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Strong Convergence Theorems for Hybrid Mixed Type Nonlinear Mappings in Banach Spaces

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Abstract. In this paper, we introduce a new two-step iteration scheme of hybrid mixed type for two asymptotically nonexpansive self mappings and two asymptotically nonexpansive non-self mappings in the intermediate sense and establish some strong convergence theorems for mentioned scheme and mappings in Banach spaces. Our results extend and generalize the corresponding results recently announced by Wei and Guo [16] (Comm. Math. Res. 31(2015), 149-160) and many others.

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1 Introduction and Preliminaries

Let K be a nonempty subset of a real Banach space E. Let $T\colon K \to K$ be a nonlinear mapping, then we denote the set of all fixed points of T by $F(T)$.

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The set of common fixed points of four mappings S_1, S_2, T_1 and T_2 will be denoted by $F = F(S_1) \bigcap F(S_2) \bigcap F(T_1) \bigcap F(T_2)$ and $\mathbf{N}$ denotes the set of all positive integers.

A mapping $T\colon K \to K$ is said to be asymptotically nonexpansive [3] if there exists a sequence $\{k_n\}$ in $[1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n(x) - T^n(y)\| \leq k_n\|x - y\|,\ \forall\, x, y \in K,\ n \in \mathbf{N}. \tag{1.1}$$

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [3] as a generalization of the class of nonxpansive mappings. They proved that if K is a nonempty closed convex subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive mapping on K, then T has a fixed point.

Example 1.1. (See [3]) Let D be the closed unit ball in the Hilbert space $H = \ell_2$ and $T\colon D \to D$ a mapping defined by

$$T(x_1, x_2, x_3, \dots) = (0, x_1^2, a_2x_2, a_3x_3, \dots)$$

where $\{a_i\}$ is a sequence of real numbers such that $0 < a_i < 1$ and $\prod_{i=2}^{\infty} a_i = \frac{1}{2}$.

Then

$$\|T(x) - T(y)\| \leq 2\|x - y\| \text{ for all } x, y \in D,$$

that is, T is Lipschitzian, but not nonexpansive.

Observe that

$$\|T^n(x) - T^n(y)\| \leq 2\Big(\prod_{i=2}^{n} a_i\Big)\|x - y\| \text{ for all } x, y \in D, n \geq 2.$$

Here $k_n = 2\prod_{i=2}^{n} a_i \to 1$ as $n \to \infty$. Therefore T is an asymptotically nonexpansive mapping.

Definition 1.1. *A subset K of a Banach space E is said to be a retract of E if there exists a continuous mapping $P\colon E \to K$ (called a retraction) such that $P(x) = x$ for all $x \in K$. If, in addition P is nonexpansive, then P is said to be a nonexpansive retract of E.*

If $P\colon E \to K$ is a retraction, then $P^2 = P$. A retract of a Hausdorff space must be a closed subset. Every closed convex subset of a uniformly convex Banach space is a retract.

Definition 1.2. *([1]) Let K be a nonempty subset of a real Banach space E and let $P\colon E \to K$ be a nonexpansive retraction of E onto K. A non-self mapping $T\colon K \to E$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that*

$$\|T(PT)^{n-1}(x) - T(PT)^{n-1}(y)\| \leq k_n\|x - y\|$$

for all $x, y \in K$ and $n \in \mathbf{N}$.

In 2004, Chidume, Shahazad and Zegeye [2] introduced the concept of non-self asymptotically nonexpansive mappings in the intermediate sense as follows.

Definition 1.3. *Let K be a nonempty subset of a real Banach space E and let $P\colon E \to K$ be a nonexpansive retraction of E onto K. A non-self mapping $T\colon K \to E$ is said to be asymptotically nonexpansive in the intermediate sense if T is uniformly continuous and*

$$\limsup_{n\to\infty} \sup_{x,y\in K} \Big(\|T(PT)^{n-1}(x) - T(PT)^{n-1}(y)\| - \|x - y\|\Big) \leq 0. \tag{1.2}$$

Example 1.2. Let $X = \mathbb{R}$ be a normed linear space, $K = [0,1]$ and P be the identity mapping. For each $x \in K$, we define

$$T(x) = \begin{cases} kx, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

where $0 < k < 1$. Then

$$|T^n x - T^n y| = k^n|x - y| \leq |x - y|$$

for all $x, y \in K$ and $n \in \mathbf{N}$.

Thus T is an asymptotically nonexpansive mapping with constant sequence $\{1\}$ and

$$\begin{aligned} \limsup_{n\to\infty}\{|T^n x - T^n y| - |x - y|\} &= \limsup_{n\to\infty}\{k^n|x - y| - |x - y|\} \\ &\leq 0 \end{aligned}$$

because $\lim_{n\to\infty} k^n = 0$ as $0 < k < 1$, for all $x, y \in K$, $n \in \mathbf{N}$ and T is continuous. Hence T is an asymptotically nonexpansive mapping in the intermediate sense.

Example 1.3. Let $X = \mathbb{R}$, $K = [-\frac{1}{\pi}, \frac{1}{\pi}]$, $|\lambda| < 1$ and P be the identity mapping. For each $x \in K$, define

$$T(x) = \begin{cases} \lambda\, x\, sin(1/x), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Clearly $F(T) = \{0\}$ and $\{T^n x\} \to 0$ uniformly on K as $n \to \infty$. Thus

$$\limsup_{n\to\infty} \Big\{ |T^n x - T^n y| - |x - y| \vee 0 \Big\} = 0$$

for all $x, y \in K$. Hence T is an asymptotically nonexpansive mapping in the intermediate sense (ANI in short), but it is not a Lipschitz mapping. In fact, suppose that there exists $\lambda > 0$ such that $|Tx - Ty| \leq \lambda |x - y|$ for all $x, y \in K$. If we take $x = \frac{2}{5\pi}$ and $y = \frac{2}{3\pi}$, then

$$|Tx - Ty| = \Big|\lambda \frac{2}{5\pi} sin\Big(\frac{5\pi}{2}\Big) - \lambda \frac{2}{3\pi} sin\Big(\frac{3\pi}{2}\Big)\Big| = \frac{16\lambda}{15\pi},$$

whereas

$$\lambda |x - y| = \lambda \Big|\frac{2}{5\pi} - \frac{2}{3\pi}\Big| = \frac{4\lambda}{15\pi},$$

and hence it is not an asymptotically nonexpansive mapping.

In 2004, Chidume et al [2] studied the following iteration scheme:

$$\begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= P((1-\alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n),\ n \geq 1, \end{aligned} \tag{1.3}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$, and K is a nonempty closed convex subset of a real uniformly convex Banach space E, P is a nonexpansive retraction of E onto K, and proved some strong and weak convergence theorems for asymptotically nonexpansive non-self mappings in the intermediate sense in the framework of uniformly convex Banach spaces.

In 2006, Wang [15] generalized the iteration process (1.3) as follows:

$$\begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= P((1-\alpha_n)x_n + \alpha_n T_1(PT_1)^{n-1}y_n), \\ y_n &= P((1-\beta_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n),\ n \geq 1, \end{aligned} \tag{1.4}$$

where T_1, $T_2 \colon K \to E$ are two asymptotically nonexpansive non-self mappings and $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $[0, 1)$, and proved some strong and weak convergence theorems for asymptotically nonexpansive non-self mappings. Convergence results have been given by numerous authors in different spaces using different iterative schemes for asymptotically nonexpansive self (or non-self) mappings. For example, see [4–8] among others.

In 2012, Guo et al. [9] generalized the iteration process (1.4) as follows:

$$\begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= P((1-\alpha_n)S_1^n x_n + \alpha_n T_1(PT_1)^{n-1} y_n), \\ y_n &= P((1-\beta_n)S_2^n x_n + \beta_n T_2(PT_2)^{n-1} x_n),\ n \geq 1, \end{aligned} \tag{1.5}$$

where $S_1, S_2\colon K \to K$ are two asymptotically nonexpansive self mappings and $T_1, T_2\colon K \to E$ are two asymptotically nonexpansive non-self mappings and $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $[0, 1)$, and proved some strong and weak convergence theorems for mixed type asymptotically nonexpansive mappings.

Now, we define the hybrid mixed type iteration scheme as follows.

Let E be a real Banach space, K be a nonempty closed convex subset of E and $P\colon E \to K$ be a nonexpansive retraction of E onto K. Let $S_1, S_2\colon K \to K$ be two asymptotically nonexpansive self mappings and $T_1, T_2\colon K \to E$ be two asymptotically nonexpansive non-self mappings in the intermediate sense. Then the hybrid mixed type iteration scheme for the mentioned mappings is as follows:

$$\begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= P((1-\alpha_n)S_1^n x_n + \alpha_n T_1(PT_1)^{n-1} y_n), \\ y_n &= P((1-\beta_n)S_2^n x_n + \beta_n T_2(PT_2)^{n-1} x_n),\ n \geq 1, \end{aligned} \tag{1.6}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1)$.

The purpose of this paper is to study newly defined hybrid mixed type iteration scheme (1.6) and establish some strong convergence theorems in the setting of real Banach spaces.

Next, we need the following useful lemma to prove our main results.

Lemma 1.1. *(See [14]) Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ be sequences of nonnegative numbers satisfying the inequality*

$$\alpha_{n+1} \leq (1+\beta_n)\alpha_n + r_n,\ \forall\, n \geq 1.$$

If $\sum_{n=1}^{\infty} \beta_n < \infty$ *and* $\sum_{n=1}^{\infty} r_n < \infty$*, then*

(i) $\lim_{n\to\infty} \alpha_n$ *exists;*

(ii) In particular, if $\{\alpha_n\}_{n=1}^{\infty}$ *has a subsequence which converges strongly to zero, then* $\lim_{n\to\infty} \alpha_n = 0$*.*

2 Main Results

In this section, we prove some strong convergence theorems of iteration scheme (1.6) for two asymptotically nonexpansive mappings and two asymptotically nonexpansive non-self mappings in the intermediate sense in real Banach spaces. First, we shall need the following lemma.

Lemma 2.1. *Let* E *be a real Banach space,* K *be a nonempty closed convex subset of* E*. Let* $S_1, S_2\colon K \to K$ *be two asymptotically nonexpansive self mappings with sequences* $\{k_n\}, \{l_n\} \subset [1, \infty)$ *such that* $\sum_{n=1}^{\infty}(k_n l_n - 1) < \infty$ *and* $T_1, T_2\colon K \to E$ *be two asymptotically nonexpansive non-self mappings in the intermediate sense. Put*

$$\begin{aligned} h_n \;=\; \max\Big\{0, &\sup_{x,\,y\in K,\, n\geq 1} \Big(\|T_1(PT_1)^{n-1}(x) - T_1(PT_1)^{n-1}(y)\| - \|x-y\|\Big), \\ &\sup_{x,\,y\in K,\, n\geq 1} \Big(\|T_2(PT_2)^{n-1}(x) - T_2(PT_2)^{n-1}(y)\| - \|x-y\|\Big)\Big\} \end{aligned} \tag{2.1}$$

such that $\sum_{n=1}^{\infty} h_n < \infty$*. Let* $\{x_n\}$ *be the sequence defined by (1.6), where* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are real sequences in* $[0, 1)$*. Suppose that* $F = F(S_1) \bigcap F(S_2) \bigcap F(T_1) \bigcap F(T_2) \neq \emptyset$*. Then* $\lim_{n\to\infty} \|x_n - q\|$ *and* $\lim_{n\to\infty} d(x_n, F)$ *both exist for any* $q \in F$*.*

Proof. Let $q \in F$. From (1.6) and (2.1), we have

$$\begin{aligned} \|y_n - q\| &= \|P((1-\beta_n)S_2^n x_n + \beta_n T_2(PT_2)^{n-1}x_n) - P(q)\| \\ &\leq \|(1-\beta_n)S_2^n x_n + \beta_n T_2(PT_2)^{n-1}x_n - q\| \\ &= \|(1-\beta_n)(S_2^n x_n - q) + \beta_n(T_2(PT_2)^{n-1}x_n - q)\| \\ &\leq (1-\beta_n)\|S_2^n x_n - q\| + \beta_n\|T_2(PT_2)^{n-1}x_n - q\| \\ &\leq (1-\beta_n)l_n\|x_n - q\| + \beta_n[\|x_n - q\| + h_n] \\ &\leq (1-\beta_n)l_n\|x_n - q\| + \beta_n l_n\|x_n - q\| + \beta_n h_n \\ &\leq l_n\|x_n - q\| + h_n. \end{aligned} \tag{2.2}$$

Again from (1.6) and (2.1), we have

$$\begin{aligned}\|x_{n+1}-q\| &= \|P((1-\alpha_n)S_1^n x_n+\alpha_n T_1(PT_1)^{n-1}y_n)-P(q)\| \\ &\le \|(1-\alpha_n)S_1^n x_n+\alpha_n T_1(PT_1)^{n-1}y_n-q\| \\ &= \|(1-\alpha_n)(S_1^n x_n-q)+\alpha_n(T_1(PT_1)^{n-1}y_n-q)\| \\ &\le (1-\alpha_n)\|S_1^n x_n-q\|+\alpha_n\|T_1(PT_1)^{n-1}y_n-q\| \\ &\le (1-\alpha_n)k_n\|x_n-q\|+\alpha_n[\|y_n-q\|+h_n] \\ &\le (1-\alpha_n)k_n\|x_n-q\|+\alpha_n\|y_n-q\|+h_n. \qquad (2.3)\end{aligned}$$

Using equation (2.2) in (2.3), we obtain

$$\begin{aligned}\|x_{n+1}-q\| &\le (1-\alpha_n)k_n\|x_n-q\|+\alpha_n[l_n\|x_n-q\|+h_n] \\ &\quad +h_n \\ &\le (1-\alpha_n)k_n\|x_n-q\|+\alpha_n l_n\|x_n-q\|+h_n \\ &\quad +h_n \\ &\le (1-\alpha_n)k_n l_n\|x_n-q\|+\alpha_n k_n l_n\|x_n-q\| \\ &\quad +2h_n \\ &= k_n l_n\|x_n-q\|+2h_n \\ &= [1+(k_n l_n-1)]\|x_n-q\|+2h_n. \qquad (2.4)\end{aligned}$$

Since $\sum_{n=1}^{\infty}(k_n l_n-1)<\infty$ and $\sum_{n=1}^{\infty}h_n<\infty$, it follows from Lemma 1.1 that $\lim_{n\to\infty}\|x_n-q\|$ exists.

Now, taking the infimum over all $q\in F$ in (2.4), we have

$$d(x_{n+1},F) \le [1+(k_n l_n-1)]d(x_n,F)+2h_n \qquad (2.5)$$

for all $n\in\mathbf{N}$, it follows from $\sum_{n=1}^{\infty}(k_n l_n-1)<\infty$, $\sum_{n=1}^{\infty}h_n<\infty$ and Lemma 1.1 that $\lim_{n\to\infty}d(x_n,F)$ exists. This completes the proof. □

Theorem 2.2. *Let E be a real Banach space, K be a nonempty closed convex subset of E. Let S_1, $S_2\colon K\to K$ be two asymptotically nonexpansive self mappings with sequences $\{k_n\},\{l_n\}\subset[1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n l_n-1)<\infty$ and T_1, $T_2\colon K\to E$ be two asymptotically nonexpansive non-self mappings in the intermediate sense. Put*

$$\begin{aligned}h_n &= \max\Big\{0, \sup_{x,\,y\in K,\,n\ge1}\Big(\|T_1(PT_1)^{n-1}(x)-T_1(PT_1)^{n-1}(y)\|-\|x-y\|\Big), \\ &\qquad \sup_{x,\,y\in K,\,n\ge1}\Big(\|T_2(PT_2)^{n-1}(x)-T_2(PT_2)^{n-1}(y)\|-\|x-y\|\Big)\Big\}\end{aligned}$$

such that $\sum_{n=1}^{\infty} h_n < \infty$. *Let* $\{x_n\}$ *be the sequence defined by (1.6), where* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are real sequences in* $[0,1)$. *Suppose that* $F = F(S_1) \bigcap F(S_2) \bigcap F(T_1) \bigcap F(T_2)$ *is nonempty and closed. Then* $\{x_n\}$ *converges strongly to a common fixed point of* S_1, S_2, T_1 *and* T_2 *if and only if* $\liminf_{n\to\infty} d(x_n, F) = 0$, *where* $d(x, F) = \inf\{\|x - p\| : p \in F\}$.

Proof. The necessity is obvious. Indeed, if $x_n \to q \in F$ as $n \to \infty$, then

$$d(x_n, F) = \inf_{q\in F} d(x_n, q) \le \|x_n - q\| \to 0\, (n \to \infty).$$

Thus $\liminf_{n\to\infty} d(x_n, F) = 0$.

Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F) = 0$. By Lemma 2.1, we have that $\lim_{n\to\infty} d(x_n, F)$ exists. Further, by assumption $\liminf_{n\to\infty} d(x_n, F) = 0$, from (2.5) and Lemma 1.1(ii), we conclude that $\lim_{n\to\infty} d(x_n, F) = 0$. Now, we show that $\{x_n\}$ is a Cauchy sequence in E. Indeed, from (2.4), we have

$$\|x_{n+1} - q\| \le [1 + (k_n l_n - 1)]\|x_n - q\| + 2h_n$$

for each $n \in \mathbf{N}$ with $\sum_{n=1}^{\infty}(k_n l_n - 1) < \infty$, $\sum_{n=1}^{\infty} h_n < \infty$ and $q \in F$. For any $m, n, m > n \in \mathbf{N}$, we have

$$\begin{aligned}
\|x_m - q\| &\le [1 + (k_{m-1}l_{m-1} - 1)]\|x_{m-1} - q\| + 2h_{m-1} \\
&\le e^{(k_{m-1}l_{m-1}-1)}\|x_{m-1} - q\| + 2h_{m-1} \\
&\vdots \\
&\le \Big(e^{\sum_{i=n}^{m-1}(k_i l_i - 1)}\Big)\|x_n - q\| + 2\Big(e^{\sum_{i=n+1}^{m-1}(k_i l_i - 1)}\Big)\sum_{i=n}^{m-1} h_i \\
&\le K'\|x_n - q\| + 2K'\sum_{i=n}^{m-1} h_i
\end{aligned}$$

where $K' = e^{\sum_{i=n}^{\infty}(k_i l_i - 1)}$.

Thus for any $q \in F$, we have

$$\begin{aligned}
\|x_n - x_m\| &\le \|x_n - q\| + \|x_m - q\| \\
&\le \|x_n - q\| + K'\|x_n - q\| + 2K'\sum_{i=n}^{m-1} h_i \\
&= (K' + 1)\|x_n - q\| + 2K'\sum_{i=n}^{m-1} h_i.
\end{aligned}$$

Taking the infimum over all $q \in F$, we obtain

$$\|x_n - x_m\| \leq (K'+1)d(x_n, F) + 2K' \sum_{i=n}^{m-1} h_i.$$

Thus it follows from $\lim_{n\to\infty} d(x_n, F) = 0$ and $h_n \to 0$ as $n \to \infty$ that $\{x_n\}$ is a Cauchy sequence in K. Since K is closed subset of E, the sequence $\{x_n\}$ converges strongly to some $q^* \in K$. Next, we show that $q^* \in F$. Now, $\lim_{n\to\infty} d(x_n, F) = 0$ gives that $d(q^*, F) = 0$. Since F is closed, $q^* \in F$. Thus q^* is a common fixed point of S_1, S_2, T_1 and T_2. This completes the proof. □

Theorem 2.3. *Let E be a real Banach space, K be a nonempty closed convex subset of E. Let $S_1, S_2 \colon K \to K$ be two asymptotically nonexpansive self mappings with sequences $\{k_n\}, \{l_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n l_n - 1) < \infty$ and $T_1, T_2 \colon K \to E$ be two asymptotically nonexpansive non-self mappings in the intermediate sense and h_n be taken as in Lemma 2.1. Let $\{x_n\}$ be the sequence defined by (1.6), where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1)$. Suppose that $F = F(S_1) \bigcap F(S_2) \bigcap F(T_1) \bigcap F(T_2)$ is nonempty. If one of S_1, S_2, T_1 and T_2 is completely continuous and $\lim_{n\to\infty} \|x_n - S_i x_n\| = \lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for $i = 1, 2$, then the sequence $\{x_n\}$ converges strongly to a common fixed point of S_1, S_2, T_1 and T_2.*

Proof. Without loss of generality we can assume that S_1 is completely continuous. Since $\{x_n\}$ is bounded by Lemma 2.1, there exists a subsequence $\{S_1 x_{n_k}\}$ of $\{S_1 x_n\}$ such that $\{S_1 x_{n_k}\}$ converges strongly to some $q_* \in K$. Moreover, by hypothesis of the theorem we know that

$$\lim_{k\to\infty} \|x_{n_k} - S_1 x_{n_k}\| = \lim_{k\to\infty} \|x_{n_k} - S_2 x_{n_k}\| = 0$$

and

$$\lim_{k\to\infty} \|x_{n_k} - T_1 x_{n_k}\| = \lim_{k\to\infty} \|x_{n_k} - T_2 x_{n_k}\| = 0$$

which implies that

$$\|x_{n_k} - q_*\| \leq \|x_{n_k} - S_1 x_{n_k}\| + \|S_1 x_{n_k} - q_*\| \to 0$$

as $k \to \infty$ and so $x_{n_k} \to q_* \in K$. Thus, by the continuity of S_1, S_2, T_1 and T_2, we have

$$\|q_* - S_i q_*\| = \lim_{k\to\infty} \|x_{n_k} - S_i x_{n_k}\| = 0$$

and

$$\|q_* - T_i q_*\| = \lim_{k\to\infty} \|x_{n_k} - T_i x_{n_k}\| = 0$$

for $i = 1, 2$. Thus it follows that $q_* \in F = F(S_1)\bigcap F(S_2)\bigcap F(T_1)\bigcap F(T_2)$. Again, since $\lim_{n\to\infty} \|x_n - q_*\|$ exists by Lemma 2.1, we have $\lim_{n\to\infty} \|x_n - q_*\| = 0$. This shows that the sequence $\{x_n\}$ converges strongly to a common fixed point of S_1, S_2, T_1 and T_2. This completes the proof. □

Theorem 2.4. *Let E be a real Banach space, K be a nonempty closed convex subset of E. Let $S_1, S_2\colon K \to K$ be two asymptotically nonexpansive self mappings with sequences $\{k_n\}, \{l_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n l_n - 1) < \infty$ and $T_1, T_2\colon K \to E$ be two asymptotically nonexpansive non-self mappings in the intermediate sense and h_n be taken as in Lemma 2.1. Let $\{x_n\}$ be the sequence defined by (1.6), where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1)$. Suppose that $F = F(S_1)\bigcap F(S_2)\bigcap F(T_1)\bigcap F(T_2)$ is nonempty. If one of S_1, S_2, T_1 and T_2 is semi-compact and $\lim_{n\to\infty} \|x_n - S_i x_n\| = \lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for $i = 1, 2$, then the sequence $\{x_n\}$ converges strongly to a common fixed point of S_1, S_2, T_1 and T_2.*

Proof. Since by hypothesis $\lim_{n\to\infty} \|x_n - S_i x_n\| = \lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for $i = 1, 2$ and one of S_1, S_2, T_1 and T_2 is semi-compact, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to some $p_* \in K$. Moreover, by the continuity of S_1, S_2, T_1 and T_2, we have $\|p_* - S_i p_*\| = \lim_{j\to\infty} \|x_{n_j} - S_i x_{n_j}\| = 0$ and $\|p_* - T_i p_*\| = \lim_{j\to\infty} \|x_{n_j} - T_i x_{n_j}\| = 0$ for $i = 1, 2$. Thus it follows that $p_* \in F = F(S_1)\bigcap F(S_2)\bigcap F(T_1)\bigcap F(T_2)$. Since $\lim_{n\to\infty} \|x_n - p_*\|$ exists by Lemma 2.1, we have $\lim_{n\to\infty} \|x_n - p_*\| = 0$. This shows that the sequence $\{x_n\}$ converges strongly to a common fixed point of S_1, S_2, T_1 and T_2. This completes the proof. □

Theorem 2.5. *Let E be a real Banach space, K be a nonempty closed convex subset of E. Let $S_1, S_2\colon K \to K$ be two asymptotically nonexpansive self mappings with sequences $\{k_n\}, \{l_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n l_n - 1) < \infty$ and $T_1, T_2\colon K \to E$ be two asymptotically nonexpansive non-self mappings in the intermediate sense and h_n be taken as in Lemma 2.1. Let $\{x_n\}$ be the sequence defined by (1.6), where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1)$. Suppose that $F = F(S_1)\bigcap F(S_2)\bigcap F(T_1)\bigcap F(T_2)$ is nonempty. If S_1, S_2, T_1 and T_2 satisfy the following conditions:*

(i) $\lim_{n\to\infty} \|x_n - S_i x_n\| = \lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ *for $i = 1, 2$;*

(ii) there exists a continuous function $f\colon [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0, \infty)$ such that

$$f(d(x, F)) \le a_1\|x - S_1 x\| + a_2\|x - S_2 x\| + a_3\|x - T_1 x\| + a_4\|x - T_2 x\|$$

for all $x \in K$, and a_1, a_2, a_3, a_4 are nonnegative real numbers such that $a_1 + a_2 + a_3 + a_4 = 1$, where $d(x, F) = \inf\{\|x - p\| : p \in F\}$.

Then the sequence $\{x_n\}$ converges strongly to a common fixed point of S_1, S_2, T_1 and T_2.

Proof. It follows from the hypothesis that

$$\begin{aligned}\lim_{n\to\infty} f(d(x_n,F)) &\leq a_1.\|x_n - S_1x_n\| + a_2.\|x_n - S_2x_n\| \\ &\quad + a_3.\|x_n - T_1x_n\| + a_4.\|x_n - T_2x_n\| \\ &= 0.\end{aligned}$$

That is,

$$\lim_{n\to\infty} f(d(x_n,F)) = 0.$$

Since $f\colon [0,\infty) \to [0,\infty)$ is a continuous function and $f(0) = 0$, therefore we have

$$\lim_{n\to\infty} d(x_n,F) = 0.$$

Therefore, Theorem 2.2 implies that $\{x_n\}$ must converges strongly to a common fixed point of S_1, S_2, T_1 and T_2. This completes the proof. $\square$

Example 2.1. Let $X = \mathbb{R}$, $K = [-\frac{1}{\pi}, \frac{1}{\pi}]$, $|\lambda| < 1$ and P be the identity mapping. For each $x \in K$, define two mappings $S, T\colon K \to K$ by

$$T(x) = \begin{cases} \lambda\, x\, sin(1/x), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

and

$$S(x) = \begin{cases} x, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Then $\{T^n x\} \to 0$ uniformly on K as $n \to \infty$. Thus

$$\limsup_{n\to\infty} \big\{|T^n x - T^n y| - |x - y| \vee 0\big\} = 0$$

for all $x, y \in K$. Hence T is an asymptotically nonexpansive mapping in the intermediate sense (ANI in short), but it is not a Lipschitz mapping and S is an asymptotically nonexpansive mapping with constant sequence $\{k_n\} = \{1\}$ for all $n \in \mathbf{N}$ and uniformly L-Lipschitzian with $L = \sup_{n\geq 1}\{k_n\}$. Also $F(T) = \{0\}$ is the unique fixed point of T and $F(S) = \{0\}$ is the unique fixed point of S, that is, $F = F(S) \cap F(T) = \{0\}$ is the unique hybrid common fixed point of S and T.

3 Concluding remarks

In this paper, we study hybrid mixed type iteration scheme for two asymptotically nonexpansive self mappings and two asymptotically nonexpansive non-self mappings in the intermediate sense and establish some strong convergence theorems using completely continuous and semi-compactness conditions. Our results extend and generalize the corresponding results of [1, 2, 9–16].

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