

LUDUS REGULARIS: THE CLERGY GAME

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Abstract: Ludus Regularis is a board game with dice, from the 10th century, which came to our knowledge through literary sources. The rules of the game were kept by the historian Balderic, in the 11th century, in a book edited in 1615 by Colvener: *Chronicon Cameracense et Atrebatense*, and reedited in 1834 by André Le Glay, in Paris: *Chronique d'Arras et de Cambrai*.

By studying the probabilities involved in the dynamics of the game Ludus Regularis, it is possible to advance that the author's concerns were mainly centred on presenting a game that followed Catholic doctrine.

Ludus Regularis is a carefully conceived game, using dice commonly inaccessible to members of the clergy, in a context of deep Christian religious symbolism.

Keywords: Game with dice; Medieval game; Christian virtues.

Sources

It seems natural to start the study on Ludus Regularis by the source which is temporally closest and which, at the same time, is the most divulged: the work *Chronique d'Arras et de Cambrai*, dated 1834, by the French historian Edouard André Joseph Le Glay; when analysing the work, we see that it is based on a book edited in 1615 by George Colvener (*Georgium Colvenerium*): *Chronicon Cameracense et Atrebatense*.

Ludus Regularis Seu Clericalis, the Clergy Game, was invented in the 10th century by Bishop Wibold¹ of Cambrai (France) to give clergy access to

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¹ Wibold would have been the twenty-sixth bishop of Cambrai and, according to Balderic's description, he was a man learned in the humanities as well as in divine themes (*Biographie Universelle Michaud*, 1843, p. 557). Wibold is said to have taken up the office of bishop of Cambrai in 965, holding it for less than a year. In fact, Wibold could not resist

a game of dice without the vulgar component that prohibited this type of game for members of the Church. Colvener's work includes, in respect of *Ludus Regularis*, a chapter numbered LXXXVIII which will be the edition of the work of the 11th century historian Balderic de Théroutanne (Balderico Noviomensi). This chapter, in a second part of the *Chronicle*, is supplemented by a set of notes containing extremely relevant information. The study of these two texts by Colvener (chapter LXXXVIII and its notes) involved direct translation from Latin. Although the information found was very relevant, there is a contingency that it was not possible to overcome: in many passages of the text it is not possible to know whether they belong to Colvener, to Balderic or to Wibold. In fact, there are situations which, due to the time reference after the 11th century, have to be attributed to Colvener, but as for the remaining information, there is doubt as to its author. Moreover, Colvener himself admits that, in addition to Balderic's document, he consulted three different manuscripts in his research for the publication of the work. In fact, in the notes to chapter LXXXVIII, Colvener lists the manuscripts consulted for the construction of the text he presents, on *Ludus Regularis* (all of them connected to monasteries), namely: Roodeclooste, Saint Ghislain and Arras. This information matches entirely with the information that LeGlay records in the early part of his 1834 work, in which he retrieves Colvener's records and deals with the game *Ludus Regularis*. In addition to mentioning the three sources, Colvener draws comparisons between their texts and highlights some inaccuracies in them.

the fatigue caused by the journey to Italy to meet the emperor of the time, Otto I, for his formal investiture, and died shortly after his return (Biographie Universelle M. Weiss, 1841, p. 442).

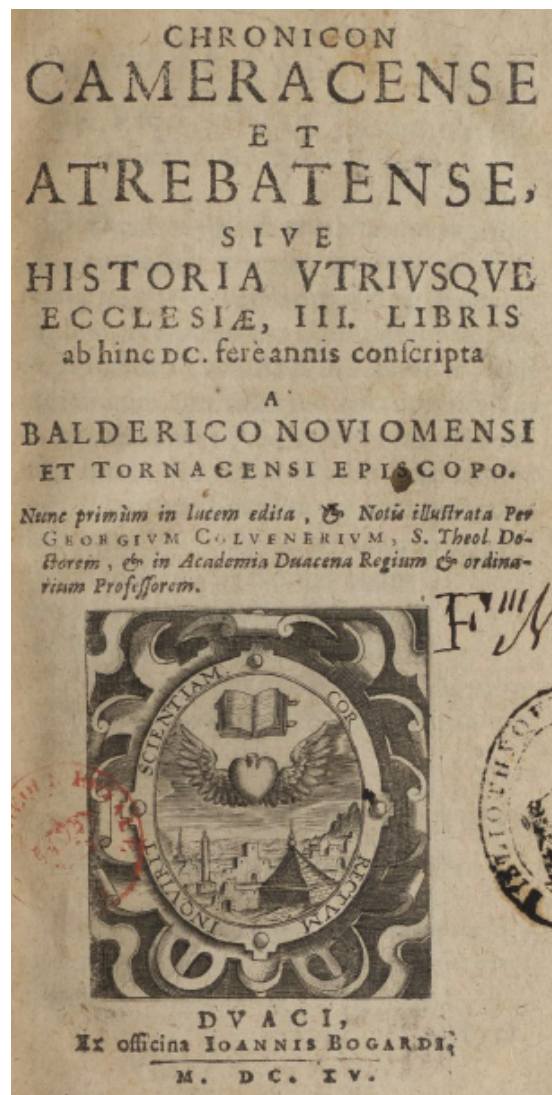


Figure 1: First page of the book *Chronicon Cameracense et Atrebatense*, edited by Georgium Colvenerium, in 1615.

Playing Ludus Regularis

In practising the *Ludus Regularis*, clerics were to work on the level of personal enrichment of the virtues they had earned in a particular match of the game (table 1). To play *Ludus Regularis*, four dice were required: three cubes and a tetrahedron; and a board with the fifty-six virtues that Wibold defined as game objectives (represented in figures 2 and 3). Each side of the

cubic dice is marked with one or more vowels representing the dots on the die: one side with one vowel, one side with two vowels, one side with three, and so on. Sixteen consonants are marked on the tetrahedron dice, divided into groups of four on the sides of the tetrahedron, in alphabetical order.

Figure 2 shows a square board for the game Ludus Regularis, with a grid of 16x16 squares. The board is divided into four quadrants by a central cross. Each quadrant contains a different set of letters or symbols. The top-left quadrant contains the letters A, B, C, D, E, F, G, H, I, K, L, M, N, O, P, Q, R, S, T, V, X, Y, Z. The top-right quadrant contains the letters A, B, C, D, E, F, G, H, I, K, L, M, N, O, P, Q, R, S, T, V, X, Y, Z. The bottom-left quadrant contains the letters A, B, C, D, E, F, G, H, I, K, L, M, N, O, P, Q, R, S, T, V, X, Y, Z. The bottom-right quadrant contains the letters A, B, C, D, E, F, G, H, I, K, L, M, N, O, P, Q, R, S, T, V, X, Y, Z. The board is surrounded by a border containing various text and symbols. The text "Pertinet ad pag. 491. & ad cap. 88. lib. 1." is written at the bottom right.

Figure 2: Square board of the game Ludus Regularis, with the dice schemes, included as a folding element in the Chronicon Cameracense et Atrebatense, edition by Georgium Colvenerium, 1615.

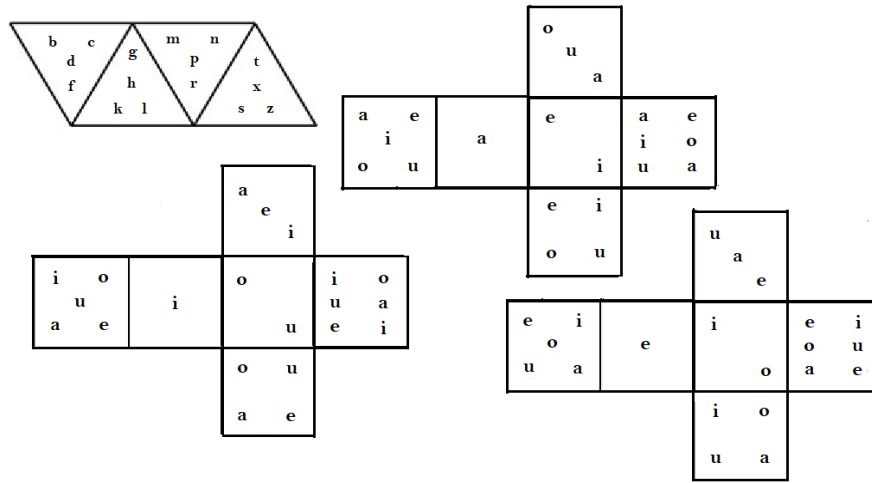


Figure 3: Planning of the dice of the game Ludus Regularis, designed from schemes and descriptions in the works of George Colvener, *Chronicon Cameracense et Atrebatense* and of Le Glay, *Chronique d'Arras et de Cambrai par Balderic*.

Table 1: The fifty-six virtues of the game Ludus Regularis; to the left of each virtue is the numerical combination of the three cubic dice (each vowel is considered as a dot) and to the right is the value resulting from the sum of the three partitions of the dice. The translation to English was done according to the information on the website of MAA (Mathematical Association of America - <https://www.maa.org/press/periodicals/convergence/wibolds-ludus-regularis-a-10th-century-board-game-virtues-outcomes>).

Event (outcome)	Virtue (Latin)	Virtue (English)	Sum (points)
I, I, I	Karitas	Charity	3
I, I, II	Fides	Faith	4
I, I, III	Spes	Hope	5
I, I, IV	Iustitia	Justice	6
I, I, V	Prudentia	Prudence	7
I, I, VI	Temperantia	Temperance	8
I, II, II	Fortitudo	Fortitude	5
I, II, III	Pax	Peace	6
I, II, IV	Castitas	Chastity	7
I, II, V	Misericordia	Mercy	8
I, II, VI	Obedientia	Obedience	9
I, III, III	Timor	Fear	7
I, III, IV	Providentia	Foresight	8
I, III, V	Discretio	Discretion	9
II, III, V	Hospitalitas	Hospitality	10
II, III, VI	Parcitas	Economy	11
II, IV, IV	Patientia	Patience	10
II, IV, V	Zelus	Zeal	11
II, IV, VI	Paupertas	Poverty	12
II, V, V	Lenitas	Mildness	12
II, V, VI	Virginitas	Virginity	13
II, VI, VI	Reverentia	Reverence	14
III, III, III	Pietas	Piety	9
III, III, IV	Indulgentia	Indulgence	10
III, III, V	Oratio	Prayerfulness	11
III, III, VI	Amor	Love	12
III, IV, IV	Judicium	Judgment	11
III, IV, V	Vigilantia	Vigilance	12

I, III, VI	Perseverantia	Perseverance	10	III, IV, VI	Mortificatio	Mortification	13
I, IV, IV	Bonitas	Goodness	9	III, V, V	Innocentia	Innocence	13
I, IV, V	Modestia	Modesty	10	III, V, VI	Contritio	Contrition	14
I, IV, VI	Longanimitas	Long-suffering	11	III, VI, VI	Confessio	Exomologesis	15
I, V, V	Mansuetudo	Gentleness	11	IV, IV, IV	Maturitas	Maturity	12
I, V, VI	Benignitas	Liberality	12	IV, IV, V	Sollicitudo	Concern	13
I, VI, VI	Sapientia	Wisdom	13	IV, IV, VI	Constantia	Constancy	14
II, II, II	Compunctio	Remorse	6	IV, V, V	Intellectus	Understanding	14
II, II, III	Gaudium	Joy	7	IV, V, VI	Suspiratio	Longing	15
II, II, IV	Sobrietas	Sobriety	8	IV, VI, VI	Fletus	Lamentation	16
II, II, V	Delectatio	Satisfaction	9	V, V, V	Hilaritas	Cheerfulness	15
II, II, VI	Suavitas	Sweetness	10	V, V, VI	Compassio	Compassion	16
II, III, III	Astutia	Cleverness	8	V, VI, VI	Continentia	Self-control	17
II, III, IV	Simplicitas	Simplicity	9	VI, VI, VI	Humilitas	Humility	18

In the context of playing the game, here's how you acquire the virtues. Each player rolls the four dice simultaneously. The combination of the points obtained on the cubic dice indicates the virtue in play. However, to acquire that same virtue, some conditions need to be fulfilled:

- The vowels obtained on the three cubic dice must match those of the virtue (for example, to acquire *Oratio*, at least two *o*, one *a* and one *i* must come out);
- At least one of the consonants coming out on the down side of the tetrahedral die must be in the word referring to the virtue in play (for example, for *Oratio* it must come out *r* or *t*);
- The virtue in play has not been previously acquired by another player;
- To acquire the virtue *Karitas*, an exception must be made: *Karitas* will be obtained when the point combination of the roll is I, I, I, that is, the faces corresponding to the vowels *a*, *e*, *i*; this situation does not allow the appearance of two *a*'s, so a second roll of the cubic dice is foreseen to find the missing letter *a*.

Wibold highlights situations in which the point sums of two virtues add up to twenty-one - these are called virtue unions (table 2). In these cases, the counting of points follows a different procedure.

Table 2: Virtue unions.

Karitas	Humilitas	3 + 18	Obedientia	Benignitas	9 + 12
Fides	Continentia	4 + 17	Discretio	Paupertas	
Spes	Fletus	5 + 16	Bonitas	Lenitas	
Fortitudo	Compassio		Delectatio	Amor	
Justitia	Confessio	6 + 15	Simplicitas	Vigilantia	
Pax	Suspiratio		Pietas	Maturitas	
Compunctio	Hilaritas		Perseverantia	Longanimitas	10 + 11
Prudentia	Reverentia	7 + 14	Modestia	Mansuetudo	
Castitas	Contritio		Suavitas	Parcitas	
Timor	Constantia		Hospitalitas	Zelus	
Gaudium	Intellectus		Patientia	Oratio	
Temperantia	Sapientia	8 + 13	Indulgentia	Judicium	
Misericordia	Virginitas				
Providentia	Mortificatio				
Sobrietas	Innocentia				
Astutia	Sollicitudo				

The game ends when all the virtues have been assigned to the players. At that point, to determine the winner, the points of each player must be counted, as follows:

- When virtue pairs make up a union (i. e., their sum is 21 points), the score will be counted as double the value of the highest individual score (e.g., Fletus, with a score of 16, and Fortitudo, with a score of 5, constitute a union - so whoever holds this pair scores 2 X 16 points);
- The virtue Karitas is worth double the points (when it is in union with Humilitas the union rule is fulfilled, i. e., the pair scores 2 X 18 points);
- The remaining virtues are counted by the value that appears in table 1 as their score.

As an alternative, there is also the possibility of playing a simpler and shorter version, with a round board (on whose perimeter are the fifty-six virtues in play) with a needle or pointer fixed in the centre, rotated by the various players (figure 4). In this situation, each player acquires the virtue

indicated by the needle, except when the virtue is already in the possession of another player (support material is needed to record which virtues come out for each player).

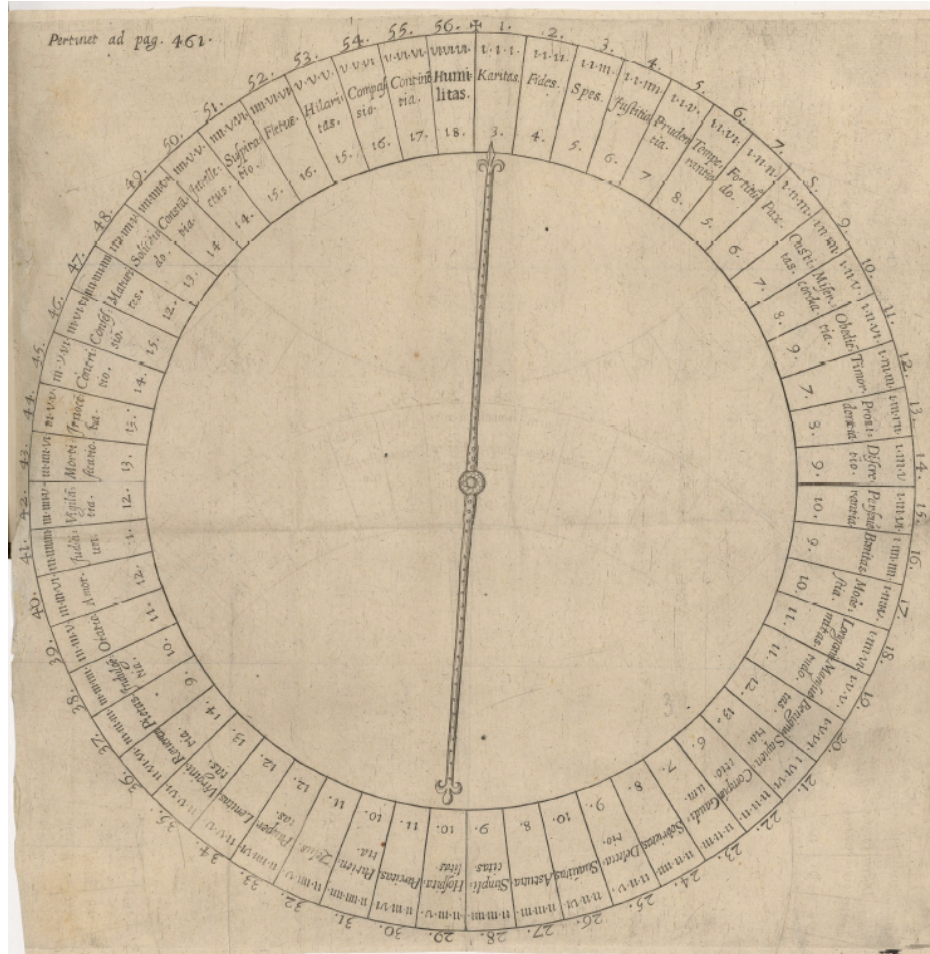


Figure 4: Round board, with pointer, from the game *Ludus Regularis*, included as a folding element in the *Chronicon Cameracense et Atrebatense*, edition by Georgium Colvenerium, 1615.

Why the choice of this characteristic of virtue unions linked to the point total of twenty-one? It is true that if you add the minimum score (three) to the maximum (eighteen) you get twenty-one; the same happens with the second and penultimate (four and seventeen) and with the other pairs already listed. Thus, the relevance of the twenty-one value can come from

that reason: it is the result of the sum of possible scores in the game, having the particularity of moving from the extreme values to the central ones ($3+18=21$; $4+17=21$; $5+16=21$; $6+15=21$; $7+14=21$; $8+13=21$; $9+12=21$ and $10+11=21$).

But in dealing with dice games, the number twenty-one has a particular context: it is the number of possible pairs of outcomes that are obtained with two (six-sided) dice without the order in which they occur being relevant. That is, twenty-one is the total number of possible occurrences with unordered pairs, just as fifty-six is the total number of possible cases with unordered triplets - working with equal dice of six different faces, of course.

As for the practical rules for implementing the game, after reading chapter LXXXVIII and its notes, the care with which they are recorded is evident. Wibold presents them in a careful way, with details such as the registration of who is winning each one of the virtues, or how to end the game without attributing all the virtues (as this would be excessively time-consuming). See chapter LXXXVIII for a suggestion on how to end the game:

And until the sixth hour [noon], those who are left in the lead call those who are left in the lower places and urge them to try to attain by good manners the virtues which the game has not given them. Those who remain in the lower places, for their part, recognizing themselves defeated in the game of dice, and trusting those who remain ahead of them as if they were pedagogues, dare not call them less than masters.

On the rule that defines that a virtue can only be conquered by a player if no other has previously done so, read in the notes to chapter LXXXVIII: “We must note the virtue which you will have on the side. (...) By no means can you claim it for yourself in the same game.”

It also says about the mode of marking the virtues already acquired in the game (in the notes to chapter LXXXVIII):

To this, the aforementioned Boethius Epo² (to whom it was clearly in his mind to bring this game into use) adds this

² Colvener refers to Boetius Epo (Boetii Eponis, 1529 - 1599, Dutch) doctor and professor of law at the University of Doauai - France (according to the author himself, on the third page of the text of the notes to chapter LXXXVIII). Colvener says: “I have

annotation: “And therefore it shall be decided that either one of those present and of the spectators shall write down on a little paper, or rather, that each receptacle of each of the virtues shall be pierced with a needle, or with little wooden sticks, prepared for each of the players, in distinct colours, which shall be pierced with little holes. Or better still, whoever is lucky enough to have these virtues, should cover them with the little wheels called Checkers, each player having his own wheels. Coins can also be used for this, or any other kind of cover”.

There is also a specific mention of the single rule on the virtue *Karitas* (Charity) which refers to its doubled value when counting each player's points, in chapter LXXXVIII: “(...) charity, which is double, in case of need, may be counted for two virtues - since by drawing a single letter, namely *a*, it will be counted as two after the first roll of dice - a rule which is entirely forbidden for the other virtues”.

In the notes to chapter LXXXVIII the subject of *Karitas* is taken up again, with an explanation of the rule involving two rolls of the cubic dice, for it is only in this way that the rule requiring all the vowels of the word in play to be inscribed on the faces of the cubes obtained on the roll can be implemented:

Karitas (Charity) has on its side three units. So, on the dice they make three units, on the first *A*, on the second *E*, on the third *I*. Nor can you chance *Karitas*, unless on its upper surface the dice have these three units. If you take all the vowels out of the word *Karitas* you will be missing an *A*. In fact, it has two *A*'s and one *I*. Wibold thus makes the power to roll again and take a chance on whether, when the dice are rolled, on their upper surface there are two *A*'s and an *I*. Or, certainly, a single die that has only the vowel *E* in the unit must be rolled again, so that if it gives the vowel *A*, you will get, as far as vowels are concerned, the word *Karitas*, without which the whole roll will have been in vain.

fortunately, however, found excellent help here. In fact, the most illustrious doctor and professor of law Boethius Epo, a singular honour of our university of Doau, having sometimes admired this game, reduced it to the boards we had engraved partly in images, partly in bronze”.

Towards the end of the notes on chapter LXXXVIII, Wibold devotes some time to the prize that the winner will receive, preferably, and in the context of the religious framework, dedication to the development of the virtues: “And they should congratulate themselves with a fraternal name. There may also be agreement about another prize for the consideration of the players, though there is advantage in being satisfied with the author's prize, to exercise themselves in virtues”. There is also reference to alternative versions of the game, as it is noted that the original rules can make the game too long. Note, in the notes to chapter LXXXVIII, that the connection with the canons of the Catholic Church is quite evident:

In the same way as if playing with ordinary dice, where only the number is seen, and so two, three or more players, throwing the dice in turn, in a short time they will distribute among themselves all those virtues; he who obtains more virtues will win in both ways of playing. (...) Indeed, whoever throws the dice a second time in the same way makes the throw invalid, just as another who has already obtained this virtue. Hence it is important to note what virtues each one obtains. And it is not always necessary that all the virtues be divided among the players (which would sometimes take many hours), but it is stipulated among the players (for example two) that if one player gets twenty-nine virtues, since he constitutes the majority of them, he is the winner when the game is over. It can also be arranged that the player who obtains first ten virtues, or fifteen, or another number, is the winner, so that only certain moves end the game. For instance, only even or only odd, like only three pieces, six [hexades], five [pentades], etc.

Probabilistic analysis of Ludus Regularis

In table 3 one can find, in increasing order of their point value, the list of virtues with the respective probability of obtaining each one of them. The case of *Karitas* stands out because it is a special case for Wibold, as we have seen in the need to present a specific rule in the output of vowels, to allow *Karitas* to correspond to the output triplet of a point (or a vowel) in each cube. It will be seen later that the reasons justifying this status of the *Karitas* virtue are clothed in religious symbolism or mysticism. For the purposes of probability, it should be noted that *Karitas* is not presented as the virtue with the lowest probability (such a title goes to *Justitia* and *Suavitas*) nor as

the virtue with the highest probability of occurrence (with this place we have Benignitas and Intellectus).

Table 3: Probability of each virtue.

Event (outcome)	Virtue	Sum (points)	Probability
I, I, I	Karitas	3	0,334%
I, I, II	Fides	4	0,231%
I, I, III	Spes	5	0,694%
I, II, II	Fortitudo	5	0,694%
I, I, IV	Justitia	6	0,116%
I, II, III	Pax	6	0,694%
II, II, II	Compunctio	6	0,347%
I, I, V	Prudentia	7	1,042%
I, II, IV	Castitas	7	0,231%
I, III, III	Timor	7	0,463%
II, II, III	Gaudium	7	0,694%
I, I, VI	Temperantia	8	0,694%
I, II, V	Misericordia	8	0,347%
I, III, IV	Providentia	8	0,347%
II, II, IV	Sobrietas	8	0,694%
II, III, III	Astutia	8	0,231%
I, II, VI	Obedientia	9	0,347%
I, III, V	Discretio	9	0,694%
I, IV, IV	Bonitas	9	1,042%
II, II, V	Delectatio	9	0,694%
II, III, IV	Simplicitas	9	0,463%
III, III, III	Pietas	9	0,231%
I, III, VI	Perseverantia	10	0,347%
I, IV, V	Modestia	10	1,042%
II, II, VI	Suavitas	10	0,116%
II, III, V	Hospitalitas	10	0,694%
II, IV, IV	Patientia	10	0,231%
III, III, IV	Indulgentia	10	0,926%

Event (outcome)	Virtue	Sum (points)	Probability
I, IV, VI	Longanimitas	11	0,347%
I, V, V	Mansuetudo	11	1,042%
II, III, VI	Parcitas	11	0,694%
II, IV, V	Zelus	11	0,694%
III, III, V	Oratio	11	0,463%
III, IV, IV	Judicium	11	0,231%
I, V, VI	Benignitas	12	1,389%
II, IV, VI	Paupertas	12	0,463%
II, V, V	Lenitas	12	1,042%
III, III, VI	Amor	12	0,347%
III, IV, V	Vigilantia	12	0,347%
IV, IV, IV	Maturitas	12	0,231%
I, VI, VI	Sapientia	13	0,694%
II, V, VI	Virginitas	13	0,694%
III, IV, VI	Mortificatio	13	0,694%
III, V, V	Innocentia	13	1,042%
IV, IV, V	Sollicitudo	13	1,042%
II, VI, VI	Reverentia	14	0,463%
III, V, VI	Contritio	14	1,042%
IV, IV, VI	Constantia	14	1,042%
IV, V, V	Intellectus	14	1,389%
III, VI, VI	Confessio	15	1,042%
IV, V, VI	Suspiratio	15	0,694%
V, V, V	Hilaritas	15	0,347%
IV, VI, VI	Fletus	16	1,042%
V, V, VI	Compassio	16	1,042%
V, VI, VI	Continentia	17	1,042%
VI, VI, VI	Humilitas	18	0,347%

A statistical analysis of the situation shows that there is no relationship between the point value and the probability of each virtue. The conclusion is the same if the cases of unions of virtues and the probability associated with them are studied.

In view of the analysis made up to this point, it will be said that Wibold did not establish a relationship between the point value he attributed to each virtue and the greater or lesser probability involved in obtaining that same virtue. In fact, as will be seen later, this will take on the same mysticism and religious concerns that were recorded earlier for the fact that *Karitas* is listed as the first virtue.

While it is true that there were no probabilistic concerns, Wibold carefully organised the list of the fifty-six virtues. In fact, the virtues could not be connected to the point total and vowel terms without paying attention to the vowels obtained with the cubic data and the spelling of each of the words (virtues). With a closer look at the letters that make up the virtues and their respective scores and probabilities, it is possible to understand a little better what was going on in Wibold's mind; here are some interesting cases.

The virtues *Fides* (Faith) and *Spes* (Hope) are the only ones that can be obtained with any triplet in cubic dice (that is, they have fifty-six favourable combinations for their occurrence) and, in tetrahedral dice, they both have two sides that allow their occurrence. Nevertheless, their probability values, depending on the triplet with which Wibold associated them, are different: *Fides* has a probability of 0.231% and four points, while *Spes* has a probability of occurrence of 0.694% and is worth five points. In fact, the places assigned to these virtues in the integral list, respectively second and third positions, seem to be related to their importance in the context of the Catholic religion, as we shall see.

About the vowels, some virtues may be considered rare in the sense that, given the cubic dice with the vowels inscribed, they have the fewest combinations of triplets that allow the acquisition of the letters: *Vigilantia* may be obtained with twenty-nine of the possible combinations; *Mortificatio* may be obtained with thirty-two possible combinations; *Perseverantia* may be obtained with thirty-three combinations.

With the consonants, an opposite situation to the previous one can be found: there are virtues whose consonants are always found on the roll of the tetrahedron, that is, with this die, their probability of occurrence takes the value one. These are the virtues: *Benignitas*, *Simplicitas*, *Indulgentia* and *Intellectus*. Nevertheless, the probabilities of occurrence and the associated scores are different, namely: *Benignitas* has a probability of 1.389% and is worth twelve points; *Simplicitas* has a probability of 0.463% and is worth nine points; *Indulgentia* has a probability of 0.926% and is worth ten points; *Intellectus* has a probability of 1.389% and is worth fourteen points. Also, within this group no relationship is revealed between the probabilistic value of each virtue and its score.

Still on consonants, the letter *Q* does not appear in the tetrahedral die, although it belonged to the Latin alphabet as a consonant. In fact, Wibold assumes, in the text of chapter LXXXVIII, this decision:

As for the consonants - since there are sixteen left in the alphabet, apart from the letter *Q* which grammarians consider superfluous and which, for this very reason, we have not counted, also because it is usually associated with another letter, the letter *U*, without which it loses the force of a letter.

The search for a connection between the vowels that make up the virtues, their probability of occurrence and their point value were paths explored. In fact, it is on the vowels that the strongest constraints regarding the virtue in play fall: if in the cubic dice the vowels necessary for the construction of the word do not appear, there is no need even to move on to the tetrahedral dice. As can be seen in the cases listed in table 4, in which sets of virtues that share the same vowels are considered, there is no relationship between the overall probability (which considers the obtaining of the consonants in each case) and the point value of each virtue. For example, *Innocentia* and *Continentia* have equal probability but different point values; *Fortitudo* has twice the probability of occurrence of *Compunctio* and yet is worth one point less; *Lenitas* has a probability about four and a half times higher than *Pietas* and its score is only three points higher, which corresponds to a one-third increase.

Table 4: Virtue analysis according to the vowels that make up words.

Virtue	Probability	Points	Common vowels
Karitas	0,334%	3	a a i
Castitas	0,231%	7	
Parcitas	0,694%	11	
Innocentia	1,042%	13	a e i i o
Continentia	1,042%	17	
Providentia	0,347%	8	
Lenitas	1,042%	12	a e i
Pietas	0,231%	9	
Fortitudo	0,694%	5	i o o u
Compunctio	0,347%	6	
Suavitas	0,116% (minimum)	10	a a i u
Astutia	0,231%	8	
Maturitas	0,231%	12	
Simplicitas	0,463%	9	a i i i
Virginitas	0,694%	13	
Sobrietas	0,694%	8	a e i o
Modestia	1,042%	10	
Fletus	1,042%	16	e u
Zelus	0,694%	11	
Oratio	0,463%	11	a i o o
Compassio	1,042%	16	
Justitia	0,116% (minimum)	6	a i i u
Humilitas	0,347%	18	
Sapientia	0,694%	13	a a e i i
Patientia	0,231%	10	
Longanimitas	0,347%	11	a a i i o
Hospitalitas	0,694%	10	

Thus, it is possible to conclude that Wibold was careful to construct the list of virtues by assigning to them triplets of pips (which are, simultaneously, triplets of vowels), which allow the construction of the word for each of the virtues. Nevertheless, there are cases that raise questions, such as the need to create a specific rule for *Karitas* in order to associate it with the first triplet (I, I, I), when with forty-nine of the remaining triplets it is possible to write *Karitas* in a simple way (it is not possible to obtain *Karitas*, in a direct way, with the triplets I,I,I; I,I,II; I,II,II; II,II,II; II,II,III; II,II,IV; II,II,V). Also, the absence of a connection between the ease or rarity with which virtues are obtained and their point value is a way to conclude that Wibold had no such concerns.

Against the backdrop of explanations (not) provided by mathematics, one must explore the path of symbolism associated with the clerical culture of the 10th century. It will be seen that, in this context, there are many of Wibold's choices for which a *raison d'être* can be found.

Symbolism

In this moment that precedes the more detailed analysis of Wibold's text, regarding the symbology it presents, and through the discourse of several biblical references, some direct others implied, it is worth remembering that the Holy Bible, the basis of the Christian faith, finds its content defined, as we know today, since the 4th century: "Which of the many original Christian writings are really inspired by the Holy Spirit has been established since the 4th century in the so-called Canon of Sacred Scriptures" (YouCat, p. 21). In view of this reality, it is possible to consult the biblical references that are given in the Latin version with a current copy of the Bible.

Wibold groups the virtues; the virtues *Karitas* (Charity) and *Humilitas* (Humility), respectively the first (corresponding to result I, I, I) and the last (relating to result VI, VI, VI) have no group; for Wibold the first is the mother of all virtues and the last is the guardian of the virtues. In Wibold's text one can read:

No one who has read the words of the Gospel (on these two commandments depend all the Law and the Prophets) can be unaware that charity has first place. It is only natural that where such a fountainhead is hidden, there should spring forth a vast and abundant ramification of virtues. Let us imagine, then, charity with its many and varied offshoots. Let us then assign

numbers to each of them in such an order that the following principle should be respected: that what this number can determine, is not contained in it, and that it is impossible for this number to reveal anything to which this number alone cannot lead.

The first group of virtues runs from *Fides* (Faith) to *Sapientia* (Wisdom), containing twenty virtues, and concerns those with the number one included in the result of the dice associated with it; in the notes on chapter LXXXVIII, Wibold calls them virtues daughters of *Karitas* (Charity). The second group of virtues contains fifteen elements, from *Compunctio* (Remorse) to *Reverentia* (Reverence), with those containing the number two as the minimum value of the dice result. The third group, with ten virtues, includes from *Pietas* (Piety) to *Confessio* (Exomologesis), the virtues whose minimum result on the dice is three points. The fourth group, from *Maturitas* (Maturity) to *Fletus* (Lamentation), with six virtues, corresponds to the results for which the minimum score is four points. The fifth group contains only three virtues, from *Hilaritas* (Cheerfulness) to *Continentia* (Self-control), being those for which the minimum value is five points. The last virtue, composed of the repetition of the number six, *Humilitas* (Humility), is the guardian of the virtues. Concerning the last virtue, *Humilitas* (Humility), one can read in the notes to chapter LXXXVIII:

Now, the last number six, as being the first to rejoice in simplicity, possesses the guard of virtues, Humility. Simplicity here does not refer to the virtue, which on the side has the numbers II, III, IV, but it means: just as the three unities (which first put the number and make up a ternary, which you will not find anywhere else but in the first number) have one single Charity, so also the three senarii (which make eighteen, and which you will not find anywhere else but in the last number) have one single Humility.

In the same notes, Colvener states that: “it is to be observed in what a beautiful order he lists the virtues: first the theological, then the cardinal, finally the others, which are added to or taken from these”. In fact, the theological virtues would be *Karitas*, *Fides* and *Spes* (respectively, Charity

or Love³, Faith and Hope), according to the New Testament [1 Cor. 13: 13]. As for the cardinal virtues, it is possible to find them in St. Augustine (in *De Moribus Ecclesiae Catholicae et de Moribus Manichaeorum* or *Of the Morals of the Catholic Church*, in chapter fifteen): Justitia, Prudentia, Temperantia and Fortitudo (respectively, Justice, Prudence, Temperance and Fortitude).

We return to the first virtue, Karitas (Charity), to note its importance. Wibold justifies his choice when he begins the text on Ludus Regularis, bridging directly with a passage from the Gospel according to Matthew: “on these two commandments depend all the Law and the Prophets”.

The Gospel according to Matthew says: [Matt. 22: 34-40] the commandment of love: But when the Pharisees had heard that he had put the Sadducees to silence, they were gathered together. Then one of them, which was a lawyer, asked him a question, tempting him and saying, ‘Master, which is the great commandment in the law?’ Jesus said unto him, Thou shalt love the Lord thy God with all thy heart, and with all thy soul, and with all thy mind. This is the first and great commandment. And the second is like unto it, Thou shalt love thy neighbour as thyself. On these two commandments hang all the law and the prophets.

In a similar way to the Gospel, where Jesus Christ concentrates the entire content of his Law (the Christian Law) on these two commandments, Wibold also makes the other virtues spring from charity.

In this section of the text, in which a framework is given of the point value of the virtues, Wibold mentions that Karitas (Charity), with three points, fits the connection to the trinity, in the sense of the Holy Trinity, sanctifying, principle of its unity: “What do we symbolize by the three if not the name of the sanctifying Trinity principle of its unity?” In the YouCat catechism (pp. 33-34) one can read:

We believe in one God in three persons (Trinity). (...) Christians do not worship three different Gods, but one single Being that is threefold and yet remains one. We know that God is triune from

³ It should be noted that Caritas will be related to the ancient Greek, Agape, which, according to the online encyclopaedia britannica.com, is: “is the highest form of love, charity, the unconditional love of God”. In Greek there are other distinct forms: Eros, physical or sexual love, and Philos, friendship or affection. (<https://www.britannica.com/topic/charity-Christian-concept>).

Jesus Christ: He, the Son, speaks about his *Father in heaven* (“I and the Father are one”, Jonh 10:30). He prays to him and sends us the *Holy Spirit*, who is the love of the Father and the Son. That is why we are baptized “in name of the Father and of the Son and of the Holy Spirit”. [Matt. 28:19.]

For the symbolism of the number four, applied to the virtue Fides (Faith), one finds reference to the four evangelists: St. Matthew, St. Mark, St. Luke and St. John. Then Wibold introduces the number seven on the basis of the sum of three with four, associating it to the Bible in an indirect way, leaving an opening to be able to think of the seven gifts of the Holy Spirit, the seven sacraments, among other references less relevant to the Catholic Church⁴. See what Wibold says in chapter LXXXVIII: “What do we symbolize by three if not the name of the sanctifying Trinity, the principle of its unity? What do we symbolize by the four pipes but the four evangelists? These two together indicate the most holy septiform grace, full of the gifts of charisms”.

The concept of charism has, in the biblical context, a direct link to the Holy Spirit, in the sense of the transformations they produce in human beings. Let us look at the seven gifts of the Holy Spirit (according to YouCat⁵, p. 176, which cites 1 Cor. 12: 8-10 as its source in the Bible): Wisdom, Understanding, Counsel, Fortitude, Knowledge, Piety and Fear of the Lord. Many of these characteristics are connected with virtues that Wibold uses directly in the game; however, cross-referencing this information with the point value he assigns to them, there is no indication of a relationship of interest. In fact, Sapientia (Wisdom) and Intellectus (Understanding), appear, respectively, with thirteen and fourteen points, while Fortitudo (Fortitude) has only five and Pietas (Piety) is worth nine points. The framing of the virtues in the groups that Wibold defines at the beginning of the text also has no effect, since Sapientia and Fortitudo belong to the first group of virtues, Pietas to the third and Intellectus to the fourth.

Wibold does not present his reflections on the symbology of the point valuation of the virtues in an increasing manner of their value, on the

⁴ The seven sacraments of the Catholic Church are: Baptism, Confirmation, Eucharist, Penance, Anointing of the Sick, Matrimony and Holy Orders (YouCat, p. 105).

⁵ In this catechism (p. 176) an explanation is given of the gifts of the Holy Spirit: “With these the Holy Spirit “endows” Christians, in other words, he grants them particular powers that go beyond their natural aptitudes and gives them the opportunity to become God’s special instruments in this world.”

contrary: he makes several cross-references, as he presents his reasoning in text. Thus, after treating the virtues with scores three and four, and from these having also commented on seven, Wibold deals with the scores that constitute unions of virtues, that is, the point values eighteen and seventeen, respectively linked to three and four points. See what Wibold records in the text of chapter LXXXVIII: “The seven added twice, and if we add three, it gives seventeen; if we add four, eighteen - there it is how three, seven, and four come back to themselves”. It follows that in uniting three with eighteen there is a union between *Karitas* (Charity) and *Humilitas* (Humility); in the case of four with seventeen, *Fides* (Faith) and *Continentia* (Self-control) will be united. There are strong indications that, in fact, these virtues have a particular importance for Wibold and are very closely linked to each other. Notice another passage in Wibold's text, after he has treated the eight and the ten:

(...) seek [the player] to obtain the clamour of the tubas of the holy Gospel, which with the ear of the heart loves the holy Trinity with the deepest love, and wishes to be guarded to the end with septiform grace; charity, in which the whole law is fulfilled; faith, without which it is impossible to please God; continence [self-control], to abstain from vices; and humility to be able to maintain the virtues (trying to gather virtues without this one, is like holding dust in the midst of the wind). Once in possession of these virtues, let [the player] strive to practice them with charity, faith, continence [self-control] and humility to reach their summits.

Regarding the number ten, Wibold presents three brief references, all of which point in the same direction - the ten commandments explained in the Old Testament: “(...) attributing the ten to the commandment of the decalogue that once was in force as law and now as grace”; “(...) the ten words of the Law (...)”; “(...) going back through the ten commandments (...)”.

Immediately following his first reference to the number ten, Wibold refers to the number eight, and this happens twice (a second occurrence a little later), the general idea being that of an eighth age or phase of life:

We may also attribute the ten to the commandment of the Decalogue that once was in force as law and now as grace; and the eighth to the eighth age into which we have not yet entered.

In fact, there is evidence that Wibold, in referring to the “eighth age” would have in mind life after death, which, according to the Catholic Church, will be eternal life. In this context, the number eight represents a new beginning, a new order or creation, the rebirth of the human being as resurrection from death and passage to eternal life.

In the context of medieval works that influenced the themes to be broached in the seven liberal arts, particularly with regard to the quadrivium, the article *The influence of quadrivium studies in the eleventh - and twelfth - century schools* reveals a work that is shown to have influenced Wibold in his symbolic description of Ludus Regularis: *Moralia in Job*, or *The Books of the Morals of St. Gregory the Pope, or an Exposition on the Book of Blessed Job*. It should be noted that this work was written by a Pope, Gregory I (between 590 and 604)⁶, who was later sanctified by the Catholic Church. In fact, it is possible to read passages that make evident some common features with the text on Ludus Regularis: the Catholic importance of the number seven and the association of the number eight with the idea of eternal life. One can also find a passage in which the number seven is decoded as the sum of three and four, something that is reproduced in Wibold's text: “But the number seven is among the wise of this world considered to be perfect on some special grounds of its own, because it is the sum of the first even, and the first uneven number. For the first uneven number is three, and the first even number is four” (The Books of the Morals of St. Gregory, 1844, p. 1118).

The number five is treated, by Wibold, in the same paragraph as the number that consolidates a union with it - the number sixteen:

To the number five we can associate the five senses, because if we multiply it by three and add one unit, it is present in the sixteen. If we repress these five senses through Physics, Logic and Ethics in order to return always to the unity of charity which is God, then we will possess naturally, rationally and morally these virtues which are designated by the two numbers.

As a note to this passage a reference from the Gospel according to John is attached, namely John 4: 8, which, in fact, only makes sense if one reads

⁶ Official information published by the Vatican:
<https://www.vatican.va/content/vatican/en/holy-father/gregorio-i--magno.html>

almost the whole of chapter four. If the number five is understood as an allusion to the five senses characteristic of the human being, which, according to Wibold, when repressed, lead to God, the Gospel passage referred to can be understood. In fact, the woman mentioned in the Gospel lived a non-Christian life, proving to be a liar and adulteress, having already lived with five different men. A parallel can be drawn between the five men and the five senses which, when overcome by the woman, or by the human being in general, lead to the discovery of the Christian life.

In the text of notes regarding chapter LXXXVIII, Colvener presents a paragraph which, although somewhat enigmatic, may help to clarify the relationship between the five human senses (referred to at the time of the meaning of the number five) and the eternal life of the soul (with connection to the symbolism of the number eight): “He means that not only the beatitude of body and soul, which one will have in the eighth age, but not even that of the soul (as the saints have until the day of judgment) is acquired unless the senses are restrained from the excesses of voluptuousness, so that the word 'one deserves', which is at the end of the sentence, is understood passively”.

Regarding the connection established between the number five and the number sixteen, this reveals characteristics of Boethius' arithmetic⁷. According to the author Espallargas (2004, p. 290), one of the arithmetical properties that Boethius explores is that of superparticular multiples⁸, it can be read that $16 = \left(3 + \frac{1}{5}\right) \times 5$. This reference offers arguments in favour of Wibold's background in Boethius' arithmetic, i.e., it is evidence of education within the quadrivium, as one would expect in the Early Middle Ages.

Wibold recreates the reasoning of explaining the meaning of some numbers from others. This feature can be confirmed with respect to the reference to the number fourteen (based on the number seven, because it is presented as its double) and the number thirteen (as the sum of five with eight). See, in chapter LXXXVIII:

⁷ Anicius Manlius Torquatus Severinus Boethius, lived between 480 and 525.

⁸ Superparticular multiples will be numbers in which one corresponds to the totality of the other more than once and also to its part. In the case in which 16 is a superparticular multiple of 5 we can say, according to the author Espallargas, that “16 is triplus sesquiquintus of 5”, noting the reference to the three units and the fraction of numerator one and denominator five. (Espallargas, 2004, p. 290.)

There again is the number seven, overflowing with spiritual grace, pouring out for two the number fourteen. (...) However, since we have mentioned eternity above, about which the title of the Psalm says: “to the end over the eighth life”, we can confer on the eight, just as we thought that the ten would be inferior; if we add five to it, it will explain the thirteen.

The number six is explained simultaneously with the number fifteen, with whom it can unite. In the whole chapter LXXXVIII, one finds here the only potentially Pythagorean reference (the question of whether the game *Ludus Regularis* is considered Pythagorean will be dealt with later):

To the number six whose sixth part has the one, and which in the three has its half, since it is a perfect number, well constituted by its parts, we can attribute the perfection of the work. Doubling this number [the number six], and adding three, we obtain fifteen. We shall then be perfect because of the good deed, and we can try to ascend the fifteen degrees mentioned by the author of the Psalm⁹ to deserve to contemplate the Holy Trinity in Zion, and without doubt we shall not be deprived of the virtues which these numbers express.

Regarding the number fifteen, and the reference to fifteen degrees that would be travelled (or ascended) there is a set of biblical texts, in the book of Psalms, which perfectly fits what Wibold's text refers to. In fact, the Psalms numbered one hundred and twenty to one hundred and thirty-four, a total of fifteen texts, constitute the songs of pilgrimage (or songs of degrees); they call for the pilgrimage of the faithful to the city of Zion, or Jerusalem, as the city of God, celebrated as king (Bible, pp. 839-840).

About the number six it is worth reflecting a little more. In mathematics, a perfect number is equal to the sum of all its proper divisors, that is, all

⁹ In the Bible used for this study (p. 838) we read: “The Hebrew and Christian tradition has always attributed great importance to David [King of Israel according to the Old Testament: 2 Sam. 5:1-5] as being at the origin of the Psalms. (...) For the Hebrews, the Psalms were not as important as the books attributed to Moses, for example. (...) but in religious life, the Psalms represented a much-used heritage and a fundamental link in the transmission of faith; some of them are certainly among the most repeated texts in the whole Bible”.

divisors of the number except itself (Katz, 2010, p. 579). In fact, six is a perfect number since its proper divisors are 1, 2 and 3, so $1 + 2 + 3 = 6$.

In the Bible, the number six is related to man's work, and we can read in the book of Exodus [Ex. 20: 9]: “Six days shalt thou labour and do all thy work.” In the book of Genesis, it is explicitly stated that the world was created in six days [Gen. 1: 31 and Gen. 2: 1-2]; the number six can be linked to the work of God, which is, by definition, perfect [Deut. 32: 3-4; Is. 25: 1; Ps. 96: 4-6; Ps. 19: 8-9].

Let us return to *The Books of the Morals of St Gregory*, with some passages relevant to Wibold's understanding:

But if we are now asked to discuss the number of the animals, why a thousand yoke of oxen, or a thousand she asses, and six thousand camels, and fourteen thousand sheep, are mentioned; we can state briefly, that in secular knowledge the number thousand is considered perfect, because it is the solid square of the number ten. For ten times ten are a hundred, which though a square, is a plane figure. But in order that it may rise in height and become solid, the hundred is again multiplied by ten, and becomes a thousand. But the number six is perfect, because it is the first number which is made up of its several parts, that is, its sixth, its third, and its half, which are one, and two, and three, and these added together become six. Nor is any other number found before six, which, when it is divided into its several parts, has its whole amount made up. But because we transcend all this knowledge, by advancing through the loftiness of Holy Scripture, we there find the reason why the numbers six, seven, ten, and a thousand, are perfect. For the number six is perfect in Holy Scripture, because in the beginning of the world God completed on the sixth day those works which He began on the first. The number seven is perfect therein, because every good work is performed with seven virtues through the Spirit, in order that both faith and works may be perfected at the same time. The number ten is perfect therein, because the Law is included in ten precepts, and no fault is forbidden further than by the ten words, and as the Truth relates, the labourers in the vineyard are rewarded with a denarius. (The Books of the Morals of St. Gregory, 1844, pp. 1129-1130.)

In the transcribed paragraph, the notion of perfect number is not the one attributed to Pythagoras; it is also notable the similarity of reasoning with the text of Wibold when it explores the number six according to “the parts” into which it is divided. Transversal to the entire paragraph are the biblical references, also common in the text of *Ludus Regularis*.

By searching for information in the texts of the Bible it is possible to find a reference which, not being explicit in Wibold, is worth looking at. In fact, in the context of the characteristics that the Church associates with the Holy Spirit (fruits of the Holy Spirit), one can find virtues that are part of the *Ludus Regularis* game board. The subject at hand are the fruits of the Holy Spirit, and they constitute a list of nine qualities. It would be licit to think that there might be a connection with some significance in respect of the number nine, but this does not seem to be the case; in fact, concerning the number nine, Wibold says that: “There now remain the larger numbers nine, ten, eleven and twelve. The first is adorned with the nine celestial hierarchies”; adding the enigmatic passage: “whoever, after passing the ten commandments, stumbles over the eleventh, adorned with the splendid lustre of the twelve stones, and to repair the previous nine damages, passes again through the ten commandments”.

See what the catechism *YouCat* (p. 177) says about the nine fruits of the Holy Spirit [Gal 5: 22-23]:

The fruits of the Holy Spirit are charity, joy, peace, patience, kindness, goodness, generosity, gentleness, faithfulness, modesty, self-control, and chastity. In the fruits of the Holy Spirit the world can see what becomes of people who let themselves be adopted, led, and completely formed by God. The fruits of the Holy Spirit show that God really plays a role in the life of Christians.

As with the seven gifts of the Holy Spirit, framed within the interpretation of the number seven, the nine fruits recorded in the Letter to the Galatians overlap with the virtues of the game *Ludus Regularis* (the translation may not be exactly coincident, but it is acceptable that these are qualities that Wibold was aiming at). Nevertheless, in the face of such diversity and broadness, it is inconceivable to achieve a pattern that relates the score of the virtues, the associated probability or the triples that correspond to them.

Regarding the reference to nine heavenly spheres, one finds in St. Thomas Aquinas's¹⁰ *Summa Theologica* (pp. 710-711) a religious interpretation of the entities he calls angels that meets with biblical texts. Nine hierarchies of heavenly creatures are presented. See:

On the contrary, Dionysius, places in the highest hierarchy the “Seraphim” as the first, the “Cherubim” as the middle, the “Thrones” as the last; in the middle hierarchy he places the “Dominations,” as the first, the “Virtues” in the middle, the “Powers” last; in the lowest hierarchy the “Principalities” first, then the “Archangels,” and lastly the “Angels”.

I answer that, the grades of the angelic orders are assigned by Gregory and Dionysius, who agree as regards all except the “Principalities” and “Virtues.” For Dionysius places the “Virtues” beneath the “Dominations,” and above the “Powers”; the “Principalities” beneath the “Powers” and above the “Archangels.” Gregory, however, places the “Principalities” between the “Dominations” and the “Powers”; and the “Virtues” between the “Powers” and the “Archangels.” Each of these placings may claim authority from the words of the Apostle, who [Eph. 1: 20, 21] enumerates the middle orders, beginning from the lowest saying that “God set Him,” i.e., Christ, “on His right hand in the heavenly places above all Principality and Power, and Virtue, and Dominion.” Here he places “Virtues” between “Powers” and “Dominations,” according to the placing of Dionysius. Writing however to the Colossians (1: 16), numbering the same orders from the highest, he says: “Whether Thrones, or Dominations, or Principalities, or Powers, all things were created by Him and in Him.” Here he places the “Principalities” between “Dominations” and “Powers,” as does also Gregory.

In the Bible we can find passages that corroborate this situation, either in the books mentioned by St. Thomas Aquinas, or with the same type of

¹⁰ The chronological line of St. Thomas Aquinas, 1225 - 1274, does not intersect with that of Wibold or Balderic; nevertheless, it was through records of his work that it became possible to frame, from the perspective of the Catholic Church, some of the biblical references in the approach to the nine heavenly hierarchies.

denomination, under the hand of a different author, as Eph. 1: 20-21; Col. 1: 16; Gen. 3: 24; Ex. 25: 18-20; Is. 6: 2; Gen. 16: 7; Ps. 34: 7; Dan. 12: 1.

The number eleven appears, in the text of chapter LXXXVIII, associated with sin. Indeed, in St. Augustine (*Sermons on Selected Lessons of The New Testament*) one can read:

Now, why the number seventy-seven should contain all sins which are remitted in Baptism, there occurs this probable reason, for that the number ten implies the perfection of all righteousness, and blessedness, when the creature denoted by seven cleaves to the Trinity of the Creator; whence also the Decalogue of the Law was consecrated in ten precepts. Now the “transgression” of the number ten is signified by the number eleven; and sin is known to be transgression, when a man, in seeking something “more,” exceeds the rule of justice (p. 436).

A final comment on the book by Pope Gregory I, which immediately refers to the text edited by Colvener: the meaning of the number eleven and its identification with sin. One can also note the connotation of the number ten with the ten Christian commandments.

For every sin belongs to the number eleven, because while it does perverse things, it goes beyond the precepts of the decalogue. And because sin is bewailed in goats’ hair, hence it is that in the Tabernacle there are made eleven veils of goats’ hair [Ex. 26: 7]. Hence it is said in the eleventh Psalm, *Save me, Lord, for the godly man hath ceased* [Ps. 12: 1]. Hence Peter, being afraid of the Apostles continuing in the number eleven, sought, by casting lots, for Matthias as the twelfth [Acts 1: 15-26]. For unless he observed that fault was signified by the number eleven, he would not be so hastily anxious for the number of the Apostles to be completed to that of twelve.

In view of the way Wibold organises the presentation of the various numbers and their symbolism, it makes sense to end the reasoning regarding the number eleven almost simultaneously with the treatment of the number twelve. Regarding the meaning of the number twelve, the statements appear cross-referenced with those of other numbers, in sentences already transcribed in this text; thus, it is possible to read in chapter LXXXVIII:

There now remain the larger numbers nine, ten, eleven and twelve. The first is adorned with the nine heavenly hierarchies; the second with the ten words of the Law; the third with the fruits worthy of penance; the fourth with the twelve chosen heralds of the New Testament.

The fifth, because, as it is written – “We all stumble over many commandments” - whoever, after going through the ten commandments, stumbles over the eleventh, adorned with the splendid lustre of the twelve stones and to repair the nine previous damages, going through the ten commandments again, cover the Church with the eleven curtains of goat's hair, so that someday he may say with the psalmist: “You have turned my weeping into joy” [Ps 30: 11-12].

It makes sense to consider two different references to the number twelve: one associating twelve with stones (“adorned with the splendid brightness of twelve stones”), and one referring to the twelve apostles (“the twelve chosen heralds of the New Testament”). Regarding the allusion to twelve stones, this refers to the Old Testament, starting with “the number twelve is the number of the people of God: twelve tribes [of Israel]”, in the consulted Bible (p. 2036). One can find these references in Ex. 28: 15-21 or in Josh. 4: 4-7. In the New Testament, it is possible to meet the twelve Apostles, namely: Simon (Peter), Andrew, James, John, Philip, Bartholomew, Thomas, Matthew, James, Thaddeus, Simon (the Canaanite) and Judas (can be consulted in Matthew 10: 2-4).

The use of religious symbolism is so transversal to the whole text on *Ludus Regularis* that it is possible to find it when the author explains that cubic dice are read on the upper face (the vowels), while in the tetrahedral dice (that of the consonants) the face resulting from a roll is the one on the underside, that is, the one that touches the table or the game board. In fact, Wibold comments, in chapter LXXXVIII:

It makes perfect sense for the body - of which these, the consonants, are an image, while those, the vowels, represent the image of the soul - be it conveniently endowed with its natural composition, be it deprived of some part, or even added to with some superfluity, it will either have its spirit in its entirety or have none at all. And when that spirit departs, this, the body, returns

to the lowest element from which it was formed, while that, the spirit, returns to God who gave it to it.

Parallel to this rhetoric, and entirely in its context, the biblical passage Eccl. 12: 7 is quoted: “Then shall the dust return to the earth as it was: and the spirit shall return unto God who gave it”.

The Pythagorean character, the case of Rithmomachia and the difficulty of Ludus Regularis

At the beginning of the notes to chapter LXXXVIII we can read some statements that deserve a closer look (from the chronology, they seem to be by Colvener). The first statement to be highlighted is: “Wibold philosophises very ingeniously on the comparison of both numbers among themselves and also of their relation to others. He followed, so it seems, at a distance the Jew Philo¹¹, Plato and the Pythagoreans”. Wibold's knowledge of Plato's works and the concepts attributed to Pythagoras seems unavoidable, since the context that surrounds him is that of an educated personality who, for that reason, will have received an education in the liberal arts, common to people of that class in the Early Middle Ages. But the situation is not so clear when it is stated in the same notes that the game has a markedly Pythagorean character: “(...) this game is similar to the Pythagorean game, or philosophical game, of which there is a booklet printed in Paris, in-octavo format, in the year 1556, with this title: 'Noble and most ancient Pythagorean game, which is called Rithmomachia (...)’”. Thus, we will try to verify if the reality of Ludus Regularis is, in fact, of markedly Pythagorean characteristics and of strong similarities with the game Rithmomachia. Also, Le Glay, in the 19th century (in the work of 1828, p. 191; and in the Chronicle of 1834, p. 465¹²) states that Wibold resorts to characteristics typical of the Pythagorean school. Thus, let us analyse the association of Ludus Regularis with Pythagorean thought.

¹¹ According to the online encyclopaedia

<https://www.britannica.com/biography/Philo-Judaeus>, Philo of Alexandria lived from 15-10 BC to 45-50 AD, playing an important role as a philosopher linked to Judaism.

¹² Le Glay, 1834, p. 465: “It can be seen from this work that Wibold was imbued with the ideas of Pythagoras, and that, like this Greek philosopher, he attached a mysterious significance to numbers.”

As regards the theory of numbers, what mainly occupied the Pythagoreans were the properties and relationships they noted between numbers, which for Pythagoras represent not only the form that governs the combination of things, but also the very matter of these things. (Vasconcelos, 2009, pp. 126-127.)

In the *História da Matemática* [History of Mathematics] (Katz, 2010, pp. 62-64) one can read:

One of these important [Pythagorean] mathematical doctrines was that 'number was the essence of all things' - that numbers, that is, positive integers, formed the basic organizing principle of the universe. (...)

The starting point of this theory was the dichotomy between odd and even. The Pythagoreans probably represented numbers with points or, more concretely, with pebbles (...) It was quite easy, using pebbles, to verify some simple theorems. [The Pythagoreans worked, in this context the figurate numbers: triangular, square, oblong, among others.]

(...)

Another theorem of number theory of particular interest to the Pythagoreans concerns the construction of Pythagorean triples. (...)

The geometrical theorem from which the study of Pythagorean triples grew, namely that in any right triangle the square of the hypotenuse is equal to the sum of the squares of the cathetus, has long been attributed to Pythagoras himself, but there is no direct proof to support this attribution.

Everything indicates that the main results attributed to Pythagoras were part of the cultural context of the scholars of the Early Middle Ages; see:

According to Proclus, Pythagoras divided the science of mathematics into four parts: two corresponding to discrete quantities, considered in themselves or in relation to others, and the other two corresponding to continuous quantities, in the state of rest or movement. Hence, the division of this science into

arithmetic, music, geometry and astronomy, which was adopted almost to the letter by Plato, and then, under the name of quadrivium, with which Boethius (470-526) designated it, followed throughout the Middle Ages, this quadrivium constituting for a long time a course of studies necessary and sufficient for a liberal education. (Vasconcelos, 2009, p. 126.)

By the hand of Weisheipl (1965, pp. 55-65) it is possible to understand the seven liberal arts as the basis of Roman education which aimed to prepare young people for the specialised branches of philosophy, medicine and law. The seven arts were thus divided into trivium (grammar, dialectics and rhetoric) and quadrivium (geometry, arithmetic, astrology and music). Clement of Alexandria (Titus Flavius Clemens, 354-430), St. Augustine (Augustinus Hipponensis, 354-430), Cassiodorus (Aurelius Cassiodorus, 487-585), Isidore (St Isidore of Seville, 560-636) and, above all, Boethius (Manlius Severinus Boethius, 475-524¹³) contributed to the assimilation of these arts in medieval Europe.

The conditions seem to be met for the argument that *Ludus Regularis* is a game with markedly Pythagorean influences, as stated by Colvener in the notes to Chapter LXXXVIII and Le Glay in his work of 1834. After the brief background regarding mathematics education in the context of the Early Middle Ages and the ideas surrounding the Greek Pythagorean, some conclusions may be drawn. In fact, if on the one hand, it is unavoidable that Wibold received influences from Antiquity, on the other hand the Pythagorean characteristics pointed out by Colvener (or perhaps by Balderic, who preceded him) and by Le Glay are not visible. With the exception of the reference to the number six as perfect (a Pythagorean concept within the framework of what is now called number theory), no other mentions can be found that can be linked to Pythagoras. On the contrary, the exploration of the remaining symbolism that Wibold presents falls on the edge of the Holy Bible or of philosophers and scholars of the same, such as St. Augustine (354-430) or St. Gregory (540-604) - both of whom have been sanctified by the Catholic Church.

¹³ These dates differ, slightly, from those presented by Vasconcelos; nevertheless, the reference made by the authors is kept, having in mind that, regarding such a distant past, small historical differences are common.

In the work by Masi and Soubiran, evidence is presented regarding knowledge of the notion of perfect number, contextualized in the spirit of the quadrivium and not necessarily with a direct link to Pythagoras. It is possible to find references to moral and virtuous behaviour that cannot fail to suggest the description of Ludus Regularis:

The ethical implications of the study of the quadrivium become apparent from the beginning of the *De Institutione Arithmetica* [one of Boethius' works]. The quadrivium is a four-fold path to the study of the moral truths of the Consolation [*De Consolatione Philosophiae*, by Boethius], and Boethius is insistent that these steps be taken carefully and methodically. (...) In chapter nineteen of Book 1 [*De Institutione Arithmetica*], for example, Boethius discusses perfect numbers, that is numbers equal to the sum of all their possible divisors, e. g. $1 + 2 + 3 = 6$; $1 + 2 + 4 + 7 + 14 = 28$. These he compares to virtues and vices. As with perfect numbers, virtue is rare, and most moral behaviour is short or in excess of the virtuous middle. (Masi & Soubiran, 1983, p. 40.)

Following the authors' comments, we can see directly what Boethius said (in *De Institutione Arithmetica*), through the translation provided in the same work:

Between the two kinds of number [superfluous and diminished¹⁴], as if between two elements unequal and intemperate, is put a number which holds the middle place between the extremes like one who seeks virtue. That number is called perfect and it does not extend in a superfluous progression nor is it reduced in a contracted reduction, but it maintains the place of the middle; the sum of its parts is not more than the total nor does it suffer from a lack in comparison with the total, as are 6 and 28. (Masi & Soubiran, 1983, p. 97.)

Let us take a counterexample concerning the Pythagorean influence on Wibold. Among the triplets in Wibold's game is the best-known

¹⁴ Boethius explains, in the same section called chapter nine, that a superfluous number is one whose sum of its proper divisors is greater than the number (for example, 12); a diminished number will be one whose sum of its proper divisors is less than the number (for example, 8).

Pythagorean triplet, since it is the only one made up of three consecutive positive integers: 3, 4 and 5¹⁵. However, Wibold, who associates this result with the virtue *Vigilantia* (Vigilance), bypasses all this interpretation. In fact, the virtue he associates with this triplet is not even worthy of any prominence in the documents surrounding Colvener's publications, just as it is not a prominent quality within biblical studies. If the relationship between Wibold and the studies of the school of Pythagoras were noteworthy, everything indicates that this result would be highlighted.

On the result known as the Pythagorean theorem and the Pythagorean triplet 3, 4, 5 one can read in *The Exact Sciences in Antiquity*:

(...) it was known during the whole duration of Babylonian mathematics that the sum of the squares of the lengths of the sides of a right triangle equals the square of the length of the hypotenuse. This geometrical fact having once been discovered, it is quite natural to assume that all triples of numbers l , b and d which satisfy the relation $l^2 + b^2 = d^2$ can be used as sides of a right triangle. It is furthermore a normal step to ask the question: When do numbers l , b , d satisfy the above relation? Consequently it is not too surprising that we find the Babylonian mathematicians investigating the number-theoretical problem of producing "Pythagorean numbers". It has often been suggested that the Pythagorean theorem originated from the discovery that 3, 4 and 5 satisfy the Pythagorean relation. I see no motive which would lead to the idea of forming triangles with these sides and to investigate whether they are right triangles or not. It is only on the basis of our education in the Greek approach to mathematics that we immediately think of the possibility of a geometric representation of arithmetical or algebraic relations. (Neugebauer, 1969, p. 36.)

Another very simple situation, which can serve as a counterexample, in the context of the Pythagorean influence in the *Ludus Regularis* game (or rather, of the absence of evidence of such a connection) may come from the figurate numbers (Katz, 2010, pp. 62-64). In fact, in the sums of the values

¹⁵ In fact, it is easy to prove that, since the values that make up the triplet are called $n, n + 1, n + 2$ (three consecutive integers), the only positive value of n that verifies the Pythagorean Theorem $n^2 + (n + 1)^2 = (n + 2)^2$ is $n = 3$, which means that the triplet will be made up of the integers 3, 4 and 5.

obtained in the dice it is possible to find square numbers (such as four, nine and sixteen) and triangular numbers (such as three, six, ten and fifteen), whereas Wibold makes no connection between these Pythagorean characteristics and the meaning he attributes to any of these numbers.

It is clear that figurate numbers were known in the Middle Ages; reference to them can be found in the work of Boethius. Images in the republication of the work *De Institutione Arithmetica* that have been used (Masi & Soubiran, 1983, pp. 133-135), illustrate triangular numbers (1, 3, 6, 10, 15, 21 ...), square numbers (1, 4, 9, 16, 25 ...) and others are listed, as pentagonal, hexagonal or heptagonal.

One last example of Pythagorean mysticism that Boethius mentions and which, by not having any reflection in Wibold's text, contributes to the strengthening of the thesis of the distance between the game *Ludus Regularis* and the Pythagorean line: the number ten is now in question. Wibold unequivocally associates it with the Judaeo-Christian ten commandments. The Pythagoreans, on the other hand, here through Boethius, present the importance of the same number ten in a totally different symbolic context:

It is testified to and known among the ancients who have studied the learning of Pythagoras, or Plato, or Aristotle, that these are the three ways to knowledge: arithmetic, geometric, harmonic. (...) Then later thinkers, on account of the perfection of the number ten, which was pleasing to Pythagoras, added four other kinds, so that in these proportionalities they brought together a body of proportions ten in number. According to this number we describe the prior relationships and comparisons where there are five in the major proportions, which we call leaders, and with them we put five others, minor terms, which we call followers. Also in Aristotle's and Archytas' description of the ten predicaments, the Pythagorean ten is manifestly found. (...) So also there are ten parts among these groups and many others tens, and it is not necessary to pursue all of these here. (Masi & Soubiran, 1983, p. 165.)

In the notes on the transcribed paragraph it reads:

The mystical significance of the number ten among the Pythagoreans was well known to the ancients. It symbolized the

totality of universal existence and by the four numbers of the tetraktys ($1 + 2 + 3 + 4 = 10$) the Pythagoreans swore their most sacred oath. (Masi & Soubiran, 1983, p. 165.)

With regard to the game Rithmomachia, the statement recorded in the notes to chapter LXXXVIII¹⁶ is evidently Colvener's, for only in this way is the allusion to a work published in the 16th century justified (Le Glay, in 1834, also refers to Rithmomachia, reproducing the information recorded by Colvener - p. 465). However, the connection that Colvener sketches between Rithmomachia and Ludus Regularis does not seem to have much foundation.

The game Rithmomachia¹⁷, invented in the 11th century¹⁸, enjoyed great popularity in the 15th and 16th centuries, with several manuals published about it by authors from areas as diverse as the humanities, mathematics and education (Moyer, 2001, p. 1-2). As it seems to be a game of European origin¹⁹, it is not possible to attribute a specific origin to Rithmomachia, and there is more than one possible line of study:

The conceptual origins of Rithmomachia are twofold. One lies in a tradition of philosophical numerology dating back to the school of Pythagoras, and transmitted during the Dark Ages by Nicomachus of Gerasa (c. AD 100) and Boethius (c. AD 500), whose *Arithmetica* is largely a translation of Nicomachus. The Pythagorean arithmetic of Rithmomachia is essentially integral and proportional. (...)

¹⁶ Let us remember: “(...) this game is similar to the Pythagorean game, or philosophical game, of which there is a booklet printed in Paris, in inoctavo format, in the year 1556, with this title: 'Noble and ancient Pythagorean game, which is called Rithmomachia (...)’”.

¹⁷ “The nature of the game is indicated by its several names, of which the chief is Rithmomachia, a quasiLatin word derived from Greek *rithmos* ‘number’ and *mache* ‘batle’. (...) also known as the Philosophers’ Game – in the medieval sense of the word, that is, equivalent to what we might nowadays call scientists, or perhaps mathematicians.” (Parlett, 2018, p. 332.)

¹⁸ Ann Moyer dates the game to the 11th century; authors such as David Parlett, José Espallargas, David Smith and Clara Eaton state that it is not possible to assign, with certainty, a date (or author) for the birth of the Rithmomachia game; the fact that there is no evidence or documents about the game prior to the 11th century is an accepted one.

¹⁹ The author Espallargas (2004, p. 282) states that the oldest documents to which we had access originated in Central Europe and all related to monastic religious life.

The second element relates to the initial array. Rithmomachia is not only a game of numbers but also an abstract representation of warfare, like Chess. The authors of a German study²⁰ of the game point to a parallel between the forces and initial disposition of the game with those of the army of classical antiquity.

(...)

What light this throws on the origin of Rithmomachy remains to be explored. It is clearly an European invention, and probably inspired by the example of Chess; yet this points to a time around the end of the first millennium when few European scholars and clerics (synonymous terms, in effect) can have had either the mathematical competence or the practical knowledge of classical warfare to have devised so complicated a game. Scarcely more credible, and certainly unevidenced, is the possibility of its origin in late antiquity and transmission through the European Dark Ages. (Parlett, 2018, pp. 339-340.)

In the last paragraph transcribed above, the author David Parlett underlines two points that can be considered common between Rithmomachia and Ludus Regularis: the origin in Medieval Europe and the context of intellectually evolved players belonging to the religious class.

With the author Ann Moyer it is possible to find an argument that brings the games Ludus Regularis and Rithmomachia closer together – the ability to raise the human spirit to a state where the character of the individual can grow qualitatively, in an almost religious perspective:

Some of the claims made through the years for the value of the game [Rithmomachia] found their basis in general arguments about the value of game playing or recreation in society. Other claims had broader implications; they referred to the importance of arithmetic, and the Boethian arithmetic in particular. They help to impress upon the modern reader just how important were the arguments also made by the authors of textbooks and by educational authorities about the subject's value. Arithmetic

²⁰ Author David Parlett refers to D. Illmer et al., *Rhythmomachia* (Munich, 1987), 'based partly on oldest game descriptions, partly on French and German treatises from the Renaissance and C17'.

deserved study, they asserted, not simply because it served as a useful tool for the solving of computational problems. It improved the character of the person who studied it; further, it offered insight into religious truth. The philosopher's game [Rithmomachia] put these principles into practice. The player benefited by coming to master the calculatory skills needed to win the game (skills that could be applied to other problems outside the game as well). More important, in contemplating and practicing the principles of arithmetic he thereby improved his soul. (Moyer, 2001, p. 3.)

The author Espallargas offers a perspective of the game Rithmomachia as an intellectual activity suitable for members of the clergy, as opposed to other games (including chess) that are not in line with the possible virtuous development of the players; this reading leads to points of contact between the games Rithmomachia and Ludus Regularis. See:

(...) the way rhythmomachia was perceived by his contemporaries: it was a game that allowed the pleasant learning of arithmetic and, especially, of the way it was explained by Boethius, but it was also a profitable intellectual entertainment in the eyes of Catholic religiosity, so critical, on the other hand, of any other type of pastime, since even chess, which was not explicitly forbidden, was not included in the category of virtuous entertainments either. (Espallargas, 2004, p. 283.)

Nevertheless, both Espallargas and Ann Moyer assume that the main aim of Rithmomachia is mathematical education; it was intended to develop specific learning, that which would be included in the quadrivium (with mathematics, or arithmetic, which is considered to be Boethius'). Wibold, on the other hand, assumes, in the first paragraph that Balderic has recorded in chapter LXXXVIII, that Ludus Regularis aims at the development of virtues as opposed to vices (in particular those arising from dice games): "By means of this game, which they would practise naturally in the schools, they could exchange vice for charity and avoid the profane - and feuding - game of dice".

In addition to the differences in the objectives pursued with the construction of each of the games, the way they work is very different. Rithmomachia is played on a rectangular board (eight by sixteen) with one

hundred and twenty-eight squares (in a pattern common to chess and checkers boards), where both players move their black and white pieces (circular, triangular, square or pyramid-shaped, all with numbers on them) and make captures using rules that involve mathematical concepts, until the winner is found (Moyer, 2001, pp. 143-145. Smith & Eaton, 1911, p. 74). In the game *Ludus Regularis* the operation is quite different: it is a dice game, with an unlimited number of players, in which the existence of a board is not imperative (the players can simply be guided by a list of virtues and triples, writing down the ones they acquire during the game); there are no moves or captures, also there are no pieces.

One of the first characteristics that Colvener, in his notes on chapter LXXXVIII, points out to the game *Ludus Regularis* is the difficulty in its implementation. Curiously, it is in this line of reasoning that he brings to this reality of Wibold's edition in the 17th century, another personality who seems to have had relevance: Boethius Epo, already presented in this document in footnote number two. Read:

Then the clerical game of this Bishop Wibold, which is described throughout this chapter, is outside the story and so obscure that I confess I don't understand some of his points. I have fortunately, however, found excellent help here. In fact, the eminent doctor and professor of law Boethius Epo, a singular honour of our university of Doai, having sometimes admired this game, reduced it to the tablets we had engraved partly in pictures, partly in bronze. He himself also states in pages (which I received from his son, Epo Boethius, himself a Doctor of Law. and a most distinguished professor at our university) that he received the interpretation of some of the more obscure words of this author through another more recent author, to which he added some [words]. Of these, therefore, and of others that have occurred to us, we take note for the benefit of those who perhaps like these things.

In Le Glay's work, one can see that the author shares Colvener's opinion on the difficulty of *Ludus Regularis*, or possibly reproduces the opinion of the editor before him, without having looked more carefully at the question of the rules of the game. In fact, one can read (p. 466): "Without going into the inextricable maze of this game, we shall limit ourselves here to giving one of three tables that Colvener has attached to his notes, tables that offer

at least a general idea of the way in which it seems that Wibold intended his clerics to enjoy themselves.”

This idea of difficulty or obscurity of the game *Ludus Regularis*, present in Colvener and which Le Glay repeats, does not seem to have any connection with reality. As it was possible to explore in this paper, the care with which Wibold explains the rules, with details as specific as on which side of the dice one reads the result of the move or the repetition of the differentiated status of the virtue *Karitas*, absolutely contradict the idea of incomprehensibility of the game. Even in the face of the feature of the game that could cause less dissemination, namely its long duration, Wibold offers alternative versions: such as ending at a certain moment (he mentions noon) and ascertaining, there, who would have the highest score, or ending when a player has attained twenty-nine virtues, or even “it is arranged among the players that he who first obtains ten virtues, or fifteen, or another number, shall be the winner, so that only certain moves end the game” (transcription of the notes to chapter LXXXVIII). The amount and care with which comments are made regarding alternatives that make the length of the game shorter may be an indication of an affirmative answer to the question regarding the actual implementation game - whether, in practice, *Ludus Regularis* was played. In the notes to chapter LXXXVIII, regarding a very specific and practical question - how to mark, in the course of the game, the virtues that each of the players manages to acquire - one can read, again, the name of Boethius Epo; this time associated with the dissemination of the game *Ludus Regularis*. The truth is that the question of playing *Ludus Regularis* remains open.

To this, the aforementioned Boethius Epo (to whom it was clearly in his mind to bring this game into use) adds this annotation: “And therefore it shall be decided that either one of those present and of the spectators shall write down on a little paper, or rather, that each receptacle of each of the virtues shall be pierced with a needle or with little wooden sticks, prepared for each of the players, in distinct colours, that are pierced with little holes. Or better still, whoever gets lucky with these virtues should cover them with the little rings called Checkers, each player having his own rings. Coins can also be used for this, or any other kind of cover”. (In the notes to chapter LXXXVIII.)

The reference to other authors, namely Boethius Epo, deserves a few words (the fact that works by other personalities refer to the Ludus Regularis reveals a possible common ground for determining who really played it). In his work *Heroicarum et Ecclesiasticarum Quaestionum Libri VI*, Boethius Epo refers to Wibold in the context of a set of rules to be followed by members of the clergy, almost a manual of conduct. In this context, Wibold is presented as an exemplary personality who contributed to the elevation of the habits of the clergymen, building a dice game that they can use and that, in addition to recreational moments, provides spiritual development:

The priests, they say, appreciate the dice. This is a miserable plague to the souls of fortune, and often the most odious of civil laws. A certain Wiboldus, bishop of Cambrai, invented a certain spiritual game, in which there is a contest of fifty-six virtues; it was clever and pious by fate, that he might divert the clergy from other kinds of play. As the old chronicle of the bishops collects the manuscripts of Cameracesus and Arras, these are restored to the commons: we shall publish the game of Wiboldus in his notebooks, which we shall print in due course, with illustrations. Wiboldus flourished during the reign of Otton I, that is, six hundred years ago.

A few words about Boethius Epo. This personality, a native of the Netherlands in the 16th century, is found in Colvener's notes on chapter LXXXVIII. After locating the digital formats of his works (at least those that it is possible to find in Dutch, French or Belgian databases), it was in the work *Heroicarum et Ecclesiasticarum Quaestionum* that the references to Wibold and his virtue game were found. As can be read in these pages, by consolidating the information given to us by Colvener and Boethius Epo himself, this professor at the University of Douai will have had some interest and carried out research into the game Ludus Regularis. In the second footnote there is the indication, given by Colvener, that illustrations or plates about Ludus Regularis would have been produced by Boethius Epo. Despite all the efforts made, it was not possible to advance this line of research, since no other works by Boethius Epo have been found, either in book format or in any other support more connected to the graphic arts.

In the 20th century, some mathematics in connection with Wibold: the function W

In 1988, the authors Kurt Kreith and Judith Kysh published an article in *The Mathematics Teacher* magazine, in the United States of America, which refers to the game Ludus Regularis: *The Fourth Way to Sample k Objects from a Collection of n* . In this article, the authors devote some attention to mathematical techniques of counting, or combinatorial analysis, in cases in which it is intended to obtain a subset of elements from a larger set, in which it is possible to have repetitions of objects and the order in which their appearance occurs has no relevance to the situation in question.

How many possible results can be obtained if five balls are extracted from a box of eight balls? Or, in a more contextualized perspective with the historical episode that we want to follow: from cubic dice, numbered from one to six, how many results can be obtained by making three throws? In a more general way, one can ask the question: how many possibilities are there to extract sets with n elements in a process repeated k times? For a correct approach, it is necessary to take into account whether the situation occurs with or without replacement of the previously extracted object and whether the order in which the extracted elements are obtained is relevant. Since each of these scenarios can be configured in yes or no answers, there are four different situations to consider.

Performing the replacement of the extracted objects and being the order relevant, in a situation of obtaining sets of k elements, from n objects, we have n^k possible results, each of them being similar to a $1 \times k$ row matrix in which each of the entries of the k columns can assume values between 1 and n .

If the extraction of the elements is done without replacement, still having relevance the order in which the elements are considered we obtain (with $k \leq n$): $n(n-1)(n-2) \dots (n-k+1) = \frac{n!}{(n-k)!} = {}^nA_k$ possible results, each result being a $1 \times k$ row matrix, in which no two entries are equal, with values between 1 and n .

Consider that the construction of subsets is being carried out without replacement of the extracted objects and without relevance to the order in which such extraction takes place. Since the order is irrelevant, it no longer makes sense to think of matrices and start considering only unordered sets

of objects. In this situation there will be $\frac{{}^nA_k}{k!} = {}^nC_k = \frac{n!}{k!(n-k)!}$ possibilities of results.

The last situation to consider is to extract k elements from a set with a total of n , where it is possible to repeat the elements, that is, there is replacement, but the order in which the k objects are extracted is of no interest. This case can be thought of as the simultaneous throwing of k indistinguishable dice with n faces. This situation is found in the game Ludus Regularis by Wibold.

In Wibold's game three cubic dice numbered one to six were used (in fact, the faces of the dice were filled with vowels that could be interpreted as the number of points). In Ludus Regularis, it is with some ease that the fifty-six possible outcomes are enumerated, but with higher values of n and k , the existence of a generalised formula has benefits.

It is precisely in tribute to Wibold that the notation nW_k is used to refer to the number of results that can be obtained when subsets of k elements are formed from a set with n objects, there being element replacement, but without the order of extraction having any relevance. In fact, ${}^nW_k = {}^{n-1+k}C_k = \frac{(n-1+k)!}{k!(n-1+k-k)!} = \frac{(n-1+k)!}{k!(n-1)!}$. This situation can be illustrated by thinking of k dice rolls with n faces, disregarding the order in which the dice are rolled. From the analysis of Ludus Regularis it is known that ${}^6W_3 = 56$.

Indeed, ${}^6W_3 = {}^{6-1+3}C_3 = {}^8C_3 = \frac{8!}{3!5!} = 56$, or ${}^6W_3 = \frac{(6-1+3)!}{3!(6-1)!} = \frac{8!}{3!5!} = 56$.

Concluding Remarks

It is fair to conclude that Wibold's work in developing Ludus Regularis has not been forgotten. As already mentioned, the manuscripts have been preserved in at least three monasteries in the area of Arras and Cambrai, and in the 11th century they are said to have been compiled and treated by Balderic. Colvener published his work in 1615; two centuries later, Le GLay took up the subject again and, in re-editing Colvener, consulted other manuscript works that Colvener would not have known about, and made a decisive contribution to the knowledge of this game from the 19th century onwards.

At the end of the 20th century, a little over thirty years ago, two researchers in the field of mathematics education, in the United States of America, revisited Wibold's work from the perspective of counting cases associated with the throwing of dice. This reality shows the importance and the current relevance of Ludus Regularis - a religious, virtuous motivation which has indelibly marked the history of counting techniques in the context of the birth of probability theory.

The dynamics of the game and its spirituality are magnificent as soon as one takes the time to understand them. Who played Ludus Regularis, and in what time frame this might have happened, are open questions. Nevertheless, there is the possibility that they will be answered, as soon as it is possible to carry out the research on Boethius Epo – the 16th century Dutch professor, who seems to have had an interest in the game Ludus Regularis.

References

- Aquinas, Saint Thomas. *Summa Theologica*. URL: <http://www.ccel.org/ccel/aquinas/summa.html>
- Balderic. (1615). *Chronicon Cameracense et Atrebatense*. George Colvener, editor. Ioannis Bogardi.
- Biographie Universelle (Michaud) Ancienne et Moderne*, Volume 44. Paris (Madame C. Desplaces) e Leipzig (Librairie de F. A. Brockhaus), 1843.
- Biographie Universelle ou Dictionnaire Historique*. Direction by M. Weiss; volume 16. Paris: Furne et Cie Libraires-Éditeurs, 1841.
- Boécio E. (1588). *Boetti Eponis Frisii. Heroicarum et Ecclesiasticarum Quaestionum Libri VI*, Edition by Ioannis Bogardi.
- Boethius. *The theological tractates. De Trinitate*. Translated by H. F. Stewart, D. D. and E. K. Rand, Ph D.
- Chevalier, J; Gheerbrant, A. (1996). *Dictionary of Symbols*. Translated from the French by John Buchanan-Brown; Great Britain, Penguin Books, 1996.
- Espallargas, J. M. N. *La Aritmética de Boecio y la Ritmomaquia: Teoría y Práctica del Juego Medieval de los Sabios*. Anuario de Estudios Medievales, 34/1 (2004), pp. 279-306.
- Evans, G. R. (1975) *The influence of quadrivium studies in the eleventh – and twelfth – century schools*. Journal of Medieval History I (151-164), North Holland Publishing Company.
- Feller, W. (1968). *An Introduction to Probability Theory and Its Applications*, Volume I, 3rd edition. New York: John Wiley & Sons.
- Holy Bible*. Authorized King James Version; Published by Thomas Nelson; Tennessee, U. S. A., 2017.

- Katz, V. J. (2010). *História da Matemática*. Revised by Jorge Nuno Silva; Lisboa, Fundação Calouste Gulbenkian.
- Kreith, K; Kysh, J. (1988). The fourth way to sample k objects from a collection of n . *The Mathematics Teacher*, 81: 146-149.
- Le Glay, A. (1828). *Mémoires de la Société d'Emulation de Cambrai*. Cambrai: S. Berthoud, Imprimeur du Roi.
- Le Glay, A. (1843). *Chronique d'Arras et de Cambrai par Balderic*. Paris: Levrault, J.-A. Mercklein, e Téchener.
- L'Hôte, J-M. (1994). *Dictionnaire des Jeux de Société*. Paris: Flammarion.
- L'Hôte, J-M. (1994). *Histoire des Jeux de Société*. Paris: Flammarion.
- Masi, M; Soubiran, J. *Boethian number theory. A translation of the "De Institutione Arithmetica", 1983 (Studies in Classical Antiquity, vol. 6)*. *Revue des Études Anciennes*; URL: http://www.persee.fr/doc/rea_0035-2004_1984_num_86_1_5246_t1_0385_0000_3
- Moyer, A. E. (2001) *The Philosophers' Game: Rithmomachia in Medieval and Renaissance Europe*. University of Michigan Press.
- Neugebauer, O. (1969). *The Exact Sciences in Antiquity*. Dover Publications, Inc., Nova Iorque.
- Parlett, D. (2018). *Parlett's History of Board Games*. Echo Point Books & Media, Vermont.
- Sagrada Bíblia Católica: Antigo e Novo Testamentos*. Coordination by Herculano Alves. Lisboa, Fátima: Difusora Bíblica (Franciscanos Capuchinhos), 2018.
- Smith, D. E.; Eaton, C. C. *Rithmomachia, the Great Medieval Number Game*. *The American Mathematical Monthly*, Vol. 18, No. 4, (1911) pp. 73-80. Mathematical Association of America.

- Stapleton, H. E.; G. J. W. *Ancient and Modern Aspects of Pythagoreanism*. Osiris, 1958, Vol. 13 (1958), pp. 12-53. Publicado por: The University of Chicago Press on behalf of The History of Science.
- St. Augustin. *Of the Morals of the Catholic Church and On the Morals of the Manichæans. De Moribus Ecclesiae Catholicae et de Moribus Manichaeorum* Translated by the Rev. Richard Stothert, M. A.. URL: <http://www.ccel.org/ccel/schaff/npnf104.html>
- St. Augustin. *Sermons On Selected Lessons Of The New Testament*. Translated by the Rev. R. G. MacMullen, M.A.; Edited by Philip Schaff, D.D.. URL: <http://www.ccel.org/ccel/schaff/npnf106.html>
- St. Gregory the Great. *The Books of the Morals of St. Gregory the Pope, or an Exposition on the Book of Blessed Job*. Published by: Oxford, John Henty Parker, J. G. F. e J. Rivington, London, 1844. URL: <http://www.lectionarycentral.com/gregorymoraliaindex.html>
- Vasconcelos, F. A. L. (2009). *História das Matemáticas na Antiguidade*. Coordinated and edited by Augusto J. Franco de Oliveira; Lisboa, Associação Ludus.
- Weisheipl, J. A. (1965). *Medieval Studies, Volume XXVII*. Pontifical Institute of Mediaeval Studies, Toronto.
- Youcat – Youth Catechism of the Catholic Church*. Translated by Michael J. Miller; published by Incorporated Catholic Truth Society, London, 2011.