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A sufficient condition for local controllability of a Caputo type fractional differential inclusion

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Dedicated to Professor Emeritus Mihail Megan on the occasion of his 75th anniversary

Abstract We consider a Cauchy problem for a fractional differential inclusion defined by a Caputo type fractional derivative and we obtain a sufficient condition for local controllability along a reference trajectory in terms of a certain fractional variational differential inclusion associated to the initial problem.

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1 Introduction

In the last decades the literature provides a huge development of the theory of differential equations and inclusions of fractional order [5, 13, 17, 19] etc.. This is due, mainly, to the fact that fractional differential equations are very useful tools in order to model many physical phenomena. In the fractional calculus there are several fractional derivatives. From them, the fractional derivative introduced by Caputo in [6] allows to use Cauchy conditions which have physical meanings.

A Caputo type fractional derivative of a function with respect to another function [15] that extends and unifies several fractional derivatives existing in the literature like Caputo, Caputo-Hadamard, Caputo-Katugampola was intensively studied in recent years in [1–3, 9, 11, 12] etc., where existence results and qualitative properties of the solutions for fractional differential equations defined by this fractional derivative are obtained.

The present paper is concerned with the following problem

$$D_C^{\alpha,\psi}x(t) \in F(t,x(t)) \quad a.e.\ ([0,T]), \quad x(0) \in X_0, \quad x'(0) \in X_1, \tag{1.1}$$

where $\alpha \in (1, 2]$, $D_C^{\alpha,\psi}$ is the fractional derivative mentioned above, $F : [0, T] \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set-valued map and $X_0, X_1 \subset \mathbf{R}$ are closed sets.

Our goal is to obtain a sufficient condition for local controllability along a reference trajectory for problem (1.1). In order to do this we use the notion of derived cone to an arbitrary subset of a normed space introduced by M.Hestenes in [15] and successfully used to obtain necessary optimality conditions in control theory. Other properties of derived cones (e.g., [18]) are very useful to obtain several results in the qualitative theory of control systems.

We prove that the reachable set of a certain variational fractional differential inclusion is a derived cone to the reachable set of the problem (1.1). In order to deduce this property we need a continuous version of Filippov's theorem for solutions of fractional differential inclusions (1.1), recently obtained in [9].

We note that similar results for fractional differential inclusions defined by Caputo fractional derivative may be found in [7] and for fractional differential inclusions defined by Caputo-Katugampola fractional derivative are obtained in [8]; therefore, the present paper extends and unifies all these results (see also [10]).

The paper is organized as follows: in Section 2 we present the notations and the preliminary results to be used in the sequel and in Section 3 we provide our main results.

2 Preliminaries

The reachable set to a control system is, generally, neither a differentiable manifold, nor a convex set, its infinitesimal properties may be characterized only by tangent cones in a generalized sense, extending the classical concepts of tangent cones in differential geometry and convex analysis, respectively. The next definition may be found in [13].

Definition 2.1. *A subset $D \subset \mathbf{R}^n$ is said to be a derived set to $X \subset \mathbf{R}^n$ at $x \in X$ if for any finite subset $\{w_1, ..., w_k\} \subset D$, there exist $s_0 > 0$ and a continuous mapping $c(\cdot) : [0, s_0]^k \to X$ such that $c(0) = x$ and $c(\cdot)$ is (conically) differentiable at $s = 0$ with the derivative col$[w_1, ..., w_k]$ in the sense that*

$$\lim_{R_+^k \ni \tau \to 0} \frac{||c(\tau) - c(0) - \sum_{i=1}^k \tau_i w_i||}{||\tau||} = 0.$$

We shall write in this case that the derivative of $c(\cdot)$ at $s = 0$ is given by

$$Dc(0)\tau = \sum_{i=1}^{k} \tau_j w_j \quad \forall \tau = (\tau_1, ..., \tau_k) \in \mathbf{R}_+^k := [0, \infty)^k.$$

A subset $C \subset \mathbf{R}^n$ is said to be a *derived cone* of X at x if it is a derived set and also a convex cone.

For the basic properties of derived sets and cones we refer to [15]; we recall that if D is a derived set then $D \bigcup \{0\}$ as well as the convex cone generated by D, defined by $cco(D) = \{\sum_{i=1}^k \lambda_j w_j; \quad \lambda_j \geq 0,\ k \in N,\ w_j \in D,\ j = 1, ..., k\}$ is also a derived set, hence a derived cone.

The fact that the derived cone is a proper generalization of the classical concepts in differential geometry and convex analysis is illustrated by the following results in [15]: if $X \subset \mathbf{R}^n$ is a differentiable manifold and T_xX is the tangent space in the sense of differential geometry to X at x, then T_xX is a derived cone; also, if $X \subset \mathbf{R}^n$ is a convex subset then the tangent cone in the sense of convex analysis is also a derived cone. Since any convex subcone of a derived cone is also a derived cone, such an object may not be uniquely associated to a point $x \in X$; moreover, simple examples show that even a maximal with respect to set-inclusion derived cone may not be uniquely defined.

At the same time, the up-to-date experience in nonsmooth analysis shows that for some problems, the use of one of the intrinsic tangent cones may be preferable. The most known intrinsic tangent cones in the literature (e.g., [4]) are the contingent, the quasitangent (intermediate) and Clarke's tangent cones, defined, respectively, by

$$\begin{aligned}
K_xX &= \{v \in X; \quad \exists\, s_m \to 0+, \quad \exists\, x_m \to x,\ x_m \in X : \tfrac{x_m - x}{s_m} \to v\},\\
Q_xX &= \{v \in X; \quad \forall s_m \to 0+, \quad \exists\, x_m \to x,\ x_m \in X : \tfrac{x_m - x}{s_m} \to v\},\\
C_xX &= \{v \in X; \quad \forall\,(x_m, s_m) \to (x, 0+), \quad x_m \in X, \exists\, y_m \in X : \tfrac{y_m - x_m}{s_m} \to v\}
\end{aligned}$$

The next property of derived cone, obtained by Hestenes (Theorem 4.7.4 in [15]) and stated in the next lemma is essential in the proof of our main result.

Lemma 2.1. *Let $X \subset \mathbf{R}^n$. Then $x \in int(X)$ if and only if $C = \mathbf{R}^n$ is a derived cone at $x \in X$ to X.*

Corresponding to each type of tangent cone, say τ_xX one may introduce (e.g., [4]) a *set-valued directional derivative* of a multifunction $G(\cdot) : X \subset \mathbf{R}^n \to \mathcal{P}(\mathbf{R}^n)$ (in particular of a single-valued mapping) at a point $(x, y) \in Graph(G)$ as follows

$$\tau_yG(x; v) = \{w \in \mathbf{R}^n; \quad (v, w) \in \tau_{(x,y)}Graph(G)\}, \quad v \in \tau_xE.$$

We recall that a set-valued map, $A(\cdot) : \mathbf{R}^n \to \mathcal{P}(\mathbf{R}^n)$ is said to be a *convex* (respectively, closed convex) *process* if $Graph(A(\cdot)) \subset \mathbf{R}^n \times \mathbf{R}^n$ is a convex (respectively, closed convex) cone. For the basic properties of convex processes we refer to [4], but we shall use here only the above definition.

Let $T > 0$, $I := [0, T]$ and denote by $\mathcal{L}(I)$ the σ-algebra of all Lebesgue measurable subsets of I. Denote by $\mathcal{P}(\mathbf{R})$ the family of all nonempty subsets of $\mathbf{R}$ and by $\mathcal{B}(\mathbf{R})$ the family of all Borel subsets of $\mathbf{R}$.

As usual, we denote by $C(I, \mathbf{R})$ the Banach space of all continuous functions $x(\cdot) : I \to \mathbf{R}$ endowed with the norm $|x(\cdot)|_C = \sup_{t\in I}|x(t)|$ and by $L^1(I, \mathbf{R})$ the Banach space of all (Bochner) integrable functions $x(\cdot) : I \to \mathbf{R}$ endowed with the norm $|x(\cdot)|_1 = \int_0^T |x(t)|\mathrm{d}t$.

Consider $\beta > 0$, $f(\cdot) \in L^1(I, \mathbf{R})$ and $\psi(\cdot) \in C^n(I, \mathbf{R})$ such that $\psi'(t) > 0\ \forall\, t \in I$.

Definition 2.2. *a) The ψ - Riemann-Liouville fractional integral of f of order β is defined by*

$$I^{\beta,\psi} f(t) = \frac{1}{\Gamma(\beta)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\beta-1} f(s)ds,$$

where Γ is the (Euler's) Gamma function defined by $\Gamma(\beta) = \int_0^\infty t^{\beta-1}e^{-t}dt$.

b) The ψ - Riemann-Liouville fractional derivative of f of order β is defined by

$$D^{\beta,\psi}f(t)=\frac{1}{\Gamma(n-\beta)}(\frac{1}{\psi'(t)}\frac{d}{dt})^n\int_0^t\psi'(s)(\psi(t)-\psi(s))^{n-\beta-1}f(s)ds,$$

where $n=[\beta]+1$.

c) The ψ - Caputo fractional derivative of f of order β is defined by

$$D_C^{\beta,\psi}f(t)=D^{\beta,\psi}[f(t)-\sum_{k=0}^{n-1}\frac{f_\psi^{[k]}(0)}{k!}(\psi(t)-\psi(0))^k],$$

where $f_\psi^{[k]}(t)=(\frac{1}{\psi'(t)}\frac{d}{dt})^k x(t)$, $n=\beta$ if $\alpha\in\mathbf{N}$ and $n=[\beta]+1$, otherwise.

We note that if $\beta=m\in\mathbf{N}$ then $D_C^{\beta,\psi}f(t)=f_\psi^{[m]}(t)$ and if $n=[\beta]+1$ then $D_C^{\beta,\psi}f(t)=\frac{1}{\Gamma(n-\beta)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{n-\alpha-1}f_\psi^{[n]}(s)ds$. Also, if $\psi(t)\equiv t$ one obtains Caputo's fractional derivative, if $\psi(t)\equiv\ln(t)$ one obtains Caputo-Hadamard's fractional derivative ([14]) and, finally, if $\psi(t)\equiv t^\sigma$ one obtains Caputo-Katugampola's fractional derivative ([16]).

In what follows we need the following technical lemma proved in [2] (namely, Theorem 2 in [2]).

Lemma 2.2. *Let $\alpha\in[1,2)$ and $\psi(\cdot)\in C^1(I,\mathbf{R})$ with $\psi'(t)>0\ \forall\ t\in I$. For a given integrable function $h(\cdot):I\to\mathbf{R}$, the unique solution of the initial value problem*

$$D_C^{\alpha,\psi}x(t)=h(t)\quad a.e.\ (I),\quad x(0)=x_0,\quad x'(0)=x_1$$

is given by

$$x(t)=x_0+x_1(\psi(t)-\psi(0))+\frac{1}{\Gamma(\alpha)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}h(s)ds.$$

Definition 2.3. *By a solution of the problem (1.1) we mean a function $x\in C(I,\mathbf{R})$ for which there exists a function $h\in L^1(I,\mathbf{R})$ satisfying $h(t)\in F(t,x(t))$ a.e. (I), $D_C^{\alpha,\psi}x(t)=h(t)$ a.e. (I) and $x(0)=x_0$, $x'(0)=x_1$.*

In this case we say that $(x(\cdot),h(\cdot))$ is a *trajectory-selection pair* of (1.1).

Hypothesis H1. (i) $F(\cdot,\cdot):I\times\mathbf{R}\to\mathcal{P}(\mathbf{R})$ has nonempty closed values and is $\mathcal{L}(I)\otimes\mathcal{B}(\mathbf{R})$ measurable.

(ii) There exists $L(\cdot)\in L^1(I,(0,\infty))$ such that, for almost all $t\in I, F(t,\cdot)$ is $L(t)$-Lipschitz in the sense that

$$d_H(F(t,x),F(t,y))\le L(t)|x-y|\quad\forall x,y\in\mathbf{R},$$

where $d_H(\cdot,\cdot)$ is the Hausdorff distance

$$d(A,B)=\max\{d^*(A,B),d^*(B,A)\},d^*(A,B)=\sup\{d(a,B);a\in A\}.$$

In what follows $\alpha\in[1,2)$ and $\psi(\cdot)\in C^1(I,\mathbf{R})$ with $\psi'(t)>0\ \forall\ t\in I$.

Hypothesis H2. i) S is a separable metric space and $a(\cdot), b(\cdot) : S \to \mathbf{R}$, $\varepsilon(\cdot) : S \to (0, \infty)$ are continuous mappings.

ii) There exists the continuous mappings $z(\cdot) : S \to AC(I, \mathbf{R})$ and $q(\cdot) : S \to \mathbf{R}$ such that

$$(Dz(s))_C^{\alpha,\psi}(t) = h(s)(t) \quad a.e.\ t \in I, \quad \forall s \in S,$$
$$d(h(s)(t), F(t, z(s)(t)) \le q(s)(t) \quad a.e.\ t \in I,\ \forall\ s \in S.$$

Next we use the notations

$$\xi(s) = \frac{1}{1 - I^{\alpha,\psi} L}(|a(s) - z(s)(0)| + T|b(s) - (z(s))'(0)| + \varepsilon(s) + I^{\alpha,\psi} q(s)),\ s \in S,$$

where $I^{\alpha,\psi} L := \sup_{t\in I} |I^{\alpha,\psi} L(t)|$ and $I^{\alpha,\psi} q(s) := \sup_{t\in I} |I^{\alpha,\psi} q(s)(t)|$.

In order to characterize derived cones to reachable sets of problem (1.1) we need the following parametrized version of Filippov's theorem for fractional differential inclusion (1.1) proved in [9].

Theorem 2.3. *Assume that Hypotheses H1 and H2 are satisfied.*

If $|I^{\alpha,\psi} L| < 1$, then there exist a continuous mapping $x(\cdot) : S \to C(I, \mathbf{R})$ such that for any $s \in S$, $x(s)(\cdot)$ is a solution of problem

$$D_C^{\alpha,\psi} u(t) \in F(t, u(t)), \quad u(0) = a(s), \quad u'(0) = b(s)$$

and

$$|x(s)(t) - z(s)(t)| \le \xi(s) \quad \forall (t, s) \in I \times S. \tag{2.1}$$

3 The results

We study the reachable set of (1.1) defined by

$$R_F(T, X_0, X_1) := \{x(T); \quad x(\cdot) \text{ is a solution of } (1.1)\}.$$

We consider a certain variational fractional differential inclusion and we shall prove that the reachable set of this variational inclusion from derived cones $C_0 \subset \mathbf{R}$ to X_0 and $C_1 \subset \mathbf{R}$ to X_1 at time T is a derived cone to the reachable set $R_F(T, X_0, X_1)$. Throughout in this section we assume the folowing hypotheses.

Hypothesis H3. i) Hypothesis H1 is satisfied and $X_0, X_1 \subset \mathbf{R}$ are closed sets.

ii) $(z(\cdot), h(\cdot)) \in AC(I, \mathbf{R}) \times L^1(I, \mathbf{R})$ is a trajectory-selection pair of (1.1) and a family $A(t, \cdot) : \mathbf{R} \to \mathcal{P}(\mathbf{R})$, $t \in I$ of convex processes satisfying the condition

$$A(t, u) \subset Q_{h(t)} F(t, \cdot)(z(t); u) \quad \forall\, u \in dom(A(t, \cdot)),\ a.e.\ t \in I \tag{3.1}$$

is assumed to be given and defines the variational inclusion

$$D_C^{\alpha,\psi} w(t) \in A(t, w(t)). \tag{3.2}$$

Remark 3.1. We recall that for any set-valued map $F(\cdot,\cdot)$, one may find an infinite number of families of convex process $A(t,\cdot)$, $t \in I$, satisfying condition (3.1); in fact any family of closed convex subcones of the quasitangent cones, $\overline{A}(t) \subset Q_{(z(t),h(t))}graph(F(t,\cdot))$, defines the family of closed convex process

$$A(t,u) = \{v \in \mathbf{R};\ (u,v) \in \overline{A}(t)\}, \quad u,v \in \mathbf{R},\ t \in I$$

that satisfy condition (3.1). For example, we may take an "intrinsic" family of such closed convex process; namely, Clarke's convex-valued directional derivatives $C_{h(t)}F(t,\cdot)(z(t);\cdot)$.

When $F(t,\cdot)$ is assumed to be Lipschitz a.e. on I an alternative characterization of the quasitangent directional derivative is (e.g., [4])

$$Q_{h(t)}F(t,\cdot)((z(t);u)) = \left\{ w \in \mathbf{R};\ \lim_{\tau\to 0+} \frac{1}{\tau} d(h(t)+\tau w, F(t,z(t)+\tau u)) = 0 \right\}. \tag{3.3}$$

Lemma 3.1. *Assume that Hypothesis H3 is satisfied, let $C_0 \subset \mathbf{R}$ be a derived cone to X_0 at $z(0)$ and $C_1 \subset \mathbf{R}$ be a derived cone to X_1 at $z'(0)$. Then the reachable set $R_A(T,C_0,C_1)$ of (3.2) is a derived cone to $R_F(T,X_0,X_1)$ at $z(T)$.*

Proof. In view of Definition 2.1, let $\{w_1,...,w_m\} \subset R_A(T,C_0,C_1)$, hence such that there exist the trajectory-selection pairs $(v_1(\cdot),g_1(\cdot)),...,(v_m(\cdot),g_m(\cdot))$ of the variational inclusion (3.2) such that

$$v_j(T) = w_j, \quad v_j(0) \in C_0, \quad v_j'(0) \in C_1, \quad j = 1,2,...,m. \tag{3.4}$$

Since $C_0 \subset \mathbf{R}$ is a derived cone to X_0 at $z(0)$ and $C_1 \subset \mathbf{R}$ is a derived cone to X_1 at $z'(0)$, there exist the continuous mappings $c_0 : S = [0,\theta_0]^m \to X_0$, $c_1 : S \to X_1$ such that

$$\begin{aligned} c_0(0) &= z(0), \quad Dc_0(0)s = \sum_{j=1}^m s_j v_j(0) \quad \forall\, s \in \mathbf{R}_+^m, \\ c_1(0) &= z'(0), \quad Dc_1(0)s = \sum_{j=1}^m s_j v_j'(0) \quad \forall\, s \in \mathbf{R}_+^m. \end{aligned} \tag{3.5}$$

For any $s = (s_1,...,s_m) \in S$ and $t \in I$ we set

$$\begin{aligned} z(s)(t) &= z(t) + \sum_{j=1}^m s_j v_j(t), \\ h(s)(t) &= h(t) + \sum_{j=1}^m s_j g_j(t), \\ q(s)(t) &= d(h(s)(t), F(t,z(s)(t))) \end{aligned} \tag{3.6}$$

and prove that $z(\cdot)$, $q(\cdot)$ satisfy the hypothesis of Theorem 2.3.

From the lipschitzianity of $F(t,\cdot,\cdot)$ we have that for any $s \in S$, the measurable function $q(s)(\cdot)$ in (3.6) it is also integrable.

$$q(s)(t) = d(h(s)(t), F(t,z(s)(t))) \le \sum_{j=1}^m s_j |g_j(t)| + d_H(F(t,z(t)),$$

$$F(t, z(s)(t))) \leq \sum_{j=1}^{m} s_j |g_j(t)| + L(t) \sum_{j=1}^{m} s_j |v_j(t)|.$$

At the same time, the mapping $s \to q(s)(\cdot) \in L^1(I, \mathbf{R})$ is Lipschitzian (and, in particular, continuous) since for any $s, s' \in S$ one may write

$$\begin{aligned} |q(s)(\cdot) - q(s')(\cdot)|_1 &= \int_0^T |q(s)(t) - q(s')(t)| dt \\ &\leq \int_0^T [|h(s)(t) - h(s')(t)| + d_H(F(t, z(s)(t)), F(t, z(s')(t))))] dt \\ &\leq ||s - s'|| (\sum_{j=1}^{m} \int_0^T [|g_j(t)| + L(t)|v_j(t)|] dt) \end{aligned}$$

Define $S_1 := S \backslash \{(0, \ldots, 0)\}$ and $\varepsilon(\cdot) : S_1 \to (0, \infty)$, $\varepsilon(s) := ||s||^2$. It follows from Theorem 2.3 the existence of a continuous function $x(\cdot) : S_1 \to C(I, \mathbf{R})$ such that for any $s \in S_1$, $x(s)(\cdot)$ is a solution of (1.1) with the property (2.1).

For $s = 0$ we define $x(0)(t) = z(0)(t) = z(t)$ $\forall t \in I$. Obviously, $x(\cdot) : S \to C(I, \mathbf{R})$ is also continuous.

Finally, we define the function $c(\cdot) : S \to R_F(T, X_0, X_1)$ by

$$c(s) = x(s)(T) \quad \forall s \in S.$$

Obviously, $c(\cdot)$ is continuous on S and verifies $\alpha(0) = z(T)$.

In order to complete the proof we have to show that $c(\cdot)$ is differentiable at $s_0 = 0 \in S$ and its derivative is given by

$$Dc(0)(s) = \sum_{j=1}^{m} s_j w_j \quad \forall\, s \in \mathbf{R}_+^m$$

which is equivalent with the fact that

$$\lim_{s \to 0} \frac{1}{||s||} \left| c(s) - c(0) - \sum_{j=1}^{m} s_j w_j \right| = 0. \tag{3.7}$$

Taking into account (3.6) we obtain

$$\begin{aligned} &\frac{1}{||s||} \left| c(s) - c(0) - \sum_{j=1}^{m} s_j w_j \right| \leq \frac{1}{||s||} |x(s)(T) - z(s)(T)| \\ &\leq \frac{1}{1 - |I^{\alpha,\psi} L|} ||s|| + \frac{1}{1 - |I^{\alpha,\psi} L|} \frac{1}{||s||} \left| c_0(s) - z(0) - \sum_{j=1}^{m} s_j v_j(0) \right| + \frac{T}{1 - |I^{\alpha,\psi} L|} \\ &\cdot \frac{1}{||s||} \left| c_1(s) - z'(0) - \sum_{j=1}^{m} s_j v_j'(0) \right| + \frac{(\psi(T))^\alpha}{(1 - |I^{\alpha,\psi} L|)\Gamma(\alpha + 1)} \int_0^T \frac{q(s)(u)}{||s||} du \end{aligned}$$

and therefore in view of (3.5), relation (3.7) is implied by the following property of the mapping $q(\cdot)$ in (3.6)

$$\lim_{s\to 0}\frac{q(s)(t)}{||s||}=0 \quad a.e.\,(I). \tag{3.8}$$

In order to prove the last property we note since $A(t,\cdot)$ is a convex process for any $s\in S$ one has

$$\sum_{j=1}^{m}\frac{s_j}{||s||}g_j(t)\in A\left(t,\sum_{j=1}^{m}\frac{s_j}{||s||}v_j(t)\right)\subset Q_{h(t)}F(t,\cdot)\left(z(t);\sum_{j=1}^{m}\frac{s_j}{||s||}v_j(t)\right)\quad a.e.\,(I).$$

Therefore, by (3.3) we obtain

$$\lim_{u\to 0+}\frac{1}{u}d\left(h(t)+u\sum_{j=1}^{m}\frac{s_j}{||s||}g_j(t),F\left(t,z(t)+u\sum_{j=1}^{m}\frac{s_j}{||s||}v_j(t)\right)\right)=0. \tag{3.9}$$

Finally, in order to prove that (3.9) implies (3.8) we take the compact metric space $\Sigma_+^{m-1}=\{\sigma\in \mathbf{R}_+^m;||\sigma||=1\}$ and the real function $\varphi_t(\cdot,\cdot):(0,\theta_0]\times\Sigma_+^{m-1}\to\mathbf{R}_+$ defined by

$$\varphi_t(u,\sigma)=\frac{1}{u}d\left(h(t)+u\sum_{j=1}^{m}\sigma_jg_j(t),F\left(t,z(t)+u\sum_{j=1}^{m}\sigma_jv_j(t)\right)\right), \tag{3.10}$$

where $\sigma=(\sigma_1,...,\sigma_m)$ and which according to (3.9) has the property

$$\lim_{u\to 0+}\varphi_t(u,\sigma)=0\quad\forall\,\sigma\in\Sigma_+^{m-1}\ a.e.\,(I) \tag{3.11}$$

Using the fact that $\varphi_t(u,\cdot)$ is Lipschitzian and the fact that Σ_+^{m-1} is a compact metric space, from (3.11) it follows easily that

$$\lim_{u\to 0+}\max_{\sigma\in\Sigma_+^{m-1}}\varphi_t(u,\sigma)=0$$

which implies the fact that

$$\lim_{s\to 0}\varphi_t\left(||s||,\frac{s}{||s||}\right)=0\quad a.e.\,(I)$$

and the proof is complete. □

Finally, we recall that fractional differential inclusion (1.1) is said to be locally controllable along a reference trajectory $z(\cdot)$ at time T, if

$$z(T)\in int(R_F(T,X_0,X_1)).$$

Theorem 3.2. *Let $z(\cdot),F(\cdot,\cdot)$ and $A(\cdot,\cdot)$ satisfy Hypothesis H3, let $C_0\subset\mathbf{R}$ be a derived cone to X_0 at $z(0)$ and $C_1\subset\mathbf{R}$ be a derived cone to X_1 at $z'(0)$. If the variational fractional differential inclusion in (3.2) is controllable at T in the sense that $R_A(T,C_0,C_1)=\mathbf{R}$, then the differential inclusion (1.1) is locally controllable along $z(\cdot)$ at time T.*

Proof. It is enough to apply Lemma 2.1 and Lemma 3.1 in order to prove the theorem. □

Remark 3.2. If in Theorem 3.4 $\psi(t)\equiv t$ we obtain Theorem 3.4 in [7] and if in Theorem 3.4 $\psi(t)\equiv t^\sigma$ we get Theorem 3 in [8].

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