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On $L^{'}(2,1)-$Edge Coloring Number of Regular Grids

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Abstract

In this paper, we study multi-level distance edge labeling for infinite rectangular, hexagonal and triangular grids. We label the edges with non-negative integers. If the edges are adjacent, then their color difference is at least 2 and if they are separated by exactly a single edge, then their colors must be distinct. We find the edge coloring number of these grids to be 9, 7 and 16, respectively so that we could color the edges of a rectangular, hexagonal and triangular grid with at most 10, 8 and 17 colors, respectively using this coloring technique. Repeating the sequence pattern for different grids, we can color the edges of a grid of larger size.

1 Introduction

A graph G is a finite nonempty set V of objects called vertices together with a possibly empty set of 2-element subsets of V called edges. To indicate that a graph has vertex set V and edge set E, we write $G = (V, E)$ [1]. If $e = uv$ is an edge of G, then e is said to join u and v. Also, u and v are called adjacent vertices. If uv and vw are distinct edges in G, then uv and vw are adjacent edges [1]. Two adjacent vertices are referred to as neighbors of each other. The set of neighbors of a vertex v is called the open neighborhood of v and is denoted by $N_G(v)$ or $N(v)$. The set $N[v] = N(v) \cup \{v\}$ is called

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the closed neighborhood of v. Similarly we define the set of neighbors of an edge as well. The degree of a vertex in a graph G is the number of vertices in G that are adjacent to v[1]. The degree of a vertex v is denoted by $deg\ v$. Hence $deg\ v = |N(v)|$. A pendant edge in a graph is an edge that is incident to a vertex of degree 1. For a nonempty subset S of V, the subgraph H of G induced by S has S as its vertex set and two vertices u and v are adjacent in H if and only if u and v are adjacent in G [1]. An edge-induced subgraph is a subset of the edges of a graph G together with any vertices that are their end vertices.

The distance between two vertices is in a graph is the length of the shortest path connecting them. Fixing a vertex, we can find the distance from it to all other vertices in the graph. The largest such distance is known as the eccentricity of that vertex. Now, for all the vertices in a graph, the largest and the samllest eccentricities are known as the diamter and radius of the graph, respectively.

As seen in [2], if G is a connected graph and $e_1 = (u_1, v_1), e_2 = (u_2, v_2)$ are two edges of G, then the *distance between edges or edge distance* of e_1 and e_2 is defined as

$$ed(e_1, e_2) = min\{d(u_1, u_2), d(u_1, v_2), d(v_1, u_2), d(v_1, v_2)\}.$$

If $ed(e_1, e_2) = 0$, then these edges are called *neighbor edges* or adjacent edges. The $L^{'}(2,1)$ edge coloring of a graph G is defined as in [4]. For non-negative integers i and j, an $L^{'}(i,j)$ edge coloring of a graph G is an assignment of non-negative integers to the edges e_1 and e_2 of G such that $|c(e_1) - c(e_2)| \geq i$ if $ed(e_1, e_2) = 0$ and $|c(e_1) - c(e_2)| \geq j$ if $ed(e_1, e_2) = 1$. No condition is placed on colors assigned to the edges e_1 and e_2 if $ed(e_1, e_2) \geq 2$. In this paper we study the case where $i = 2$ and $j = 1$. Hence, an $L^{'}(2,1)$ edge coloring of a graph G is a function c from the edge set $E(G)$ to $\{0, 1, 2, ..., k\}$ such that $|c(e_1) - c(e_2)| \geq 2$ if $ed(e_1, e_2) = 0$ and $|c(e_1) - c(e_2)| \geq 1$ if $ed(e_1, e_2) = 1$. No restriction is placed on colors assigned to edges at distance 2 or more. In fact an $L^{'}(1,0)$ edge coloring is a proper edge coloring of G. For an $L^{'}(i,j)-$ Edge Coloring c of a graph G, the c-span of G is the maximum value of $|c(e_1) - c(e_2)|$ over all pairs of edges e_1 and e_2 of $E(G)$. It is denoted by $\lambda^{'}_{i,j}(c)$ [4]. That is,

$$\lambda^{'}_{i,j}(c) = max\{|c(e_1) - c(e_2)| : e_1, e_2 \in E(G)\}.$$

In particular, [5] for $i = 2$ and $j = 1$, we have the c-span of G with respect to the $L^{'}(2,1)-$ Edge Coloring as

$$\lambda^{'}_{2,1}(c) = max\{|c(e_1) - c(e_2)| : e_1, e_2 \in E(G)\}.$$

The $L^{'}(2,1)-$ Edge Coloring Number denoted by $\lambda^{'}(G)$ is the smallest positive integer k such that there exists an $L^{'}(2,1)-$ edge coloring $c : E(G) \to$

$\{0,1,2,...,k\}$ [4]. As we consider 0 as the smallest color in any $L^{'}(2,1)-$ edge coloring of G, we see that $\lambda^{'}(G)$ is the smallest maximum color used among $L^{'}(2,1)-$ edge coloring of G.

For any edge e in a connected graph G, the edge-to-edge eccentricity $e_3(e)$ of e is $e_3(e) = max\{d(e,f) : f \in E\}$. Any edge e for which $e_3(e)$ is minimum is called an edge-to-edge central edge of G and the set of all edge-to-edge central edges of G is the edge-to-edge center $C_3(G)$ of G [5].

By a critical graph, we mean a $\chi'-$critical graph and by a critical edge we mean a $\chi'-$critical edge [8].

2 Rectangular Grids

Rectangular grid graphs(RGG) are induced subgraphs of a rectangular grid [6]. Given a grid Z_r^2 whose vertices corresponds to the points with integer coordinates in the plane, and in which two vertices are connected by an edge whenever the corresponding points are within distance 1, a rectangle grid graph is an induced subgraph of Z_r^2. Given a RGG, we assign the colors to its edges based on $L^{'}(2,1)-$ edge coloring technique and thereby obtain the $L^{'}(2,1)-$ edge coloring number of infinite rectangular grids. For convenience we denote RGG as $G_{m,n}$, an $m \times n$ rectangular grid graph.

Theorem 1. *[6] The $L^{'}(2,1)$ edge coloring number of $G_{2,4}$ is 7.*

Theorem 2. *The $L^{'}(2,1)$ edge coloring number of $G_{3,4}$ is 8. That is, $\lambda'(G_{3,4}) = 8$.*

Proof. Let H be an $8-$*critical graph* with $C_3(H) = e'$ and $C = \{0,1,\ldots,7\}$ be the set of colors used to color H. Refer Figure 1. If we add edges to the

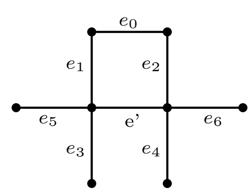


Figure 1: H: 8-critical graph.

peripheral vertices of H to obtain $G_{3,4}$, we see that $C_3(G_{3,4}) = e'$ and the optimal coloring of $G_{3,4}$ is based on the color of e', $c(e')$. As H is $8-critical$, we assign the colors to its edges, according to $L'(2,1)-$ edge coloring criteria, using the color set C.

Case 1: $c(e') = x$; *where* $0 < x < 7$; $e' \in E(H)$.

Let $c(e_0) = (x-1)$ *or* $(x+1)$. But by $L'(2,1)-$ edge coloring criteria, none of the edges of $E(H) - \{e', e_0\}$ can be colored using $x, (x-1)$ and $(x+1)$. That is, the remaining six edges need to be colored using (8-3)=5 colors, which is a contradiction, as $ed(e_i, e_j) < 2$ where $e_i,\ e_j\ \in E(H)\ \forall i, j$ and hence need distinct colors. Hence, $E(H)$ cannot be colored using the color set C. Therefore, $\lambda^{'}(H) \geq 8$ and hence $\lambda^{'}(G_{3,4}) \geq 8$.

Case 2: $c(e') = 0$ *or* 7.

Figure 2 shows an optimal coloring of H using the color set C.

(a) $c(e') = 0$ (b) $c(e') = 7$

Figure 2: An optimal coloring of H.

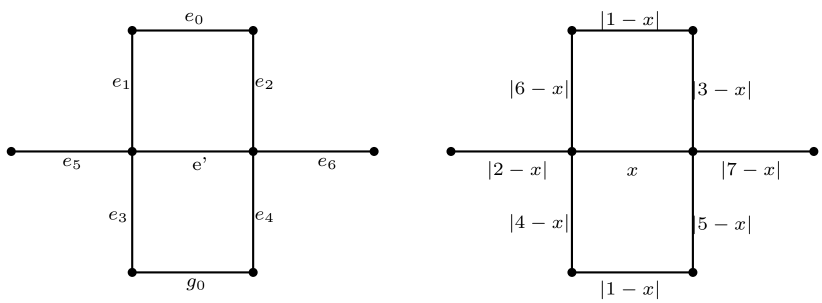
suppose $\lambda^{'}(G_{3,4}) = 7$. Also, H is an $8-$ *critical graph*. Then any superstructure of H, contained in $G_{3,4}$ too can be colored using the color set C. Consider a superstructure of H, say $H' \subseteq G_{3,4}$ such that g_0 is an edge adjacent to e_3 and e_4. Refer Figure 3(a).

Clearly, H' can be colored using the color set C. Refer Figure 3(b). We now construct a superstructure of H', say H'' by adding one more edge, g_1, adjacent to the edges having labels $|1-x|$ and $|3-x|$ such that $H'' \subseteq G_{3,4}$. We see that none of the labels from the color set C can be assigned to the edge g_1, which is a contradiction to the hypothesis that any subgraph, H_i such that $H \subseteq H_i \subseteq G_{3,4}$ can be colored using the color set C. Therefore, $\lambda'(G_{3,4}) > 7$. Hence from both the cases we have,

$$\lambda'(G_{3,4}) \geq 8$$

and by Figure 4 we conclude that $\lambda'(G_{3,4}) = 8$.

Remark 1. *The problem of solving the* $L^{'}(2,1)$ *edge coloring number of* $G_{m,n}$



(a) Subgraph $H' \subseteq G_{3,4}$ (b) An optimal coloring of H'.

Figure 3: $H'-$ A superstructure of H.

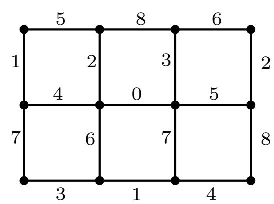


Figure 4: An optimal coloring of $G_{3,4}$

$\square$

as less than or equal to 10 *was placed as a conjecture in [4]. Here, we prove the following* $L^{'}(2,1)$ *edge coloring of* $G_{m,n}$ *equals 9 is an optimal one.*

Theorem 3. *For* $m \geq 6$ *and* $n \geq 5$*, the* $L^{'}(2,1)$ *edge coloring number of* $G_{m,n}$ *is* 9*. That is,* $\lambda'(G_{m,n}) = 9$*; where* $m \geq 6, n \geq 5$.

Proof. Suppose $\lambda'(G_{m,n}) < 9$. Obviously, by previous theorem, $\lambda'(G_{m,n}) \geq 8$; where $m, n \geq 4$. We now construct a substructure G of $G_{m,n}$ such that the edge-to-edge center of G is K_2. That is, by the $L^{'}(2,1)-$ edge coloring criteria, there exists a critical edge e' such that $C_3(G) = \{e'\}$. Refer Figure 6 to note that the color of $e^{'}$, $c(e^{'})$, cannot be repeated in G; where $e^{'} \in C_3(G)$. As $7 < \lambda'(G) < 9$, let $C = \{0, 1, \cdots, 8\}$ be the possible color set under consideration for the optimal $L^{'}(2,1)$ edge coloring of G.

But $|E(G)| = 23$ indicates that the colors has to be repeated, except the

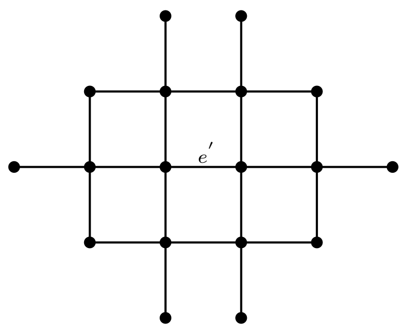


Figure 5: The substructure G

color of the critical edge, e'. Let $n(c_i)$ be the number of repetition of the color c_i, assigned to the distinct edges of G with respect to the color set. By the previous theorem, $1 \leq n(c_i) \leq 2$ for any optimal coloring of $G_{3,4}$. Hence, if any of the c_i is to be repeated in G, it can be repeated at most for some of the pendant edges of G. That is, apart from a c_i repeated twice for an edge in $G_{3,4}$, a maximum of two edges only can receive the same c_i due to the edge distance criteria. Let F be a proper subset of $E(G)$ such that F induces $G_{3,4}$. Suppose $n(c_i) = 4$; for some i. To achieve this, considering the edge distance criteria in G, we follow different cases.

Case 1. Two of the pendant edges of G and two of the edges of F induced subgraph have the same color.

Then, for $1 \leq i, j \leq 2$, there exists $f_i \in F$ and $e_j \in E/F$ such that $ed(e_i, e_j)$, $ed(f_i, e_j)$, $ed(f_i, f_j) \geq 2$ and $c(f_i) = c(e_j) = c_i$; which in turn indicates that $|(N(f_i) \cup N(e_j))/F|$=14, Which implies 14 edges cannot be colored using $c_i, c_i - 1, c_i + 1$ and $c(e^{'})$. That is, six edges in $N(e^{'})$ cannot be colored using 6 colors, including $c(e^{'})-1$ and $c(e^{'})+1$. Now if G is colored using the color class C, then six edges in $N(e^{'})$ has to be colored using $|C| - 6 = 3$ colors, which is a contradiction as edges in $N(e^{'})$ are at edge distance zero and hence need distinct colors.

Case 2. Three of the pendant edges of G have one color and the same is given to one of the edges of F induced subgraph. That is, one edge of F and three edges of E/F have the same color, say c_i.

Then, for $1 \leq i \leq 3$ there exists $e_i \in E/F$ and $f_j \in F$, for some j, such that $ed(e_i, e_j)$, $ed(e_i, f_j) \geq 2$ so that $c(e_i) = c(f_j) = c_i$. Which indicates

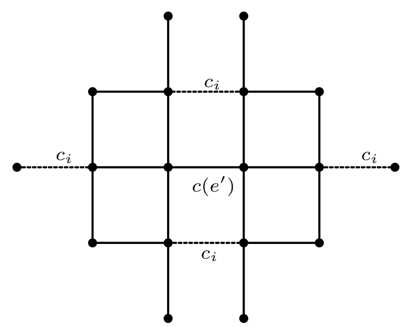


Figure 6: The substructure G under case 1

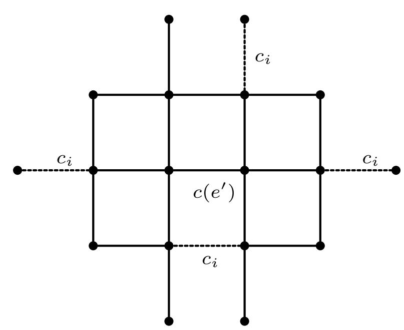


Figure 7: The substructure G under case 2

that $|N(e_i) \cup N(f_j)/F| = 13$ edges cannot be colored using $c_i, c_i - 1, c_i + 1$ and $c(e^{'})$.That is, 5 edges in $N(e^{'})$ cannot be colored using 6 colors, including $c(e^{'}) - 1$ and $c(e^{'}) + 1$. Now if color class C is considered, then 5 edges in $N(e^{'})$ is colored using $|C| - 6 = 3$ colors and we arrive at a contradiction as edges in $N(e^{'})$ are at edge distance zero and hence need distinct colors.

Case 3. Four of the pendant edges are colored same.That is, four of the edges in E/F have the same color, say c_i.

We arrive at a contradiction as in the above cases using both the color classes C and $C^{'}$. Hence we conclude that $n(c_i) < 4$.

We now suppose that a color can be repeated at most 3 times. For any optimal $L^{'}(2,1)$ edge coloring of G, we see that at most two colors can be repeated 3 times among the edges in E/F and at most one color can be repeated 3 times for the edges in F induced subgraph. That is, at most 3 colors can be repeated 3 times for any optimal $L^{'}(2,1)$ edge coloring of G. Suppose 3 colors are repeated three times in G. Then considering the color set C gives $23 = q(G) = \Sigma n(c_i) \leq 3(3) + 1 + 2(5) = 20$, which is a contradiction. Hence, at most two colors can be repeated 3 times in G. We now repeat 2 colors three times for an optimal $L^{'}(2,1)$ edge coloring of G using the color set C. This implies $23 = q(G) = \Sigma n(c_i) \leq 3(2)+1+2(6) = 19$, a contradiction.

So, we cannot color G using the color set C and at least one more color is required for the optimal coloring of G. That is, $\lambda'(G) > 8$. Any superstructure of G, say G', obtained by adding one more edge, increases the diameter which in turn implies that the $c(e')$ can be repeated. Therefore, $\lambda'(G')$ will be either equal to or greater than 9 and we can conclude that $\lambda'(G_{m,n}) \geq \lambda'(G') \geq 9$. By Figure 10 we finalize that $\lambda'(G_{m,n}) = 9$; where $m \geq 6$, $n \geq 5$. □

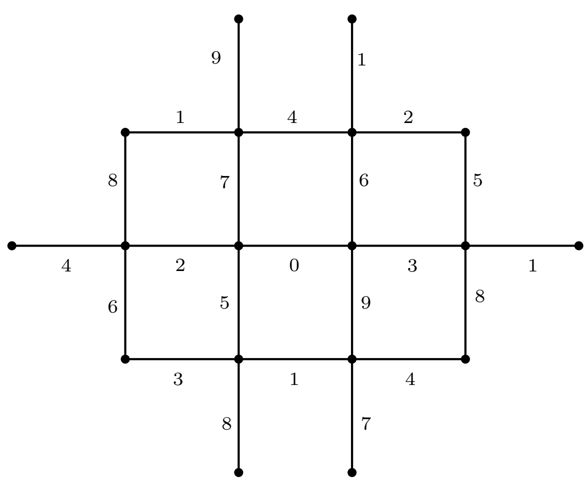


Figure 8: An optimal coloring of G

Remark 2. $L^{'}(2,1)$ *edge coloring number of any rectangular grid graph is 9.*

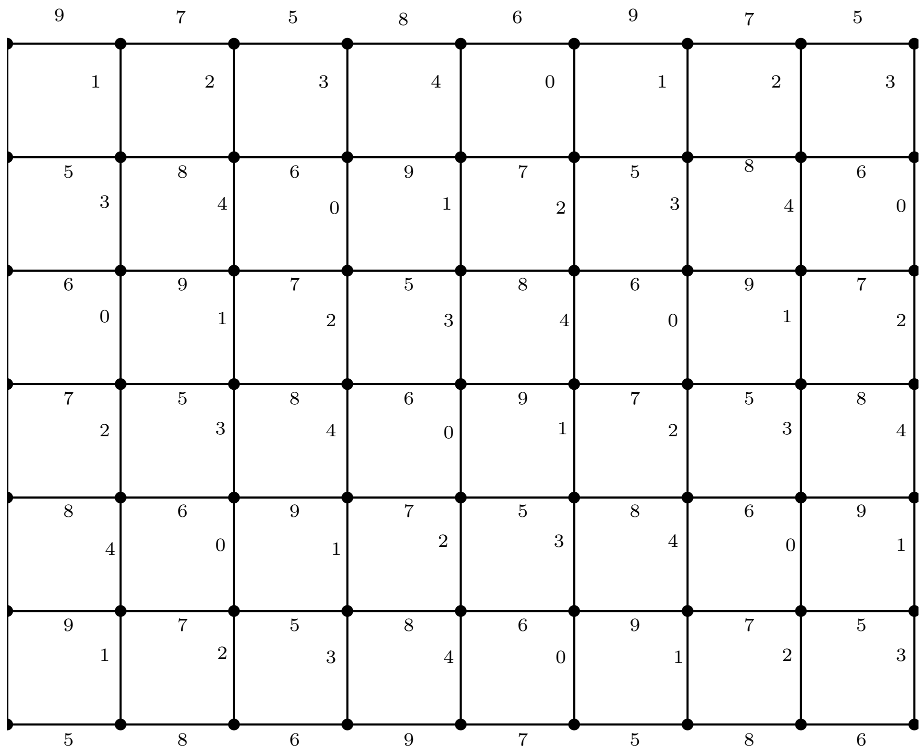


Figure 9: An optimal $L'(2,1)-$labeling of a fragment of rectangular grid

That is, we can label the edges of a rectangular grid of any size using the colors $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$.

3 Hexagonal Grids

It is evident from the following Figure that the hexagonal grid or honeycomb structure denoted by $H_{m,n}$ is a spanning subgraph of a rectangular grid with $V(H_{m,n}) = V(G_{m,n})$ and $E(H_{m,n}) \subset E(G_{m,n})$ implying that the value of Δ decreases and hence $\lambda'(H_{m,n}) < 9$.

Theorem 4. *The edge coloring number of a hexagonal grid is at most 7.*

Proof. Let H be a 5-critical subgraph of a hexagonal grid which is of diameter

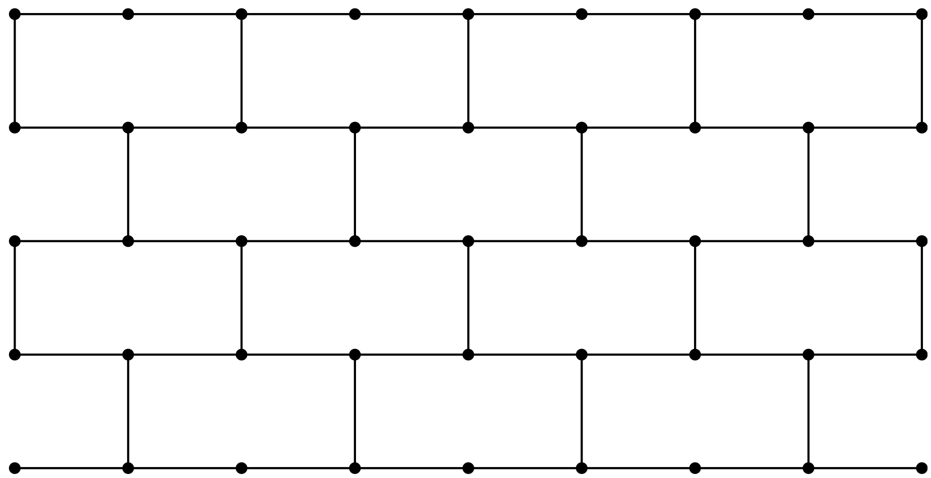


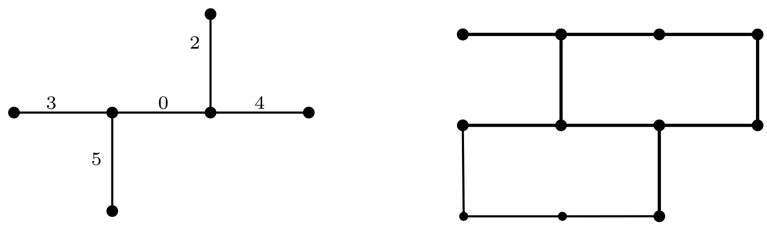
Figure 10: $H_{m,n}$- A fragment of a hexagonal grid

3. Refer Figure 11(a) for the optimal coloring of H and the $L'(2,1)-$ edge coloring number of H is given by $\lambda'(H) = 5$.

Now add edges to H to obtain a subgraph H' (figure 11(b)) of a hexagonal grid, of diameter 5. Then $\lambda'(H') \geq 5$. Suppose 0, 1, 2, 3, 4, 5 is the color set used to color the edges of H'. As the diameter is 5, none of the colors can be repeated four times. Also, no three vertical edges can be colored same. That is, at most two of the vertical edges can be colored same. As per the Figure, there exists four vertical edges in H'. If two of the vertical edges are colored same, then there exists exactly one horizontal edge which can be given the same color, which will be adjacent to any of the other two vertical edges. Hence it needs two distinct colors with difference at least two. Refer Figure 12(a). Consider the following two cases.

Case 1: $0 < x < 5$.

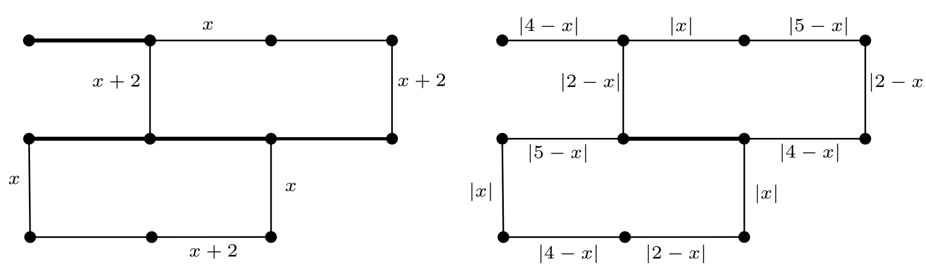
Let the two distinct colors assigned to the vertical edges be x and $x+2$. Then none of the remaining edges of H' can have the colors $(x-1)$, $(x+1)$ and $(x+3)$. Refer Figure 12(a). Which implies exactly one color is assigned to the remaining edges of H'; which is a contradiction. As there exists three edges with edge distance less than 2, we need at least 2 more colors to color the remaining edges using $L'(2,1)-$ edge coloring criteria. Hence, $\lambda'(H') = 7$.



(a) H-Diameter 3 subgraph of a hexagonal grid

(b) H'-Diameter 5 subgraph

Figure 11: substructures of a hexagonal grid



(a) coloring for $0 < x < 5$.

(b) coloring for $x = 0,\ 5$.

Figure 12: different cases of optimal labeling of H'

Case 2: $\quad x = 0$ *or 5.*

Let the two distinct colors assigned to the vertical edges be $|x|$ and $|2-x|$. Then none of the remaining edges of H' can have the colors $|1-x|$ and $|3-x|$. That is, the remaining edges can be colored using $|4-x|$ and $|5-x|$; which is a contradiction as there exists three edges with edge distance less than 2. Hence the color set is not sufficient. But the color 6 cannot be assigned to any of the edges implies that the edge coloring number is 7. Refer Figure 12(b).

Hence, from both the cases, we see that any superstructure of H' requires at least eight colors and by considering Figure 13 we can conclude that the edge coloring number of a hexagonal grid is at most 7.

Coloring technique used: Color all the horizontal edges using the sequence (7,4,6,3,5) and all 0 and all 1 for the vertical edges alternately with respect to the rows.

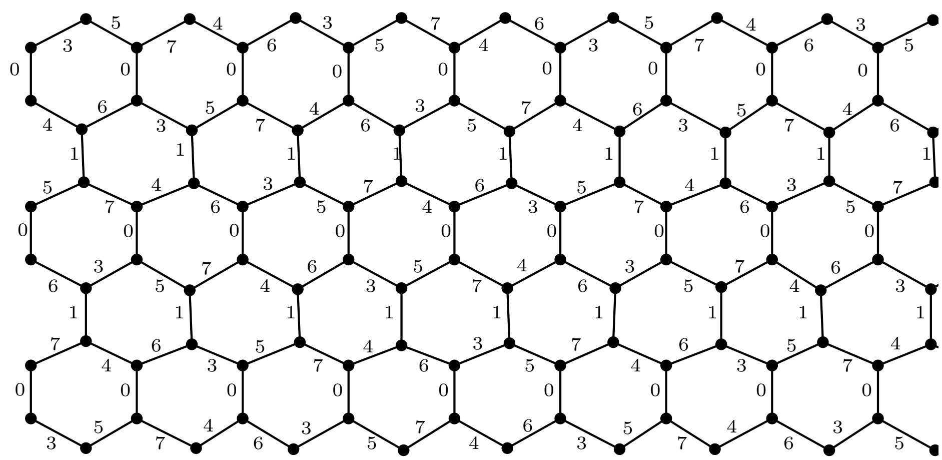


Figure 13: Optimal labeling of a fragment of a hexagonal grid

□

4 Triangular Grids

Triangular grid graphs(TGG) are induced subgraphs of a triangular grid. We observe that adding a diagonal edge to every 4-cycle in a rectangular grid leads to a triangular grid.

Obviously, the edge coloring number of a triangular grid will be much larger than that of a rectangular grid. As done earlier, we will proceed with a

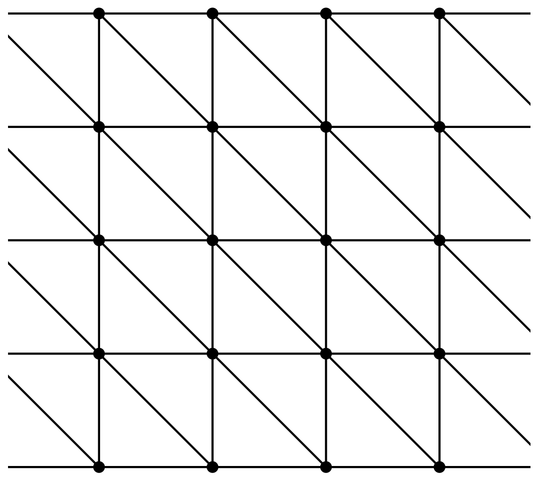
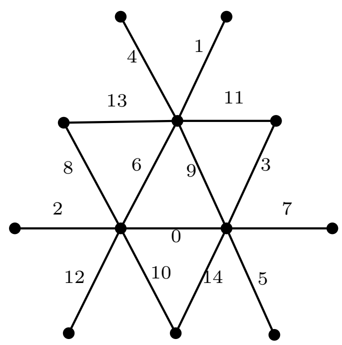


Figure 14: Triangular grid

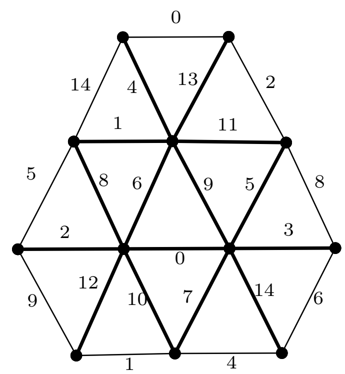
diameter 3 critical subgraph of a triangular grid.

Let H be a diameter 3 subgraph of the triangular grid which is 15 critical and this indicates that $\lambda'(H) = 14$. Refer Figure 15(a). Now add edges to the graph H to obtain an edge maximal subgraph H', of a triangular grid, of diameter 3. Note that the highlighted structure in Figure 15(b) is a subgraph isomorphic to H and hence we have that $\forall\ e \in E(H'|H), \exists$ no $f \in E(H'|H)$ such that $c(e) = c(f)\ \forall f \in E(H'|H)$ as H is 15- critical. Hence, if at all a color is repeated twice, it can be repeated only for the outer edges of H'. Hence, we can say that at most nine colors can be repeated twice and $2(9)+1(6) = 24 = q(H')$, size of the graph H'. Therefore, the optimal coloring of H' indicates that the edge coloring number of H' is also 14. Refer Figure 15(b). That is, $\lambda'(H') = 14$. This implies that any superstructure of H' will have its edge coloring number at least 14.

Construct a subgraph H'' of diameter 4, of a triangular grid, by adding edges to the subgraph H'. Refer Figure 16 for the subgraph H''. We claim



(a) H- diameter 3 subgraph



(b) H'- diameter 3 edge maximal subgraph

Figure 15: Substructures of a triangular grid of diameter 3

that $\lambda'(H'') \geq 16$.

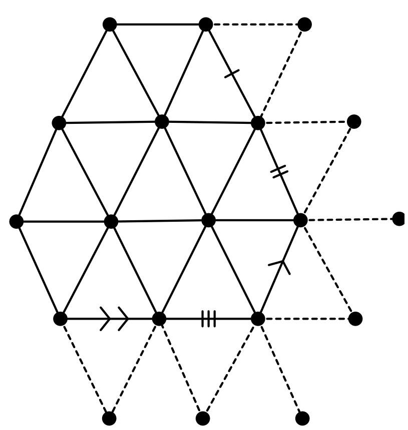


Figure 16: H''- a diameter 4 graph of size 36

For if S= 0, 1, 2, ..., 14, 15 is the color set used to color H'', at most half the edges those were colored twice in $E(H''|H')$ can be colored thrice in H''. That is, at most five colors can be repeated thrice and at most 4+6=10 colors can be repeated twice in H''. Therefore, $36 = q(H'') = \Sigma n(c_i) = 3(5) + 2(10) = 35$, which is a contradiction as the size of a diameter 4 graph is greater than 35. Therefore, at least 17 colors are required to color the edges of a diameter 4 subgraph by $L'(2,1)-$ edge coloring criteria. That is, $\lambda'(H'') \geq 16$.

Hence, by considering the Figure 17, we arrive at the following theorem.

Theorem 5. *The edge coloring number of a Triangular grid is at most 16.*

The Figure 17 shows an $L^{'}(2,1)$ edge coloring of a fragment of a Triangular Grid, which is an optimal one. That is, an $L^{'}(2,1)$ edge coloring number of any triangular grid graph is at most 16. By using the sequence (9,7,10,8,6), (0,1,2,3,4) and (12,13,14,15,16) for the horizontal, vertical and diagonal edges respectively, we can label the edges of any triangular grid using the colors $0, 1, 2, 3, \ldots, 16$.

P.S. The following inference were obtained from our study on edge coloring number of grids.

Grids	Δ	diameter=3	diameter >3
Hexagonal Grid	3	$\lambda' = 5$	$\lambda' = 7$
Rectangular Grid	4	$\lambda' = 7$	$\lambda' = 9$
Triangular Grid	6	$\lambda' = 14$	$\lambda' = 16$

Table 1: Observation on edge coloring number w.r.t. diameter

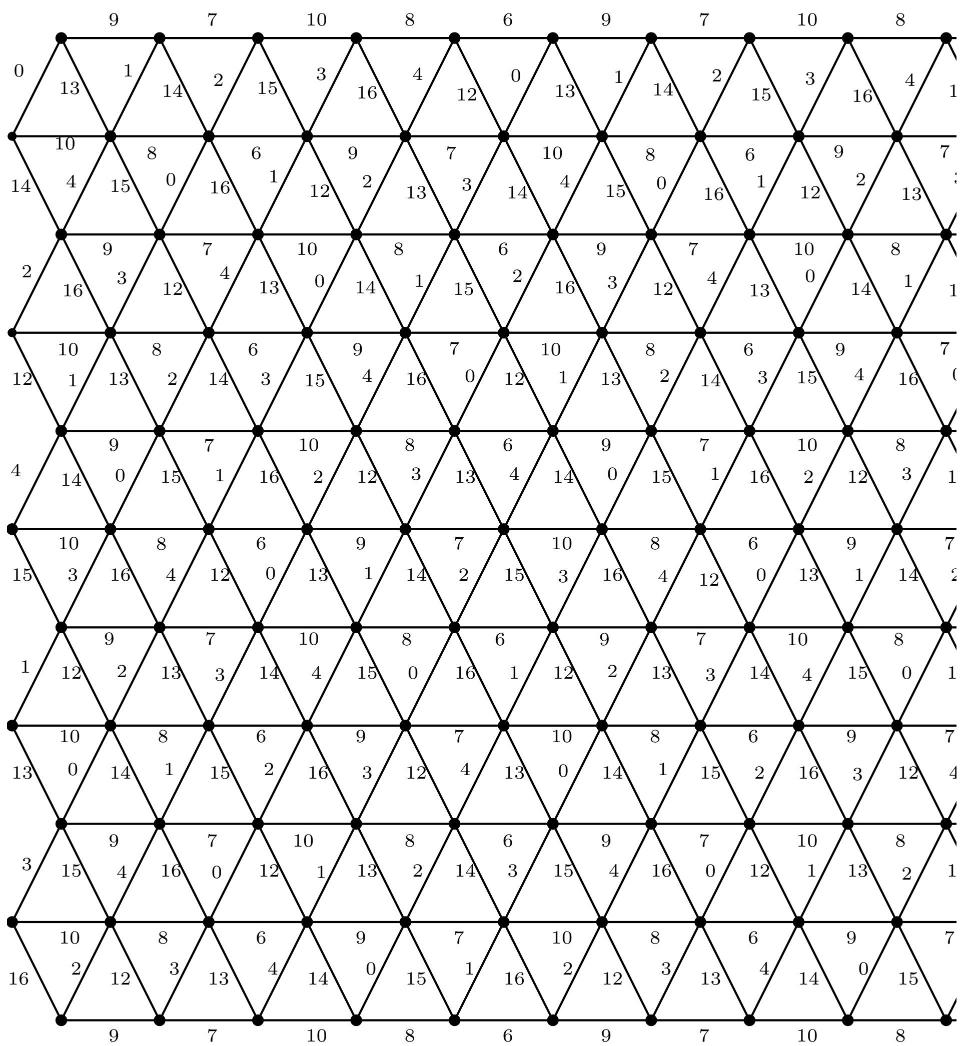


Figure 17: An optimal coloring of a fragment of a Triangular Grid

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