_{i=1}^{n} m^{1-\delta_{i1}} t_i' x_i + m t_{n+1} x_{n+1}\right) \leq \left[f\left(\sum_{i=1}^{n} m^{1-\delta_{i1}} t_i' x_i\right)\right]^{1-t_{n+1}} [f(x_{n+1})]^{m t_{n+1}}.$$

Therefore, by inductive hypothesis,

$$\begin{aligned} f\left(\sum_{i=1}^{n+1} m^{1-\delta_{i1}} t_i x_i\right) &\leq \left[\prod_{i=1}^{n} [f(x_i)]^{m^{1-\delta_{i1}} t_i'}\right]^{1-t_{n+1}} [f(x_{n+1})]^{m t_{n+1}} \\ &= \prod_{i=1}^{n} [f(x_i)]^{m^{1-\delta_{i1}} t_i} [f(x_{n+1})]^{m t_{n+1}} \\ &= \prod_{i=1}^{n+1} [f(x_i)]^{m^{1-\delta_{i1}} t_i}. \end{aligned}$$

3. Integral Inequalities

This last section is devoted to set and prove some integral inequalities, which together with the previous results, show the importance of this topic laid in our research; that is, log m-convex functions defined on intervals containing zero.

Proposition 17. *Let $f : I \to (0, +\infty)$ be a log m-convex function, $0 < m \leq 1$ and $a, b \in I, 0 < a < b$, then*

$$\frac{1}{a - mb}\int_{mb}^{a} f(x)dx \leq \frac{f(a) - [f(b)]^m}{\log f(a) - \log[f(b)]^m}.$$

Proof.

$$\begin{aligned} \frac{1}{a-mb}\int_{mb}^{a} f(x)dx &= \int_0^1 f(ta + m(1-t)b)dt \\ &\leq \int_0^1 [f(a)]^t [f(b)]^{m(1-t)} dt \\ &= [f(b)]^m \int_0^1 \left[\frac{f(a)}{[f(b)]^m}\right]^t dt, \end{aligned}$$

the conclusion is obtained by performing the above integral.

Proposition 18. *Let $f, g : I \to (0, +\infty)$ be log m-convex functions, $a, b \in (0, +\infty), a < b$ then*

$$\frac{1}{a - mb}\int_{mb}^{a} f(x)g(x)dx \leq \frac{fg(a) - [fg(b)]^m}{\log(fg(a)) - m\log(fg(b))}.$$

Proof.

$$\begin{aligned} \frac{1}{a-mb}\int_{mb}^{a} f(x)g(x)dx &= \int_0^1 f\left(ta + m(1-t)b\right) g\left(ta + m(1-t)b\right) dt \\ &\leq \int_0^1 [fg(a)]^t [fg(b)]^{m(1-t)} dt \text{ (log } m\text{-convexity)} \\ &= \frac{[fg(b)]^m}{\log(fg(a)) - m\log(fg(b))}\left[\frac{fg(a) - [fg(b)]^m}{[fg(b)]^m}\right] \end{aligned}$$

and inequality follows.

Proposition 19. *For $f:[0,+\infty)\to(0,+\infty)$ being a log m-convex function with $a,b\in I, 0<a<b$, the following inequality holds*

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)f(a+b-x)dx &\le \frac{1}{4\log\left(\dfrac{f(a)}{f^m\left(\dfrac{b}{m}\right)}\right)}\left[f^2(a)-f^{2m}\left(\frac{b}{m}\right)\right]\\ &\quad+\frac{1}{4\log\left(\dfrac{f(b)}{f^m\left(\frac{a}{m}\right)}\right)}\left[f^2(b)-f^{2m}\left(\frac{a}{m}\right)\right].\end{aligned}$$

Proof.

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)f(a+b-x)dx &= \int_0^1 f\left(ta+(1-t)b\right)f\left((1-t)a+tb\right)dt\\ &= \int_0^1 \left[f^2\left(ta+m(1-t)\left(\frac{b}{m}\right)\right)\right.\\ &\qquad\left.\times f^2\left(m(1-t)\left(\frac{a}{m}\right)+tb\right)\right]^{\frac{1}{2}}dt\\ &\le \frac{1}{2}\int_0^1 f^2\left(ta+m(1-t)\left(\frac{b}{m}\right)\right)dt\\ &\quad+\frac{1}{2}\int_0^1 f^2\left(m(1-t)\left(\frac{a}{m}\right)+tb\right)dt\\ &= \frac{f^{2m}\left(\dfrac{b}{m}\right)}{2}\int_0^1\left[\frac{f(a)}{f^m\left(\dfrac{b}{m}\right)}\right]^{2t}dt\\ &\quad+\frac{f^{2m}\left(\frac{a}{m}\right)}{2}\int_0^1\left[\frac{f(b)}{f^m\left(\frac{a}{m}\right)}\right]^{2t}dt\\ &= \frac{1}{4\log\left(\dfrac{f(a)}{f^m\left(\dfrac{b}{m}\right)}\right)}\left[f^2(a)-f^{2m}\left(\frac{b}{m}\right)\right]\\ &\quad+\frac{1}{4\log\left(\dfrac{f(b)}{f^m\left(\frac{a}{m}\right)}\right)}\left[f^2(b)-f^{2m}\left(\frac{a}{m}\right)\right]\end{aligned}$$

as desired.

Proposition 20. *If $G(x,y)=\sqrt{xy}$ with x and y positive and f,a,b as in foregoing result and $0<m<1$ then*

$$\begin{aligned}\int_0^1 G\left(f(ta+m(1-t)b),f(m(1-t)a+tb)\right)dt &\le \frac{G(f^m(a),f^m(b))}{(1-m)\log G(f(a),f(b)))}\\ &\quad\times[G(f^{1-m}(a),f^{1-m}(b))-1].\end{aligned}$$

Proof. Because of the log m-convexity of f and for any $t \in [0,1]$,

$$f(ta + m(1-t)b)f(m(1-t)a + tb) \leq [f(a)]^{t+m(1-t)}[f(b)]^{t+m(1-t)},$$

hence

$$G\left(f(ta + m(1-t)b), f(m(1-t)a + tb)\right) \leq G\left([f(a)]^{t+m(1-t)}, [f(b)]^{t+m(1-t)}\right).$$

Let us name $A = \int_0^1 G\left(f(ta + m(1-t)b), f(m(1-t)a + tb)\right) dt$, now by integrating in both sides of the late inequality we have

$$\begin{aligned} A &\leq [f(a)f(b)]^{\frac{m}{2}} \int_0^1 [f(a)f(b)]^{\frac{1-m}{2}t} dt \\ &= \frac{[f(a)f(b)]^{\frac{m}{2}}}{(1-m)\log(f(a)f(b))^{\frac{1}{2}}}[(f(a)f(b))^{\frac{1-m}{2}} - 1] \\ &= \frac{G(f^m(a), f^m(b))}{(1-m)\log G(f(a), f(b))}[G(f^{1-m}(a), f^{1-m}(b)) - 1]. \end{aligned}$$

We need the following result from [3],

Lemma 21. *Let $f : I \to (0, +\infty)$ be a differentiable function on the interior of I; $a, b \in I$ and $a < b$. If $f' \in L^1[a,b]$, then the following equality takes place*

$$\begin{aligned} (x-a)f(a) + (b-x)f(b) - \int_a^b f(u)du &= (x-a)^2 \int_0^1 (t-1)f'(tx + (1-t)a)dt \\ &+ (b-x)^2 \int_0^1 (1-t)f'(tx + (1-t)b)dt. \end{aligned}$$

Forthcoming result is inspired on one given in [6].

Theorem 22. *If $f : I \to (0, +\infty)$, $0 \in I$, is a differentiable function in the interior of I, $a, b \in I, a < b$, f' integrable on $[a,b]$ and $|f'|$ is log m-convex function, the following inequality holds*

$$\begin{aligned} &(x-a)f(a) + (b-x)f(b) - \int_a^b f(u)du \\ &\leq |f'(x)| \left[\frac{(x-a)^2}{A\log^2(A)}(A - \log(A) - 1) + \frac{(b-x)^2}{B\log^2(B)}(B - \log(B) - 1)\right], \end{aligned}$$

where $A = \dfrac{|f'(x)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}$, $B = \dfrac{|f'(x)|}{\left|f'\left(\frac{b}{m}\right)\right|^m}$ *and any $x \in [a,b]$.*

Proof. By mean of Lemma 21 (taking the modulus) and since $|f'|$ is log m-convex function,

$$\begin{aligned} &(x-a)f(a) + (b-x)f(b) - \int_a^b f(u)du \\ &\leq (x-a)^2 \int_0^1 (1-t)|f'(tx + (1-t)a)|dt + (b-x)^2 \int_0^1 (1-t)|f'(tx + (1-t)b)|dt \\ &\leq (x-a)^2 \int_0^1 (1-t)|f'(x)|^t \left|f'\left(\frac{a}{m}\right)\right|^{m(1-t)} dt \\ &\qquad + (b-x)^2 \int_0^1 (1-t)|f'(x)|^t \left|f'\left(\frac{b}{m}\right)\right|^{m(1-t)} dt \end{aligned}$$

$$= \frac{(x-a)^2 \left|f'\left(\frac{a}{m}\right)\right|^m}{\log^2\left(\frac{|f'(x)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right)} \left[\frac{|f'(x)|}{\left|f'\left(\frac{a}{m}\right)\right|^m} - \log\left(\frac{|f'(x)|}{\left|f'\left(\frac{a}{m}\right)\right|^m}\right) - 1\right]$$

$$+ \frac{(b-x)^2 \left|f'\left(\frac{b}{m}\right)\right|^m}{\log^2\left(\frac{|f'(x)|}{\left|f'\left(\frac{b}{m}\right)\right|^m}\right)} \left[\frac{|f'(x)|}{\left|f'\left(\frac{b}{m}\right)\right|^m} - \log\left(\frac{|f'(x)|}{\left|f'\left(\frac{b}{m}\right)\right|^m}\right) - 1\right],$$

proof concludes by noticing that $\left|f'\left(\frac{a}{m}\right)\right|^m = \frac{|f'(x)|}{A}$ and $\left|f'\left(\frac{b}{m}\right)\right|^m = \frac{|f'(x)|}{B}$.

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