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On the Solutions of a Class of Nonlinear Integral Equations in the Banach Algebra of the Continuous Functions and Some Examples

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Abstract. In this paper, we study the existence of the solutions of a class of functional integral equations which contain a lot of classical nonlinear integral equations as special cases. We consider the solvability of the equations in the Banach algebra of continuous functions on a closed and bounded interval. The main tools here are the measure of noncompactness and the suitable fixed point theorem for the product of two operators in the Banach algebra.

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1 Introduction

The aim of this paper is to consider the existence of the solutions for the following nonlinear integral equation

$$x(t) \;=\; f\left(t, \int_0^{\varphi(t)} v\left(t, s, x(\gamma_1(s))\right) ds, x(\alpha(t))\right) \times$$

$$\times g\left(t, x(\gamma_2(t))\int_0^1 u\left(t, s, x(\gamma_3(s))\right)ds, x(\beta(t))\right) \tag{1.1}$$

for $t \in I = [0,1]$.
In this study, we investigate a more general class of nonlinear integral equations which contain, as particular cases, a lot of integral equations, handled before. Some special cases of Eq.(1.1) have been investigated by various authors. For example, if we take $f(t,y,x) = 1$, then Eq.(1.1) can be reduced to the integral equation considered in [11] which arises in models connected with traffic and biology

$$x(t) = f(t, x(t))\int_0^1 u(t,s,x(s))ds. \tag{1.2}$$

Similarly, if

$$f(t,y,x) = 1,\ \ g(t,y,x) = 1 + y$$

and

$$u(t,s,x) = \frac{t\phi(s)x}{t+s},\ \ \gamma_2(t) = \gamma_3(t) = t,$$

then Eq.(1.1) can be reduced to the famous quadratic integral equation of Chandrasekhar type studied in many papers [2, 6, 10–12] and given of the form

$$x(t) = 1 + x(t)\int_0^1 \frac{t}{t+s}\phi(s)x(s)ds. \tag{1.3}$$

Finally, Caballero et al. in [9] studied the existence of solutions of following functional-integral equation

$$\begin{aligned} x(t) \quad &= \quad f\left(t, \int_0^t v\left(t,s,x(s)\right)ds, x(\alpha(t))\right) \times \\ &\quad\quad \times g\left(t, \int_0^a x(t)u\left(t,s,x(s)\right)ds, x(\beta(t))\right) \end{aligned} \tag{1.4}$$

under the following conditions:

(i) $f, g : [0,a] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous and there exists nonnegative constants c_i, d_i; $(i = 1,2)$ such that

$$\begin{aligned} |f(t,0,x)| \ &\leq \ c_1 + d_1|x|, \\ |g(t,0,x)| \ &\leq \ c_2 + d_2|x|, \end{aligned}$$

for all $t \in [0,a]$ and $x \in \mathbb{R}$.

(*ii*) The functions $f(t,y,x)$, $g(t,y,x)$ satisfy a Lipschitz condition with respect to the variables y and x with constants k, k' respectively, i.e.,

$$\begin{aligned}|f(t,y_1,x) - f(t,y_2,x)| &\leq k|y_1 - y_2|,\\ |g(t,y_1,x) - g(t,y_2,x)| &\leq k|y_1 - y_2|,\end{aligned}$$

for all $t \in [0,a]$ and $x, y_1, y_2 \in \mathbb{R}$ and

$$\begin{aligned}|f(t,y,x_1) - f(t,y,x_2)| &\leq k'|x_1 - x_2|,\\ |g(t,y,x_1) - g(t,y,x_2)| &\leq k'|x_1 - x_2|,\end{aligned}$$

for all $t \in [0,a]$ and $x_1, x_2, y \in \mathbb{R}$.

(*iii*) $u, v : [0,a] \times [0,a] \times \mathbb{R} \to \mathbb{R}$ are continuous.

(*iv*) $\alpha, \beta : [0,a] \to [0,a]$ are continuous and satisfy,

$$\begin{aligned}|\alpha(t_1) - \alpha(t_2)| &\leq |t_1 - t_2|,\\ |\beta(t_1) - \beta(t_2)| &\leq |t_1 - t_2|,\end{aligned}$$

for all $t_1, t_2 \in [0,a]$.

(*v*) There exist nonnegative constants α_i, β_i; $(i = 1,2)$ such that

$$\begin{aligned}&|v(t,s,x)| \leq \alpha_1 + \beta_1|x|,\\ &|u(t,s,x)| \leq \alpha_2 + \beta_2|x|,\end{aligned}$$

for all $t, s \in [0,a]$ and $x \in \mathbb{R}$.

(*vi*) The inequality

$$\Big[k(\tilde{\alpha} + \tilde{\beta}r)a + (c + dr)\Big]\Big[k(\tilde{\alpha} + \tilde{\beta}r)ra + (c + dr)\Big] \leq r$$

has a positive solution r_0, where $\tilde{\alpha} = \max\{\alpha_1, \alpha_2\}$, $\tilde{\beta} = \max\{\beta_1, \beta_2\}$, $c = \max\{c_1, c_2\}$ and $d = \max\{d_1, d_2\}$.

(*vii*) $k'\Big[k(\tilde{\alpha} + \tilde{\beta}r_0)a(1 + r_0) + 2(c + dr_0)\Big] < 1$.

Theorem 1.1. [9, Theorem 3.1] *Equation* (1.4) *has at least one solution* $x \in C[0,a]$ *under assumptions* $(i) - (vii)$.

It can be verified that if $\varphi(t) = \gamma_1(t) = \gamma_2(t) = \gamma_3(t) = t$, then Eq.(1.1) is reduced to Eq.(1.4) for $a = 1$.
Using the technique of a suitable measure of noncompactness in the Banach algebra, we prove an existence theorem for Eq.(1.1). Also, we illustrate our results by suitable examples. The results obtained in this paper generalize several ones obtained up to now. Moreover, our sufficient conditions give the results of [9] under some weaker conditions and for a rather general equation.

2 Auxiliary facts and notations

In this section, we give a collection of auxiliary facts which will be needed further on. Assume that $(E, \|.\|)$ is a real Banach space with zero element θ . Let $B(x, r)$ denote the closed ball centered at x and with radius r. The symbol B_r stands for the ball $B(\theta, r)$. If X is a subset of E, then $\overline{X}$ and Conv$X$ denote the closure and convex closure of X, respectively. With the symbols λX and $X+Y$, we denote the standard algebraic operations on sets. Moreover, we denote by $\mathfrak{M}_E$ the family of all nonempty and bounded subsets of E and $\mathfrak{N}_E$ its subfamily consisting of all relatively compact subsets. Next we give the concept of a regular measure of noncompactness.

Definition 2.1. [3] *A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}^+ = [0, \infty)$ is said to be a regular measure of noncompactness in E if it satisfies following conditions:*

(1) $\mu(X) = 0 \Leftrightarrow X \in \mathfrak{N}_E$;

(2) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$;

(3) $\mu(\overline{X}) = \mu(ConvX) = \mu(X)$;

(4) $\mu(\lambda X) = |\lambda|\mu(X),\ (\lambda \in \mathbb{R})$;

(5) $\mu(X + Y) \leq \mu(X) + \mu(Y)$;

(6) $\mu(X \cup Y) = \max\{\mu(X), \mu(Y)\}$;

(7) *If* $\{X_n\}$ *is a sequence of nonempty, bounded, closed subsets of* E *such that* $X_{n+1} \subset X_n, (n = 1, 2, ...)$ *and* $\lim_{n\to\infty} \mu(X_n) = 0$*, then the set* $X_\infty = \cap_{n=1}^{\infty} X_n$ *is nonempty.*

In the sequel, we will work in the Banach space $C(I)$ consisting of all real functions defined and continuous on $I = [0, 1]$. The space $C(I)$ is furnished with the standard norm

$$\|x\| = \max\{|x(t)| : t \in I\}.$$

Obviously, space $C(I)$ also has the structure of the Banach Algebra. We will use a measure of noncompactness in space $C(I)$ which was introduced in [3]. In order to define this measure let us fix a nonempty and bounded subset X of $C(I)$. For $x \in X$ and $\varepsilon \geq 0$ denoted by $w(x, \varepsilon)$ the modulus of continuity of function x, i.e.,

$$w(x, \varepsilon) = \sup\{|x(s) - x(t)| : t, s \in [0, 1], |t - s| \leq \varepsilon\}.$$

Further let us put

$$w(X, \varepsilon) = \sup\{w(x, \varepsilon) : x \in X\}$$

and

$$w_0(X) = \lim_{\varepsilon\to 0} w(X, \varepsilon).$$

Function w_0 is a regular measure of noncompactness in space $C(I)$, [7]. Finally, we recall the fixed point theorem of Darbo. To quote this theorem, we need the following.
Hereafter, we assume unless stated otherwise that μ is a regular measure of noncompactness in E.

Definition 2.2. [3] *Let* Ω *be a nonempty subset of a Banach space* E*, and let* $S : \Omega \to E$ *be a continuous operator that transforms bounded subsets of* Ω *onto bounded ones. We will say that* S *satisfies the Darbo condition (with a constant* $k \geq 0$*) if for any bounded subset* X *of* Ω*, we have*

$$\mu(SX) \leq k\mu(X).$$

In the case $k < 1$*, operator* S *is said to be a contraction (with respect to* μ*).*

Theorem 2.3. [7] *Let* Ω *be a nonempty, bounded, closed and convex subset of space* E *and let*

$$S : \Omega \to \Omega$$

be a continuous transformation such that $\mu(SX) \leq k\mu(X)$ *for any nonempty subset* X *of* Ω*, where* $k \in [0, 1)$ *is a constant. Then* S *has a fixed point in set* Ω*.*

The following theorem is the main tool for our proof.

Theorem 2.4. [8] *Assume that Ω is nonempty, bounded, convex and closed subset of $C(I)$ and operators F and G transform continuously set Ω into $C(I)$ in such a way that $F(\Omega)$ and $G(\Omega)$ are bounded. Moreover, assume that operator $T = F.G$ transforms Ω into itself. If operators F and G satisfy Darbo condition on set Ω, with respect to measure of noncompactness w_0, with constants k_1 and k_2, respectively, then operator T satisfies Darbo condition on Ω with constant*

$$\|F(\Omega)\|k_2 + \|G(\Omega)\|k_1.$$

In particular, if

$$\|F(\Omega)\|k_2 + \|G(\Omega)\|k_1 < 1,$$

then T is a contraction with respect to measure of noncompactness w_0 and so has at least one fixed point in set Ω, where $\|X\|$ is defined by the equality

$$\|X\| = \sup\{\|x\| : x \in X\}$$

for any nonempty and bounded subset X of $C(I)$.

3 The Main Result

We shall study the existence of the solutions of Eq.(1.1) assuming that following conditions are satisfied:

(i) $f, g : I\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are continuous. Also, $f(t, y, x)$ and $g(t, y, x)$ satisfy Lipschitz condition with respect to variables y and x with constants k, k' respectively, i.e.,

$$\begin{aligned} |f(t, y_1, x) - f(t, y_2, x)| &\leq k|y_1 - y_2|, \\ |g(t, y_1, x) - g(t, y_2, x)| &\leq k|y_1 - y_2|, \end{aligned}$$

for all $t \in I$ and $x, y_1, y_2 \in \mathbb{R}$ and

$$\begin{aligned} |f(t, y, x_1) - f(t, y, x_2)| &\leq k'|x_1 - x_2|, \\ |g(t, y, x_1) - g(t, y, x_2)| &\leq k'|x_1 - x_2|, \end{aligned}$$

for all $t \in I$ and $x_1, x_2, y \in \mathbb{R}$.

(*ii*) $u, v : I \times I \times \mathbb{R} \to \mathbb{R}$ are continuous and there exist nonnegative constants α_i, β_i, p_i; $(i = 1, 2)$ such that

$$|v(t,s,x)| \leq \alpha_1 + \beta_1 |x|^{p_1},$$
$$|u(t,s,x)| \leq \alpha_2 + \beta_2 |x|^{p_2},$$

for all $s, t \in I$ and $x \in \mathbb{R}$.

(*iii*) $\varphi, \alpha, \beta, \gamma_j : I \to I$ are the continuous functions. $(j = 1, 2, 3)$.

(*iv*)

$$(k\,\alpha_1 + m_1)\, m_2 > 0,$$

where m_1 and m_2 are the constants such that

$$|f(t, 0, 0)| \leq m_1 \text{ and } |g(t, 0, 0)| \leq m_2$$

for all $t \in I$.

(*v*)

$$[\, k(\alpha_1 + \beta_1) + m_1 + k']\, [\, k(\alpha_2 + \beta_2) + m_2 + k' \,] < 1.$$

(*vi*)

$$k' \left[(\alpha_1 + \alpha_2 + \beta_1 + \beta_2) + m_1 + m_2 + 2k' \,\right] +$$
$$+ kM \left[\, k(\alpha_1 + \beta_1) + m_1 + k' \,\right] < 1,$$

where M is the nonnegative constant such that $|u(t,s,x)| \leq M$ for all $t, s \in I$ and $x \in [-1, 1]$.

Theorem 3.1. *Under assumptions* $(i) - (vi)$, *there exists at least one* $r_0 \in (0, 1)$ *such that equation* (1.1) *has at least one solution* $x = x(t)$ *belonging to* $B_{r_0} \subset C(I)$.

Proof. We define continuous function $h : [0, 1] \to \mathbb{R}$ such that

$$h(r) = [\, k(\alpha_1 + \beta_1 r^{p_1}) + m_1 + k'r \,]\, [\, kr(\alpha_2 + \beta_2 r^{p_2}) + m_2 + k'r \,] - r,$$

where $k, k', m_i, \alpha_i, \beta_i$ and p_i for $i \in \{1, 2\}$ are the constants given in the assumptions. Then $h(0) > 0$ and $h(1) < 0$ by assumptions (*iv*) and (*v*).

The continuity of h guarantees that there exists the number r_0 such that $r_0 \in (0,1)$ and $h(r_0) = 0$.
Now, we shall prove that equation (1.1) has at least one solution $x = x(t)$ belonging to $B_{r_0} \subset C(I)$. We define operators F and G on $C(I)$ in the following way:

$$\begin{aligned}(Fx)(t) &= f\left(t, \int_0^{\varphi(t)} v\left(t,s,x(\gamma_1(s))\right)ds, x(\alpha(t))\right),\\ (Gx)(t) &= g\left(t, x(\gamma_2(t))\int_0^{1} u\left(t,s,x(\gamma_3(s))\right)ds, x(\beta(t))\right).\end{aligned}$$

From the assumptions, F and G transform space $C(I)$ into itself. Further let us define operator T on $C(I)$ by the equality

$$Tx = (Fx)(Gx).$$

Obviously, T transforms $C(I)$ into itself. Since

$$f(t,0,x) = f(t,0,x) - f(t,0,0) + f(t,0,0),$$

we have by (i) that

$$|f(t,0,x)| \le m_1 + k'|x|$$

for all $t \in I$ and $x \in \mathbb{R}$. Let us fix $x \in C(I)$. Then, using our assumptions for $t \in I$, we get

$$\begin{aligned}& |(Fx)(t)|\\ =\ & \left|f\left(t, \int_0^{\varphi(t)} v\left(t,s,x(\gamma_1(s))\right)ds, x(\alpha(t))\right)\right|\\ \le\ & \left|f\left(t, \int_0^{\varphi(t)} v\left(t,s,x(\gamma_1(s))\right)ds, x(\alpha(t))\right) - f\left(t,0,x(\alpha(t))\right)\right|\\ & + \left|f\left(t,0,x(\alpha(t))\right)\right|\\ \le\ & k\int_0^{\varphi(t)} \left|v\left(t,s,x(\gamma_1(s))\right)\right|ds + m_1 + k'\left|x(\alpha(t))\right|\\ \le\ & k\int_0^{\varphi(t)} \left(\alpha_1 + \beta_1\left|x(\gamma_1(s))\right|^{p_1}\right)ds + m_1 + k'\left|x(\alpha(t))\right|\\ \le\ & k\left(\alpha_1 + \beta_1\|x\|^{p_1}\right) + m_1 + k'\|x\|. \qquad (3.1)\end{aligned}$$

Similarly, we derive

$$|(Gx)(t)| \le k\|x\|\left(\alpha_2 + \beta_2\|x\|^{p_2}\right) + m_2 + k'\|x\| \qquad (3.2)$$

for all $x \in C(I)$ and $t \in I$. By (3.1) and (3.2), for $x \in B_{r_0}$, we obtain

$$\begin{aligned}
|(Tx)(t)| &= |(Fx)(t)||(Gx)(t)| \\
&= [\,k\,(\alpha_1 + \beta_1\|x\|^{p_1}) + m_1 + k'\|x\|\,]\times \\
&\quad [\,k\|x\|\,(\alpha_2 + \beta_2\|x\|^{p_2}) + m_2 + k'\|x\|\,] \\
&\leq [\,k\,(\alpha_1 + \beta_1 r_0^{p_1}) + m_1 + k' r_0\,]\times \\
&\quad [\,k r_0\,(\alpha_2 + \beta_2 r_0^{p_2}) + m_2 + k' r_0\,] \\
&= h(r_0) + r_0 \\
&= r_0
\end{aligned}$$

which implies that $Tx \in B_{r_0}$.
Now, we shall prove that operator F is continuous on B_{r_0}. To do this, consider $\varepsilon > 0$ and any $x, y \in B_{r_0}$ such that $\|x - y\| \leq \varepsilon$. Then,

$$\begin{aligned}
&|(Fx)(t) - (Fy)(t)| \\
= &\left| f\left(t, \int_0^{\varphi(t)} v\,(t,s,x(\gamma_1(s)))\,ds, x(\alpha(t))\right) - \right. \\
&\left. -f\left(t, \int_0^{\varphi(t)} v\,(t,s,y(\gamma_1(s)))\,ds, y(\alpha(t))\right)\right| \\
\leq &\left| f\left(t, \int_0^{\varphi(t)} v\,(t,s,x(\gamma_1(s)))\,ds, x(\alpha(t))\right) - \right. \\
&\left. -f\left(t, \int_0^{\varphi(t)} v\,(t,s,y(\gamma_1(s)))\,ds, x(\alpha(t))\right)\right| \\
&+\left| f\left(t, \int_0^{\varphi(t)} v\,(t,s,y(\gamma_1(s)))\,ds, x(\alpha(t))\right) - \right. \\
&\left. -f\left(t, \int_0^{\varphi(t)} v\,(t,s,y(\gamma_1(s)))\,ds, y(\alpha(t))\right)\right|
\end{aligned}$$

which implies that

$$\begin{aligned}
&|(Fx)(t) - (Fy)(t)| \\
\leq\ & k \int_0^{\varphi(t)} |v\,(t,s,x(\gamma_1(s))) - v\,(t,s,y(\gamma_1(s)))|\,ds + \\
&+k'\,|x(\alpha(t)) - y(\alpha(t))| \\
\leq\ & k w_{r_0}^3(v,\varepsilon) + k'\|x - y\| \\
\leq\ & k w_{r_0}^3(v,\varepsilon) + k'\varepsilon, \qquad (3.3)
\end{aligned}$$

where

$$w^3_{r_0}(v,\varepsilon) = \sup\{|v(t,s,x) - v(t,s,y)| : t,s \in I; x,y \in R; |x-y| \leq \varepsilon\}$$

such that $R = [-r_0, r_0]$. Notice that, in view of the uniform continuity of function v on set $I \times I \times [-r_0, r_0]$, we have that $w^3_{r_0}(v,\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus, above estimate (3.3) shows that operator F is continuous on ball B_{r_0}. Similarly, we can show that operator G is continuous on ball B_{r_0}. Obviously, this implies the continuity of operator T on ball B_{r_0}.

Further, we shall show that operators F and G satisfy Darbo condition on ball B_{r_0}. In order to do this, let us take a nonempty subset X of ball B_{r_0}. Fix $\varepsilon > 0$ and choose $x \in X$ and $t_1, t_2 \in I$ such that $|t_1 - t_2| \leq \varepsilon$. Without loss of generality, we may assume that $\varphi(t_2) \leq \varphi(t_1)$. Then, we obtain

$$\begin{aligned}
& |(Fx)(t_2) - (Fx)(t_1)| \\
= & \left| f\left(t_2, \int_0^{\varphi(t_2)} v\left(t_2, s, x(\gamma_1(s))\right) ds, x(\alpha(t_2))\right) - \right. \\
& \left. - f\left(t_1, \int_0^{\varphi(t_1)} v\left(t_1, s, x(\gamma_1(s))\right) ds, x(\alpha(t_1))\right) \right| \\
\leq & \left| f\left(t_2, \int_0^{\varphi(t_2)} v\left(t_2, s, x(\gamma_1(s))\right) ds, x(\alpha(t_2))\right) - \right. \\
& \left. - f\left(t_2, \int_0^{\varphi(t_1)} v\left(t_1, s, x(\gamma_1(s))\right) ds, x(\alpha(t_2))\right) \right| \\
& + \left| f\left(t_2, \int_0^{\varphi(t_1)} v\left(t_1, s, x(\gamma_1(s))\right) ds, x(\alpha(t_2))\right) - \right. \\
& \left. - f\left(t_1, \int_0^{\varphi(t_1)} v\left(t_1, s, x(\gamma_1(s))\right) ds, x(\alpha(t_2))\right) \right| \\
& + \left| f\left(t_1, \int_0^{\varphi(t_1)} v\left(t_1, s, x(\gamma_1(s))\right) ds, x(\alpha(t_2))\right) - \right. \\
& \left. - f\left(t_1, \int_0^{\varphi(t_1)} v\left(t_1, s, x(\gamma_1(s))\right) ds, x(\alpha(t_1))\right) \right|.
\end{aligned}$$

Hence,

$$\begin{aligned}
& |(Fx)(t_2)-(Fx)(t_1)| \\
\leq\ & k\left|\int_0^{\varphi(t_2)} v\left(t_2,s,x(\gamma_1(s))\right)ds-\int_0^{\varphi(t_1)} v\left(t_1,s,x(\gamma_1(s))\right)ds\right| \\
& +w_{r_0}^1(f,\varepsilon)+k'\left|x(\alpha(t_2))-x(\alpha(t_1))\right| \\
\leq\ & k\int_0^{\varphi(t_2)}\left|v\left(t_2,s,x(\gamma_1(s))\right)-v\left(t_1,s,x(\gamma_1(s))\right)\right|ds+ \\
& +k\int_{\varphi(t_2)}^{\varphi(t_1)}\left|v\left(t_1,s,x(\gamma_1(s))\right)\right|ds+w_{r_0}^1(f,\varepsilon)+ \\
& +k'\left|x(\alpha(t_2))-x(\alpha(t_1))\right|. \qquad (3.4)
\end{aligned}$$

Then, by (3.4), we have that

$$\begin{aligned}
& |(Fx)(t_2)-(Fx)(t_1)| \\
\leq\ & kw_{r_0}^1(v,\varepsilon)+kL|\varphi(t_1)-\varphi(t_2)|+w_{r_0}^1(f,\varepsilon)+ \\
& +k'\left|x(\alpha(t_2))-x(\alpha(t_1))\right| \\
\leq\ & kw_{r_0}^1(v,\varepsilon)+kLw(\varphi,\varepsilon)+w_{r_0}^1(f,\varepsilon)+k'w\left(x,w(\alpha,\varepsilon)\right), \qquad (3.5)
\end{aligned}$$

where

$$w_{r_0}^1(v,\varepsilon)=\sup\left\{|v(t_2,s,x)-v(t_1,s,x)|:t_1,t_2,s\in I,x\in R;|t_1-t_2|\leq\varepsilon\right\},$$

$$\begin{aligned}
w_{r_0}^1(f,\varepsilon)\ =\ & \sup\left\{|f(t_2,y,x)-f(t_1,y,x)|:t_1,t_2\in I,\right. \\
& \left. x\in R,y\in[-L,L];|t_1-t_2|\leq\varepsilon\right\},
\end{aligned}$$

$$L=\sup\{|v(t,s,x)|:t,s\in I;x\in R\}$$

and

$$w(\alpha_i,\varepsilon)=\sup\{|\alpha_i(t_2)-\alpha_i(t_1)|:t_1,t_2\in I;|t_1-t_2|\leq\varepsilon\}$$

for $i=1,2,3$ such that $\alpha_1=\varphi,\alpha_2=\alpha$ and $\alpha_3=x$. From (3.5), we get

$$w(Fx,\varepsilon)\leq kw_{r_0}^1(v,\varepsilon)+kLw(\varphi,\varepsilon)+w_{r_0}^1(f,\varepsilon)+k'w\left(x,w(\alpha,\varepsilon)\right). \qquad (3.6)$$

Taking into account the uniform continuity of functions f,v,α and φ on the bounded sets, we can deduce by (3.6) that

$$w_0(FX)\leq k'w_0(X). \qquad (3.7)$$

In a similar way, we have

$$\begin{aligned}
& |(Gx)(t_2)-(Gx)(t_1)| \\
= & \left|g\left(t_2, x(\gamma_2(t_2))\int_0^1 u\left(t_2,s,x(\gamma_3(s))\right)ds, x(\beta(t_2))\right)-\right. \\
& \left.- g\left(t_1, x(\gamma_2(t_1))\int_0^1 u\left(t_1,s,x(\gamma_3(s))\right)ds, x(\beta(t_1))\right)\right| \\
\leq & \left|g\left(t_2, x(\gamma_2(t_2))\int_0^1 u\left(t_2,s,x(\gamma_3(s))\right)ds, x(\beta(t_2))\right)-\right. \\
& \left.- g\left(t_2, x(\gamma_2(t_1))\int_0^1 u\left(t_1,s,x(\gamma_3(s))\right)ds, x(\beta(t_2))\right)\right| \\
& +\left|g\left(t_2, x(\gamma_2(t_1))\int_0^1 u\left(t_1,s,x(\gamma_3(s))\right)ds, x(\beta(t_2))\right)-\right. \\
& \left.- g\left(t_1, x(\gamma_2(t_1))\int_0^1 u\left(t_1,s,x(\gamma_3(s))\right)ds, x(\beta(t_2))\right)\right| \\
& +\left|g\left(t_1, x(\gamma_2(t_1))\int_0^1 u\left(t_1,s,x(\gamma_3(s))\right)ds, x(\beta(t_2))\right)-\right. \\
& \left.- g\left(t_1, x(\gamma_2(t_1))\int_0^1 u\left(t_1,s,x(\gamma_3(s))\right)ds, x(\beta(t_1))\right)\right| \\
\leq & k\left|x(\gamma_2(t_2))\int_0^1 u\left(t_2,s,x(\gamma_3(s))\right)ds - x(\gamma_2(t_1))\int_0^1 u\left(t_1,s,x(\gamma_3(s))\right)ds\right| \\
& +w^1_{r_0}(g,\varepsilon)+k'\left|x(\beta(t_2))-x(\beta(t_1))\right| \\
\leq & k\left|\left[x(\gamma_2(t_2))-x(\gamma_2(t_1))\right]\int_0^1 u\left(t_2,s,x(\gamma_3(s))\right)ds\right|+ \\
& +k\left|x(\gamma_2(t_1))\int_0^1 \left[u\left(t_2,s,x(\gamma_3(s))\right)-u\left(t_1,s,x(\gamma_3(s))\right)\right]ds\right| \\
& +w^1_{r_0}(g,\varepsilon)+k'w\left(x,w(\beta,\varepsilon)\right).
\end{aligned} \tag{3.8}$$

By (3.8), we derive that

$$\begin{aligned}
& |(Gx)(t_2)-(Gx)(t_1)| \\
\leq & kMw\left(x,w(\gamma_2,\varepsilon)\right)+kr_0w^1_{r_0}(u,\varepsilon)+w^1_{r_0}(g,\varepsilon)+k'w\left(x,w(\beta,\varepsilon)\right),
\end{aligned} \tag{3.9}$$

where

$$\begin{aligned}
w^1_{r_0}(g,\varepsilon) &= \sup\left\{|g(t_2,y,x)-g(t_1,y,x)| : t_1,t_2\in I,\right. \\
&\qquad \left. y\in[-r_0M,r_0M], x\in R; |t_1-t_2|\leq\varepsilon\right\}, \\
w^1_{r_0}(u,\varepsilon) &= \sup\left\{|u(t_2,s,x)-u(t_1,s,x)| : t_1,t_2,s\in I, x\in R; |t_1-t_2|\leq\varepsilon\right\}
\end{aligned}$$

and M is the nonnegative constant such that $|u(t,s,x)| \leq M$ for all $t, s \in I$ and $x \in [-1,1]$. Also,

$$w(\beta_j, \varepsilon) = \sup\{|\beta_j(t_2) - \beta_j(t_1)| : t_1, t_2 \in I; |t_1 - t_2| \leq \varepsilon\}$$

for $j = 1,2,3$ such that $\beta_1 = \beta, \beta_2 = \gamma_2$ and $\beta_3 = x$. From (3.9), we get

$$\begin{aligned} w(Gx,\varepsilon) \ &\leq \ kMw\left(x, w(\gamma_2,\varepsilon)\right) + kr_0 w^1_{r_0}(u,\varepsilon) + w^1_{r_0}(g,\varepsilon) + \\ &\quad + k'w\left(x, w(\beta,\varepsilon)\right). \end{aligned} \tag{3.10}$$

Since functions γ_2, u, g and β are uniform continuous on the bounded sets, we can obtain by (3.10) that

$$w_0(GX) \leq (k' + kM)\, w_0(X). \tag{3.11}$$

Finally, linking (3.1), (3.2), (3.7), (3.11) and Theorem 2.4, we get that T satisfies Darbo condition on ball B_{r_0} with constant $\tilde{k}$ such that

$$\begin{aligned} \tilde{k} \ &= \ k'\left[\, k(\alpha_1 + \beta_1 r_0^{p_1}) + m_1 + k'r_0 + kr_0(\alpha_2 + \beta_2 r_0^{p_2}) + m_2 + k'r_0\right] + \\ &\quad + kM\left[k(\alpha_1 + \beta_1 r_0^{p_1}) + m_1 + k'r_0\right]. \end{aligned}$$

Since the inequality

$$k'\left[\, k(\alpha_1 + \alpha_2 + \beta_1 + \beta_2) + m_1 + m_2 + 2k'\,\right] + kM\left[k(\alpha_1 + \beta_1) + m_1 + k'\right] < 1$$

holds by assumption (vi), $\tilde{k} < 1$ and so T is a contraction on ball B_{r_0} and has a fixed point in B_{r_0} by Theorem 2.4. Consequently, Eq.(1.1) has at least one solution in B_{r_0}. This step completes the proof of our theorem. □

Note 3.1. *Functions u, v, α and β satisfying conditions (iii), (iv) and (v) of Theorem* 3.1 *in* [9] *for $a = 1$ also satisfy conditions (ii) and (iii) of our theorem. But, converse of this may not be correct.*

4 Examples

In this section, we present some examples verifying the conditions of Theorem 3.1 and not verifying the conditions of Theorem 3.1 in [9].

Example 4.1. *Let us take functions $f, g : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by*

$$f(t,y,x) = \frac{[\,\exp(t-1) + 1\,]\sin x}{5 + (t-1)^2} + \frac{y}{5 + t^2}$$

and

$$g(t,y,x)=\frac{2\sin(|x|+t^2)}{5+t^3}+\frac{y}{5+t^2}.$$

These functions are continuous and satisfy Lipschitz condition with respect to variables y and x with constants $k=1/5$ and $k'=2/5$, respectively. Also, since

$$f(t,0,0)=0 \text{ and } g(t,0,0)=\frac{2\sin(t^2)}{5+t^3},$$

we can choose nonnegative constants m_1 and m_2 as $m_1=0$ and $m_2=2/5$. Next, we take

$$v(t,s,x)=\frac{[\,1+\cos(ts^3)\,](x^2+s)}{3+s^2}$$

and

$$u(t,s,x)=\frac{\sin\sqrt{t}+\sqrt{|x|^3}}{5+\ln(1+s)+s^3}.$$

It is easy to verify that the inequalities

$$|v(t,s,x)|\le\frac{2}{3}+\frac{2}{3}|x|^2 \text{ and } |u(t,s,x)|\le\frac{1}{5}+\frac{1}{5}|x|^{\frac{3}{2}}$$

hold for all $t,s\in I=[0,1]$ and $x\in\mathbb{R}$. So, assumption (ii) is satisfied with

$$\alpha_1=\beta_1=\frac{2}{3},\ \alpha_2=\beta_2=\frac{1}{5},\ p_1=2 \text{ and } p_2=\frac{3}{2}.$$

On the other hand, if we take

$$\alpha(t)=\beta(t)=\sqrt{t},\ \varphi(t)=t^2 \text{ and } \gamma_1(t)=\gamma_2(t)=\gamma_3(t)=\frac{t}{2},$$

assumption (iii) holds.
Furthermore, since

$$(k\,\alpha_1+m_1)\,m_2>0,$$

$$[\,k(\alpha_1+\beta_1)+m_1+k'\,]\,[\,k(\alpha_2+\beta_2)+m_2+k'\,]=\frac{44}{75}<1$$

and

$$k'\,[\,k(\alpha_1+\alpha_2+\beta_1+\beta_2)+m_1+m_2+2k'\,]+$$
$$+kM\,[k(\alpha_1+\beta_1)+m_1+k']=\frac{252}{375}<1,$$

the inequalities in assumptions (iv), (v) and (vi) hold, where

$$M=\sup\{|u(t,s,x)|:t,s\in I;x\in[-1,1]\}=\frac{2}{5}.$$

Then, Eq.(1.1) *has the form*

$$
\begin{aligned}
x(t) \quad = \quad & \left(\frac{[\,\exp(t-1)+1\,]\sin x\left(\sqrt{t}\right)}{5+(t-1)^2} + \right. \\
& \left. + \frac{1}{5+t^2}\int_0^{t^2} \frac{[\,1+\cos(ts^3)\,]\left[\left(x\left(\frac{s}{2}\right)\right)^2+s\right]}{3+s^2} ds \right) \times \\
& \left(\frac{2\sin(|x\left(\sqrt{t}\right)|+t^2)}{5+t^3} + \frac{x\left(\frac{t}{2}\right)}{5+t^2}\int_0^1 \frac{\sin\sqrt{t}+\sqrt{\left|x\left(\frac{s}{2}\right)\right|^3}}{5+\ln(1+s)+s^3} ds \right), \quad (4.1)
\end{aligned}
$$

where $t \in [0,1]$. *Consequently, by applying Theorem* 3.1, *we deduce that Eq.*(4.1) *has at least one solution* $x \in C[0,1]$.
Since there is no constants α_1, β_1, α_2 *and* β_2 *satisfying the inequalities*

$$|v(t,s,x)| \leq \alpha_1 + \beta_1|x| \text{ and } |u(t,s,x)| \leq \alpha_2 + \beta_2|x|$$

for all $t,s \in I = [0,1]$ *and* $x \in \mathbb{R}$, *if we take*

$$\gamma_1(t) = \gamma_2(t) = \gamma_3(t) = \varphi(t) = t, \ \alpha(t) = \beta(t) = \sqrt{t},$$

the result in [9] *is inapplicable to the integral equation*

$$
\begin{aligned}
x(t) \quad = \quad & \left(\frac{[\,\exp(t-1)+1\,]\sin x(\sqrt{t})}{5+(t-1)^2} + \right. \\
& \left. + \frac{1}{5+t^2}\int_0^{t} \frac{[\,1+\cos(ts^3)\,][(x(s))^2+s]}{3+s^2} ds \right) \times \\
& \times \left(\frac{2\sin(|x(\sqrt{t})|+t^2)}{5+t^3} + \frac{x(t)}{5+t^2}\int_0^1 \frac{\sin\sqrt{t}+\sqrt{|x(s)|^3}}{5+\ln(1+s)+s^3} ds \right).
\end{aligned}
$$

Also, functions α and β don't hold the inequalities in condition (iv) of Theorem 3.1 in [9].

Example 4.2. *If we take*

$$f(t,y,x) = a(t)+y, \ g(t,y,x) = 1, \ \varphi(t) = \gamma_1(t) = \alpha(t) = t \text{ and } u(t,s,x) = 0$$

for all $t,s \in I$; $x,y \in \mathbb{R}$, *then Eq.*(1.1) *is reduced to well known nonlinear Volterra integral equation*

$$x(t) = a(t) + \int_0^t v(t,s,x(s))ds, \tag{4.2}$$

where function $a : I \to \mathbb{R}$ *is continuous. Also, we assume that there exist nonnegative constants* α_1, β_1 *and* p_1 *such that the inequality*

$$|v(t,s,x)| \leq \alpha_1 + \beta_1 |x|^{p_1} \tag{4.3}$$

holds for all $t, s \in I$ *and* $x \in \mathbb{R}$.
The theory of above equation (4.2) *is well developed in* [1, 4, 5, 13, 14]. *For this specific choice of* f, g, φ, γ_1, α *and* u, *assumptions* (i), (ii) *and* (vi) *are satisfied with*

$$k = 1,\ k' = 0,\ \alpha_1,\ \beta_1,\ \alpha_2 = \beta_2 = 0,\ m_1 = \|a\|,\ m_2 = 1 \text{ and } M = 0.$$

Assumption (iii) *already holds for* φ, γ_1, α *and any continuous functions* $\beta, \gamma_2, \gamma_3 : I \to I$.
Additionally, we assume that the inequalities in conditions (iv) *and* (v) *which are equivalent to*

$$\alpha_1 + \|a\| > 0 \tag{4.4}$$

and

$$\alpha_1 + \beta_1 + \|a\| < 1, \tag{4.5}$$

respectively are verified.
Then, we deduce from Theorem 3.1 *that there exists at least one number* $r_0 \in (0,1)$ *and Eq.*(4.2) *has at least one solution* $x = x(t)$ *belonging to* $B_{r_0} \subset C[0,1]$.
We put

$$v(t,s,x) = \frac{15t^2 + 4\sin\left(\frac{x}{1+x^2}\right)}{19 + \exp(1-t)},\ a(t) = \frac{1}{t^2 + 25}.$$

The inequality

$$\begin{aligned} |v(t,s,x)| &\leq \frac{15}{20} + \frac{4}{20}\left|\sin\left(\frac{x}{1+x^2}\right)\right| \\ &\leq \frac{3}{4} + \frac{1}{5}\left|\frac{x}{1+x^2}\right| \\ &\leq \frac{3}{4} + \frac{1}{5}|x| \end{aligned}$$

holds for all $t, s \in I$ *and* $x \in \mathbb{R}$.
So (4.3), (4.4) *and* (4.5) *are satisfied with*

$$\alpha_1 = \frac{3}{4},\ \beta_1 = \frac{1}{5},\ p_1 = 1 \text{ and } \|a\| = \frac{1}{25}.$$

Then, our integral equation takes the form

$$x(t) = \frac{1}{t^2+25} + \int_0^t \frac{15t^2 + 4\sin\left(\frac{x(s)}{1+x^2(s)}\right)}{19+\exp(1-t)} ds \tag{4.6}$$

which has at least one solution $x \in B_{r_0} \subset C[0,1]$.
On the other hand, the inequality given in assumption (vi) *of Theorem* 3.1 *in* [9] *doesn't hold with*

$$k = 1,\ \tilde{\alpha} = \frac{3}{4},\ \tilde{\beta} = \frac{1}{5},\ a = 1,\ c = 1 \ \textit{and}\ d = 0,$$

since

$$[\,k(\tilde{\alpha} + \tilde{\beta}r)a + (c+dr)][\,k(\tilde{\alpha} + \tilde{\beta}r)ra + (c+dr)] > r$$

for all $r \in (0,\infty)$. *Hence, the result in* [9] *is inapplicable to integral equation* (4.6).

Example 4.3. *Let us define*

$$f(t,y,x) = 1,\ g(t,y,x) = a(t) + y,\ \gamma_2(t) = \gamma_3(t) = \beta(t) = t \ \textit{and}\ v(t,s,x) = 0$$

for all $t, s \in I$ *and* $x, y \in \mathbb{R}$, *where function* $a : I \to \mathbb{R}$ *is continuous.*
It is known from [9] *that function* u *given as*

$$u(t,s,x) = \begin{cases} 0, & s = 0,\ t \geq 0,\ x \in \mathbb{R} \\ \frac{t}{t+s}\phi(s)x, & s \neq 0,\ t \geq 0,\ x \in \mathbb{R} \end{cases}$$

is continuous on $I \times I \times \mathbb{R}$. *Here,* $\phi : I \to \mathbb{R}$ *is continuous and* $\phi(0) = 0$.
For these functions, Eq.(1.1) *has the form*

$$x(t) = a(t) + x(t) \int_0^1 \frac{t}{t+s}\phi(s)x(s) ds \tag{4.7}$$

which is related with the Chandrasekhar equation considered in [2, 6, 10–12]. *In this example,* f *and* g *satisfy Lipschitz condition with respect to variables* y *and* x *with constants* $k = 1$ *and* $k' = 0$, *respectively. Also, since*

$$f(t,0,0) = 1,\ g(t,0,0) = a(t),\ v(t,s,x) = 0 \ \textit{and}\ |u(t,s,x)| \leq \|\phi\|\,|x|$$

for all $t, s \in I$ *and* $x \in \mathbb{R}$, *we can choose nonnegative constants* m_1, m_2, α_1, β_1, α_2, β_2 *and* M *as*

$$m_1 = 1,\ m_2 = \|a\|,\ \alpha_1 = \beta_1 = \alpha_2 = 0,\ \beta_2 = \|\phi\| \ \textit{and}\ M = \|\phi\|.$$

Therefore, conditions (i) and (ii) are satisfied. It is obvious that condition (iii) is satisfied for $\beta, \gamma_2, \gamma_3$ and any continuous functions $\varphi, \gamma_1, \alpha : I \to I$. The inequalities in conditions (iv), (v) and (vi) can be expressed as

$$\|a\| > 0, \tag{4.8}$$

$$\|a\| + \|\phi\| < 1 \tag{4.9}$$

and

$$\|\phi\| < 1, \tag{4.10}$$

respectively. If (4.8) *and* (4.9) *which implies* (4.10) *hold, we have by Theorem* 3.1 *that there exists at least one number $r_0 \in (0,1)$ and Eq.*(4.7) *has at least one solution $x = x(t)$ belonging to $B_{r_0} \subset C[0,1]$.*
If we take

$$a(t) = \frac{e^{t-1}}{\sin^2(t-1)+25}, \ \ \phi(t) = \frac{7t}{10},$$

then (4.8) *and* (4.9) *hold which imply that the equation*

$$x(t) = \frac{e^{t-1}}{\sin^2(t-1)+25} + \frac{7t}{10} x(t) \int_0^1 \frac{s}{t+s} x(s) ds \tag{4.11}$$

has at least one solution $x \in C[0,1]$.
But, the inequality given in assumption (vi) of Theorem 3.1 *in* [9] *doesn't hold for constants*

$$k = 1, \ \tilde{\alpha} = 0, \ \tilde{\beta} = \frac{7}{10}, \ a = 1, \ c = 1 \text{ and } d = 0,$$

since

$$[\,k(\tilde{\alpha} + \tilde{\beta} r)a + (c + dr)][\,k(\tilde{\alpha} + \tilde{\beta} r)ra + (c + dr)] > r$$

for all $r \in (0, \infty)$. So, the result in [9] *is inapplicable to integral equation* (4.11)*.*

Example 4.4. *Let us consider the following nonlinear integral equation of the form*

$$\begin{aligned} x(t) &= \frac{\ln(t+e-1)\sin x(t^3)}{t+19} + \\ &+ \frac{1}{t+4} \int_0^t \left(\frac{4s \sin x(s)}{20s^2+5} + \frac{10 \ln(s+e-1)}{s+3} \right) ds. \end{aligned} \tag{4.12}$$

Put

$$f(t,y,x)=\frac{\ln(t+e-1)\sin x}{t+19}+\frac{y}{t+4},\ g(t,y,x)=1,$$
$$v(t,s,x)=\frac{4s\sin x}{20s^2+5}+\frac{10\ln(s+e-1)}{s+3},\ u(t,s,x)=0,$$
$$\gamma_1(t)=\varphi(t)=t,\ \alpha(t)=t^3$$

and

$$k=\frac{1}{4},k'=\frac{1}{20},\ m_1=0,\ m_2=1,\ \alpha_1=\frac{5}{2},\ \beta_1=\frac{1}{5},\ \alpha_2=0,\ \beta_2=0,M=0.$$

It is easy to show that all of the conditions of Theorem 3.1 *hold. Therefore, Theorem* 3.1 *guarantees that there exists at least one* $r_0\in(0,1)$ *such that Eq.*(4.12) *has at least one solution* $x=x(t)$ *belonging to* $B_{r_0}\subset C[0,1]$.
On the other hand, the inequality given in assumption (vi) *of Theorem* 3.1 *in* [9] *doesn't hold for constants*

$$k=\frac{1}{4},\ \alpha_1=\frac{5}{2},\ \alpha_2=0,\ \beta_1=\frac{1}{5},\ \beta_2=0,\ a=1,$$
$$c_1=0,\ c_2=1,\ d_1=\frac{1}{20}\ \textit{and}\ d_2=0.$$

Hence, the result in [9] *is inapplicable to integral equation* (4.12).

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