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Uniqueness of $p(f)$ and $P[f]$ concerning weakly weighted-sharing [1]

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Abstract

In the year 2006, S. Lin and W. Lin introduced the definition of weakly weighted-sharing of meromorphic functions which is between "CM" and "IM". In this paper, using the notion of weakly weighted-sharing, we study the uniqueness of a polynomial function $p(f)$ of f and a homogeneous differential polynomial $P[f]$ generated by f. Our results improve and generalizes the results due to Charak and Lal, S. Lin and W. Lin, and H-Y Xu and Y Hu.

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1 Introduction and main results

Let $\mathbb{C}$ denote the complex plane and let f be a non-constant meromorphic function defined on $\mathbb{C}$. We assume that the reader is familiar with the standard definitions and notations used in the Nevanlinna value distribution theory, such as $T(r, f), m(r, f), N(r, f)$(see [3, 8, 9]). By $S(r, f)$ we denote any quantity satisfying the condition $S(r, f) = \circ(T(r, f))$ as $r \to \infty$ possibly outside an exceptional set of finite linear measure. A meromorphic function a is called a small function with respect to f if either $a \equiv \infty$ or $T(r, a) = S(r, f)$. We denote by $S(f)$ the collection

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of all small functions with respect to f. Clearly $\mathbb{C}\cup\{\infty\}\subset S(f)$ and $S(f)$ is a field over the set of complex numbers. For $a\in\mathbb{C}\cup\{\infty\}$ the quantities

$$\delta(a,f)=1-\limsup_{r\to\infty}\frac{N(r,a;f)}{T(r,f)}$$

and

$$\Theta(a,f)=1-\limsup_{r\to\infty}\frac{\overline{N}(r,a;f)}{T(r,f)}.$$

are respectively called the deficiency and ramification index of a for the function f.

For any two non-constant meromorphic functions f and g, and $a\in S(f)\cap S(g)$, we say that f and g share a IM(CM) provided that $f-a$ and $g-a$ have the same zeros ignoring(counting) multiplicities. If $\frac{1}{f}$ and $\frac{1}{g}$ share 0 IM(CM), we say that f and g share ∞ IM(CM).

Definition 1 *Let k be a non-negative integer or infinity and $a\in S(f)$. We denote by $E_k(a,f)$ the set of all zeros of $f-a$, where a zero of multiplicity m is counted m times if $m\le k$ and $k+1$ times if $m>k$. If $E_k(a,f)=E_k(a,g)$, we say that f, g share the function a with weight k. We write f and g share (a,k) to mean that f and g share the function a with weight k. Since $E_k(a,f)=E_k(a,g)$ implies that $E_l(a,f)=E_l(a,g)$ for any integer $l\,(0\le l<k)$, if f, g share (a,k), then f, g share (a,l). Moreover, we note that f and g share the function a IM or CM if and only if f and g share $(a,0)$ or (a,∞) respectively.*

Definition 2 *[5] Let $N_E(r,a)$ be the counting function of all common zeros of $f-a$ and $g-a$ with the same multiplicities, and $N_0(r,a)$ be the counting function of all common zeros of $f-a$ and $g-a$ ignoring multiplicities. Denote by $\overline{N}_E(r,a)$ and $\overline{N}_0(r,a)$ the reduced counting functions of f and g corresponding to the counting functions $N_E(r,a)$ and $N_0(r,a)$ respectively. If*

$$\overline{N}(r,a;f)+\overline{N}(r,a;g)-2\overline{N}_E(r,a)=S(r,f)+S(r,g)$$

then we say that f and g share a "CM". If

$$\overline{N}(r,a;f)+\overline{N}(r,a;g)-2\overline{N}_0(r,a)=S(r,f)+S(r,g)$$

then we say that f and g share a "IM".

Definition 3 *Let k be a positive integer, and let f be a meromorphic function and $a\in S(f)$.*
(i) $\overline{N}_{k)}(r,a;f)$ denotes the counting function of those a-points of f whose multiplicities are not greater than k, where each a-point is counted only once.

(ii) $\overline{N}_{(k}(r, a; f)$ denotes the counting function of those a-points of f whose multiplicities are not less than k, where each a-point is counted only once.

(iii) $N_p(r, a; f)$ denotes the counting function of those a-points of f , where an a-point of f with multiplicity m counted m times if $m \leq p$ and p times if $m > p$.

We denote by $\delta_p(a, f)$ the quantity

$$\delta_p(a, f) = 1 - \limsup_{r\to\infty} \frac{N_p(r, a; f)}{T(r, f)},$$

where p is a positive integer. Clearly $\delta_p(a, f) \geq \delta(a, f)$.

Definition 4 *Let f and g be two non-constant meromorphic functions sharing a "IM", for $a \in S(f) \cap S(g)$, and a positive integer k or ∞.*

(i) $\overline{N}^{E}_{k)}(r, a)$ denotes the counting function of those a-points of f whose multiplicities are equal to the corresponding a-points of g, both of their multiplicities are not greater than k, where each a-point is counted only once.

(ii) $\overline{N}^{0}_{(k}(r, a)$ denotes the reduced counting function of those a-points of f which are a-points of g, both of their multiplicities are not less than k, where each a-point is counted only once.

Definition 5 *[5] For $a \in S(f) \cap S(g)$, if k is a positive integer or ∞, and*

$$\overline{N}_{k)}(r, a; f) + \overline{N}_{k)}(r, a; g) - 2\overline{N}^{E}_{k)}(r, a) = S(r, f) + S(r, g)$$

$$\overline{N}_{(k+1}(r, a; f) + \overline{N}_{(k+1}(r, a; g) - 2\overline{N}^{0}_{(k+1}(r, a) = S(r, f) + S(r, g)$$

or if $k = 0$ and

$$\overline{N}(r, a; f) + \overline{N}(r, a; g) - 2\overline{N}_0(r, a) = S(r, f) + S(r, g)$$

then we say f and g weakly share a with weight k. Here we write f, g share "(a, k)" to mean that f, g weakly share a with weight k.

Obviously if f and g share "(a, k)", then f and g share "(a, p)" for any p $(0 \leq p < k)$. Also, we note that f and g share a "IM" or "CM" if and only if f and g share "$(a, 0)$" or "(a, ∞)" respectively.

Definition 6 *Suppose F and G share 1 "IM" and let z_0 be a zero of $F - 1$ of multiplicity r and a zero of $G - 1$ of multiplicity s.*

(i) *By $\overline{N}_L(r, 1; F)$ we denotes the reduced counting function of those a-points of F and G where $r > s \geq 1$; $\overline{N}_L(r, 1; G)$ is defined similarly.*

(ii) *By* $N_E^{1)}(r,1;F)$ *the counting function of those 1-points of* F *and* G *where* $r = s = 1$ *and*

(iii) *by* $\overline{N}_E^{(2}(r,1;F)$ *the counting function of those 1-points of* F *and* G *where* $r = s \geq 2$*, where each such zero is counted only once.*

Definition 7 *Let* n_{0j}, n_{1j}, n_{2j}, ..., n_{qj} *are non-negative integers. The expression*

$$M_j[f] = (f)^{n_{0j}}(f^{(1)})^{n_{1j}}(f^{(2)})^{n_{2j}}...(f^{(q)})^{n_{qj}}$$

is called a differential monomial generated by f *of degree* $d(M_j) = \sum_{i=0}^{q} n_{ij}$ *and weight* $\Gamma_{M_j} = \sum_{i=0}^{q}(i+1)n_{ij}$. *Let* $a_j \in S(f)$ *and* $a_j \not\equiv 0 (j = 1, 2, ..., t)$. *The sum* $P[f] = \sum_{j=1}^{t} a_j M_j[f]$ *is called a differential polynomial generated by* f *of degree* $\overline{d}(P) = \max\{d(M_j) : 1 \leq j \leq t\}$ *and weight* $\Gamma_P = \max\{\Gamma_{M_j} : 1 \leq j \leq t\}$. *The numbers* $\underline{d}(P) = \min\{d(M_j) : 1 \leq j \leq t\}$ *and* q *(the highest order of the derivative of* f *in* $P[f]$*) are called respectively the lower degree and the order of* $P[f]$. $P[f]$ *is said to be homogeneous differential polynomial of degree* d *if* $\overline{d}(P) = \underline{d}(P) = d$. $P[f]$ *is called a linear differential Polynomial generated by* f *if* $\overline{d}(P) = 1$. *Otherwise,* $P[f]$ *is called non-linear differential polynomial. Also, we denote by* Q *the quantity* $Q = \max_{1 \leq j \leq t} \sum_{i=0}^{q} i.n_{ij}$.

In 2006 S. Lin and W. Lin [5] first defined and used the concept of weakly-weighted sharing of functions to prove the uniqueness of a meromorphic function and its derivative and proved the following theorems:

Theorem 1 *Let* $n \geq 1$ *and* $2 \leq k \leq \infty$, *let* f *be a non-constant meromorphic function,* $a \in S(f)$ *and* $a \not\equiv 0, \infty$. *If* f *and* $f^{(n)}$ *share "*(a,k)*" and*

$$4\Theta(\infty, f) + 2\delta_{2+n}(0, f) > 5,$$

then $f \equiv f^{(n)}$.

Theorem 2 *Let* $n \geq 1$ *and let* f *be a non-constant meromorphic function,* $a \in S(f)$ *and* $a \not\equiv 0, \infty$. *If* f *and* $f^{(n)}$ *share "*$(a,1)$*" and*

$$\left(\frac{n+9}{2}\right)\Theta(\infty, f) + \frac{5}{2}\delta_{2+n}(0, f) > \frac{n}{2} + 6,$$

then $f \equiv f^{(n)}$.

Theorem 3 *Let* $n \geq 1$ *and let* f *be a non-constant meromorphic function,* $a \in S(f)$ *and* $a \not\equiv 0, \infty$. *If* f *and* $f^{(n)}$ *share "*$(a,0)$*" and*

$$(7+2n)\Theta(\infty, f) + 5\delta_{2+n}(0, f) > 2n + 11,$$

then $f \equiv f^{(n)}$.

Later in 2016 Charak and Lal [2] proved the uniqueness of a polynomial $p(f)$ in f and a differential polynomial $P[f]$ of f, using the concept of weighted sharing.

Theorem 4 *[2] Let f be a non-constant meromorphic function and $a \in S(f)$, $a \not\equiv 0, \infty$, and $p(z)$ be a polynomial of degree $n \geq 1$ with $p(0) = 0$. Let $P[f]$ be a non-constant differential polynomial of f. Suppose $p(f)$ and $P[f]$ share (a, k) with one of the following conditions:*

(i) $k \geq 2$ and

$$(Q+3)\Theta(\infty, f) + 2n\Theta(0, p(f)) + \overline{d}(P)\delta(0, f) > Q + 3 + 2\overline{d}(P) - \underline{d}(P) + n,$$

(ii) $k = 1$ and

$$(Q+\frac{7}{2})\Theta(\infty, f) + \frac{5n}{2}\Theta(0, p(f)) + \overline{d}(P)\delta(0, f) > Q + \frac{7}{2} + 2\overline{d}(P) - \underline{d}(P) + \frac{3n}{2},$$

(iii) $k = 0$ and

$$(2Q+6)\Theta(\infty, f) + 4n\Theta(0, p(f)) + 2\overline{d}(P)\delta(0, f) > 2Q + 6 + 4\overline{d}(P) - 2\underline{d}(P) + 3n.$$

Then $p(f) \equiv P[f]$.

In this paper we prove the uniqueness of $p(f)$ and $P(f)$ mentioned in Theorem 4 with the notion of weakly weighted sharing which is between "CM" and "IM" and measures how close a share value is share "CM" or share "IM". Here we prove the following theorems:

Theorem 5 *Let $2 \leq k \leq \infty$, f be a non-constant meromorphic function, $a \in S(f)$, $a \not\equiv 0, \infty$ and $p(z)$ be a polynomial of degree $n \geq 1$ with $p(0) = 0$. Let $P[f]$ be a homogeneous differential polynomial of degree d generated by f defined as in Definition 7. If $p(f)$ and $P[f]$ share "(a, k)" and*

$$4\Theta(\infty, f) + \delta_{2+q}(0, f) + n\delta_2(0, p(f)) > 5 + n - d, \tag{1}$$

then $p(f) \equiv P[f]$.

Theorem 6 *Let f be a non-constant meromorphic function and $a \in S(f)$, $a \not\equiv 0, \infty$ and $p(z)$ be a polynomial of degree $n \geq 1$ with $p(0) = 0$. Let $P[f]$ be same as in Theorem 5. If $p(f)$ and $P[f]$ share "$(a, 1)$" and*

$$\left(\frac{7}{2} + Q\right)\Theta(\infty; f) + \frac{3n}{2}\delta_2(0; p(f)) + \delta_{2+q}(0; f) > \frac{n+9}{2} + Q, \tag{2}$$

then $p(f) \equiv P[f]$.

Theorem 7 *Let f be a non-constant meromorphic function and $a \in S(f)$, $a \not\equiv 0, \infty$, and $p(z)$ be a polynomial of degree $n \geq 1$ with $p(0) = 0$. Let $P[f]$ be same as in Theorem 5. If $p(f)$ and $P[f]$ share "$(a, 0)$" and*

$$(6 + 2Q)\Theta(\infty, f) + n\delta_2(0, p(f)) + 2n\Theta(0, p(f)) + 2\delta_{2+q}(0, f) > 2Q + 8 + 2n, \tag{3}$$

then $p(f) \equiv P[f]$.

2 Lemmas

To prove our theorems, we will require some lemmas as follows.

Lemma 1 *[4] Let f be a non-constant meromorphic function, and $P[f]$ be same as in Theorem 5. Then*

$$\begin{aligned}(i)\ T(r,P) &\le dT(r,f)+Q\overline{N}(r,\infty;f)+S(r,f).\\ (ii)\ N(r,0;P) &\le T(r,P)-dT(r,f)+dN(r,0;f)+S(r,f)\\ &\le Q\overline{N}(r,\infty;f)+dN(r,0;f)+S(r,f).\end{aligned}$$

Lemma 2 *Let f be a transcendental meromorphic function and $P[f]$ be same as in Lemma 1. If $P[f]\not\equiv 0$ then we have*
(i) $N_2(r,0;P)\le N_{2+q}(r,0;f)+Q\overline{N}(r,\infty;f)+S(r,f)$,
(ii) $N_2(r,0;P)\le N_{2+q}(r,0;f)+T(r,P)-dT(r,f)+S(r,f)$.

Proof.

$$\begin{aligned}N_2(r,0;P) &\le N(r,0;P)-\sum_{k=3}^{\infty}\overline{N}(r,0;P|\ge k)\\ &= T(r,P)-m(r,0;P)-\sum_{k=3}^{\infty}\overline{N}(r,0;P|\ge k)+O(1)\\ &\le T(r,P)+m(r,\infty;\frac{P}{f^d})-m(r,0;f^d)-\sum_{k=3}^{\infty}\overline{N}(r,0;P|\ge k)+O(1)\\ &\le T(r,P)-dT(r,f)+N(r,0;f^d)-\sum_{k=3}^{\infty}\overline{N}(r,0;P|\ge k)+S(r,f)\\ &\le T(r,P)-dT(r,f)+N_{2+q}(r,0;f^d)+\sum_{k=3+q}^{\infty}\overline{N}(r,0;f^d|\ge k)\\ &- \sum_{k=3}^{\infty}\overline{N}(r,0;P|\ge k)+S(r,f)\\ &\le T(r,P)-dT(r,f)+N_{2+q}(r,0;f)+S(r,f).\end{aligned}$$

This proves (ii).

Now,

$$\begin{aligned}T(r,P) &= N(r,\infty;P)+m(r,\infty;P)\\ &\le m(r,\infty;f^d)+m(r,\infty;\frac{P}{f^d})+N(r,\infty;P)\\ &= dm(r,\infty;f)+N(r,\infty;P)+S(r,f)\\ &\le dm(r,\infty;f)+dN(r,\infty;f)+Q\overline{N}(r,\infty;f)+S(r,f)\\ &= dT(r,f)+Q\overline{N}(r,\infty;f)+S(r,f)\end{aligned}$$

Therefore $N_2(r,0;P) \leq N_{2+q}(r,0;f) + Q\overline{N}(r,\infty;f) + S(r,f)$.

Lemma 3 *[1] Let f be a non-constant meromorphic function and $P[f]$ be as in Lemma 1. Then*

$$N\left(r, \frac{P[f]}{f^d}\right) \leq Q\left(\overline{N}(r,0;f) + \overline{N}(r,\infty;f)\right) + S(r,f).$$

Lemma 4 *[5] Let k be a non-negative integer or infinity. Let F and G be non-constant meromorphic functions, and F, G share "$(1,k)$". Let*

$$H = \left(\frac{F^{(2)}}{F^{(1)}} - 2\frac{F^{(1)}}{F-1}\right) - \left(\frac{G^{(2)}}{G^{(1)}} - 2\frac{G^{(1)}}{G-1}\right).$$

If $H \not\equiv 0$, $2 \leq k \leq \infty$, then

$$T(r,F) \leq N_2(r,\infty;F) + N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;G) + S(r,F) + S(r,G).$$

The same inequality hold for $T(r,G)$.

Lemma 5 *[7] If F and G be non-constant meromorphic functions sharing "$(1,1)$", then*
$2\overline{N}_L(r,1;F) + 2\overline{N}_L(r,1;G) + \overline{N}_E^{(2}(r,1;F) - \overline{N}_{F>2}(r,1;G) \leq N(r,1;G) - \overline{N}(r,1;G)$.

Lemma 6 *[7] If F and G be non-constant meromorphic functions sharing "$(1,1)$", then*
$\overline{N}_{F>2}(r,1;G) \leq \frac{1}{2}\overline{N}(r,0;F) + \frac{1}{2}\overline{N}(r,\infty;F) - \frac{1}{2}\overline{N}_0(r,0;F^{(1)}) + S(r,F)$.

Lemma 7 *[7] If F and G be non-constant meromorphic functions sharing "$(1,0)$", then*

$$\overline{N}_L(r,1;F) + 2\overline{N}_L(r,1;G) + N_E^{(2}(r,1;F) - \overline{N}_{F>1}(r,1;G) - \overline{N}_{G>1}(r,1;F)$$
$$\leq N(r,1;G) - \overline{N}(r,1;G).$$

Lemma 8 *[7] If F and G be non-constant meromorphic functions sharing "$(1,0)$", then*

$$\overline{N}_L(r,1;F) \leq \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + S(r,F).$$

Lemma 9 *[7] If F and G be non-constant meromorphic functions sharing "$(1,0)$", then*

$$(i)\ \overline{N}_{F>1}(r,1;G) \leq \overline{N}(r,0;F) + \overline{N}(r,\infty;F) - \overline{N}_0(r,0;F^{(1)}) + S(r,F);$$

$$(ii)\ \overline{N}_{G>1}(r,1;F) \leq \overline{N}(r,0;G) + \overline{N}(r,\infty;G) - \overline{N}_0(r,0;G^{(1)}) + S(r,G).$$

Lemma 10 *[6] Let f be a non-constant meromorphic function and let*

$$R(f) = \frac{\sum_{k=0}^{n} a_k f^k}{\sum_{j=0}^{m} b_j f^j}$$

be an irreducible rational function in f with constant coefficients a_k and b_j where $a_n \neq 0$ and $b_m \neq 0$. Then

$$T(r, R(f)) = dT(r, f) + S(r, f)$$

.
where $d = \max\{n, m\}$

3 Proof of the main Theorems

Proof of Theorem 5. Let $p(z) = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + ... + a_1 z$, where $a_1, a_2, ...a_{n-1}$ are constants,

$$F = \frac{p(f)}{a}, \ G = \frac{P[f]}{a}.$$

Since $p(f)$ and $P[f]$ share "(a, k)", it follows that F, G share "$(1, k)$" except at the zeros and poles of a.
Also note that

$$T(r, F) = O(T(r, f)) + S(r, f)$$

$$T(r, G) = O(T(r, f)) + S(r, f)$$

$$\overline{N}(r, \infty; F) = \overline{N}(r, \infty; G) + S(r, f).$$

Let H be defined as in Lemma 4. Suppose that $H \not\equiv 0$, it follows that

$$T(r, G) \leq N_2(r, \infty; F) + N_2(r, 0; F) + N_2(r, \infty; G) + N_2(r, 0; G) + S(r, F) + S(r, G),$$

$$\begin{aligned}\Rightarrow T(r, P) &\leq N_2(r, \infty; p(f)) + N_2(r, 0; p(f)) + N_2(r, \infty; P) + N_2(r, 0; P) + S(r, f)\\ &\leq N_2(r, \infty; f) + N_2(r, 0; p(f)) + N_2(r, \infty; f) + N_2(r, 0; P) + S(r, f)\\ &\leq 4\overline{N}(r, \infty; f) + N_2(r, 0; p(f)) + N_{2+q}(r, 0; f) + T(r, P) - dT(r, f) + S(r, f),\end{aligned}$$

$dT(r, f) \leq 4\overline{N}(r, \infty; f) + N_{2+q}(r, 0; f) + N_2(r, 0; p(f)) + S(r, f)$
$\Rightarrow 4\Theta(\infty, f) + \delta_{2+q}(0, f) + n\delta_2(0, p(f)) \leq 5 + n - d$,
which contradicts (1). Thus $H \equiv 0$.

That is

$$\left(\frac{F^{(2)}}{F^{(1)}} - 2\frac{F^{(1)}}{F - 1}\right) = \left(\frac{G^{(2)}}{G^{(1)}} - 2\frac{G^{(1)}}{G - 1}\right).$$

Integrating twice we get

$$\frac{1}{F-1}=\frac{A}{G-1}+B,$$

where $A\neq 0$ and B are constants.
Thus

$$F=\frac{(B+1)G+(A-B-1)}{BG+(A-B)}. \tag{4}$$

Next we consider the following three cases:
Case 1. $B\neq 0,-1$

If $A-B-1\neq 0$ then by (4)

$$\overline{N}(r,\frac{-A+B+1}{B+1};G)=\overline{N}(r,0;F).$$

By Nevanlinna second fundamental theorem and (ii) of Lemma 2 we have

$$\begin{aligned} T(r,G) &< \overline{N}(r,\infty;G)+\overline{N}(r,0;G)+\overline{N}(r,\frac{-A+B+1}{B+1};G)+S(r,G) \\ &= \overline{N}(r,\infty;G)+\overline{N}(r,0;G)+\overline{N}(r,0;F)+S(r,G) \end{aligned}$$

i.e

$$\begin{aligned} T(r,P) &< \overline{N}(r,\infty;f)+\overline{N}(r,0;P)+\overline{N}(r,0;p(f))+S(r,f) \\ &\leq \overline{N}(r,\infty;f)+T(r,P)-dT(r,f)+N_{2+q}(r,0;f)+\overline{N}(r,0;p(f))+S(r,f) \\ dT(r,f) &\leq \overline{N}(r,\infty;f)+N_{2+q}(r,0;f)+N_2(r,0;p(f))+S(r,f), \end{aligned}$$

which gives

$$\Theta(\infty,f)+\delta_{2+q}(0,f)+n\delta_2(0,p(f))\leq 2+n-d$$

which violates our assumption (1).

If $A-B-1=0$ then by (4)

$$\overline{N}(r,\frac{-1}{B};G)=\overline{N}(r,\infty;F).$$

By similar argument as above we get a contradiction.

Case 2. $B=-1$. Then $F=\frac{A}{A+1-G}$.
If $A+1\neq 0$, $\overline{N}(r,A+1;G)=\overline{N}(r,\infty;F)$.
Proceeding similarly as in Case 1 we get a contradiction.

If $A+1=0$ then $FG=1$.

$$\Rightarrow p(f).P[f] \equiv a^2$$

It is clear from above that $\overline{N}(r,0;f)+\overline{N}(r,\infty;f)=S(r,f)$.

By Lemma 3 we have $N\left(r,\frac{P[f]}{f^d}\right)=S(r,f)$ and $m\left(r,\frac{P[f]}{f^d}\right)=S(r,f)$. Again using Lemma 10 we get

$$\begin{aligned}
(d+n)T(r,f) &\leq T(r,\frac{a^2}{f^{d+n}})+O(1)\\
&\leq T\left(r,(1+\frac{a_{n-1}}{f}+.......+\frac{a_1}{f^{n-1}}).\frac{P[f]}{f^d}\right)\\
&\leq (n-1)T(r,f)+T(r,\frac{P[f]}{f^d})+S(r,f)\\
&\leq (n-1)T(r,f)+S(r,f)\\
i.e.,\ (1+d)T(r,f) &\leq S(r,f),\\
T(r,f) &= S(r,f).
\end{aligned}$$

which is a contradiction.

Case 3. $B=0$. Then (4) gives $F=\frac{G+A-1}{A}$.
If $A-1\neq 0$, $\overline{N}(r,1-A;G)=\overline{N}(r,0;F)$
which again contradict our assumption (1). Therefore $A-1=0$. Then $F=G$ i.e., $p(f)\equiv P[f]$. This completes the proof.

Proof of Theorem 6. Let $p(z)=z^n+a_{n-1}z^{n-1}+a_{n-2}z^{n-2}+........+a_1z$, where $a_1,a_2,......a_{n-1}$ are constants,

$$F=\frac{p(f)}{a},\ G=\frac{P[f]}{a}.$$

Since $p(f)$ and $P[f]$ share "$(a,1)$", it follows that F, G share "$(1,1)$" except at the zeros and poles of a.

Also H be defined as in Lemma 4. Suppose that $H\not\equiv 0$. Since F and G share "$(1,1)$", we can get

$$\begin{aligned}
N(r,\infty;H) &\leq \overline{N}(r,\infty;F)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+\overline{N}_0(r,0;F^{(1)})\\
&+ \overline{N}_0(r,0;G^{(1)})+S(r,F)+S(r,G), \qquad (5)
\end{aligned}$$

and

$$N(r,1;F|=1)\leq N(r,0;H)+S(r,F)\leq N(r,\infty;H)+S(r,F), \qquad (6)$$

where $\overline{N}_0(r,0;F^{(1)})$ is the reduce counting function of zeros of $F^{(1)}$ which are not the zeros of $F(F-1)$ and $\overline{N}_0(r,0;G^{(1)})$ is similarly defined.

By Nevanlinna second fundamental theorem, we have

$$\begin{aligned} T(r,F)+T(r,G) &\le \overline{N}(r,\infty;F)+\overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;G) \\ &+ \overline{N}(r,1;F)+\overline{N}(r,1;G)-\overline{N}_0(r,0;F^{(1)}) \\ &- \overline{N}_0(r,0;G^{(1)})+S(r,F)+S(r,G). \end{aligned} \tag{7}$$

By (5), (6) and Lemmas 5, 6 we have

$$\begin{aligned} \overline{N}(r,1;F)+\overline{N}(r,1;G) &\le N(r,1;F|=1)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G) \\ &+\overline{N}_E^{(2}(r,1;F)+\overline{N}(r,1;G)+S(r,F)+S(r,G) \\ &\le N(r,1;F|=1)-\overline{N}_L(r,1;F)-\overline{N}_L(r,1;G)+\overline{N}_{F>2}(r,1;G) \\ &+N(r,1;G)+S(r,F)+S(r,G) \\ &\le N(r,1;F|=1)-\overline{N}_L(r,1;F)-\overline{N}_L(r,1;G) \\ &+\frac{1}{2}\overline{N}(r,0;F)+\frac{1}{2}\overline{N}(r,\infty;F)-\frac{1}{2}\overline{N}_0(r,0;F^{(1)}) \\ &+N(r,1;G)+S(r,F)+S(r,G) \\ &\le \overline{N}(r,\infty;F)+\overline{N}(r,0;G|\ge 2)+\overline{N}(r,0;F|\ge 2) \\ &+\overline{N}_0(r,0;F^{(1)})+\overline{N}_0(r,0;G^{(1)}) \\ &+\frac{1}{2}\overline{N}(r,0;F)+\frac{1}{2}\overline{N}(r,\infty;F)-\frac{1}{2}\overline{N}_0(r,0;F^{(1)})+N(r,1;G) \\ &-\overline{N}_L(r,1;F)-\overline{N}_L(r,1;G)+S(r,F)+S(r,G). \end{aligned}$$

Using above inequality in (7) we get

$$\begin{aligned} T(r,F)+T(r,G) &\le \frac{3}{2}\overline{N}(r,0;F)+\frac{5}{2}\overline{N}(r,\infty;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;G) \\ &+ \overline{N}(r,0;F|\ge 2)+\overline{N}(r,0;G|\ge 2)+T(r,G)+S(r,F)+S(r,G). \end{aligned}$$

Therefore

$$\begin{aligned} T(r,F) &\le \frac{1}{2}\overline{N}(r,0;F)+\frac{5}{2}\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+N_2(r,0;F) \\ &+N_2(r,0;G)+S(r,F)+S(r,G) \\ &\le \frac{1}{2}\overline{N}(r,0;p(f))+\frac{7}{2}\overline{N}(r,\infty;f)+N_2(r,0;p(f))+N_2(r,0;P)+S(r,F)+S(r,G). \end{aligned}$$

By (i) of Lemma 2, we have

$$\begin{aligned} nT(r,f) &\le \frac{7}{2}\overline{N}(r,\infty;f)+\frac{3}{2}N_2(r,0;p(f))+N_2(r,0;P)+S(r,F) \\ &\le (\frac{7}{2}+Q)\overline{N}(r,\infty;f)+\frac{3}{2}N_2(r,0;p(f))+N_{2+q}(r,0;f)+S(r,f). \end{aligned}$$

So

$$(\frac{7}{2}+Q)\Theta(\infty,f)+\frac{3n}{2}\delta_2(0,f)+\delta_{2+q}(0,f)\leq\frac{n+9}{2}+Q,$$

which contradicts the assumption of Theorem 6. Thus $H\equiv 0$.
By similar arguments as in Theorem 5, we can prove that the conclusion of Theorem 6 holds.

Proof of Theorem 7. Let F, G and $p(f)$ be same as in Theorem 5. From given condition of Theorem 7, F, G share "$(1,0)$". Also H be defined as in Lemma 4.

Suppose that $H\not\equiv 0$. Since F and G share "$(1,0)$", we can get

$$\begin{aligned}N(r,\infty;H) &\leq \overline{N}(r,\infty;F)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)\\ &+ \overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_0(r,0;F^{(1)})\\ &+ \overline{N}_0(r,0;G^{(1)})+S(r,F)+S(r,G).\end{aligned}\tag{8}$$

and

$$N_E^{1)}(r,1;F)=N_E^{1)}(r,1;G)+S(r,F),\ N_E^{(2}(r,1;F)=N_E^{(2}(r,1;G)+S(r,F),$$

where $\overline{N}_0(r,0;F^{(1)})$ is the reduce counting function of zeros of $F^{(1)}$ which are not the zeros of $F(F-1)$ and $\overline{N}_0(r,0;G^{(1)})$ is similarly defined.

Also we have

$$N_E^{1)}(r,1;F)\leq N(r,\infty;H)+S(r,F).\tag{9}$$

By (8), (9) and Lemma 7 we have

$$\begin{aligned}\overline{N}(r,1;F)+\overline{N}(r,1;G) &\leq N_E^{1)}(r,1;F)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)\\ &+ \overline{N}(r,1;G)+S(r,F)+S(r,G).\\ &\leq N_E^{1)}(r,1;F)+N(r,1;G)-\overline{N}_L(r,1;G)+\overline{N}_{F>1}(r,1;G)\\ &+ \overline{N}_{G>1}(r,1;F)+S(r,F)+S(r,G)\\ &\leq \overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+\overline{N}(r,\infty;F)+T(r,G)\\ &+ \overline{N}_L(r,1;F)+\overline{N}_{F>1}(r,1;G)+\overline{N}_{G>1}(r,1;F)\\ &+ \overline{N}_0(r,0;F^{(1)})+\overline{N}_0(r,0;G^{(1)})+S(r,F)+S(r,G).\end{aligned}$$

Using above inequality and Lemmas 8, 9 in (7) we get

$$\begin{aligned} T(r,F) &\leq 4\overline{N}(r,\infty;F) + 2\overline{N}(r,\infty;G) + 3\overline{N}(r,0;F) \\ &+ 2\overline{N}(r,0;G) + \overline{N}(r,0;F| \geq 2) + \overline{N}(r,0;G| \geq 2) + S(r,F) + S(r,G) \\ &\leq 4\overline{N}(r,\infty;F) + 2\overline{N}(r,\infty;G) + 2\overline{N}(r,0;F) \\ &+ N_2(r,0;F) + 2N_2(r,0;G) + S(r,F) + S(r,G) \\ \textit{i.e.,}\ nT(r,f) &\leq 6\overline{N}(r,\infty;f) + 2\overline{N}(r,0;p(f)) + N_2(r,0;p(f)) + 2N_2(r,0;P) + S(r,f) \\ &\leq (6+2Q)\overline{N}(r,\infty;f) + 2\overline{N}(r,0;p(f)) + N_2(r,0;p(f)) \\ &+ 2N_{2+q}(r,0;f) + S(r,f). \end{aligned}$$

So,

$$(6+2Q)\Theta(\infty,f) + \delta_2(0,p(f)) + 2\Theta(0,p(f)) + 2\delta_{2+q}(0,f) \leq 2Q + 8 + 2n,$$

which contradicts the assumption (3) of Theorem 7. Thus $H \equiv 0$

Proceeding similarly as in Theorem 5, we can prove that the conclusion of Theorem 7 holds.

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