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A Handy Technique for Fundamental Unit in Specific Type of Real Quadratic Fields

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Abstract

Different types of number theories such as elementary number theory, algebraic number theory and computational number theory; algebra; cryptology; security and also other scientific fields like artificial intelligence use applications of quadratic fields. Quadratic fields can be separated into two parts such as imaginary quadratic fields and real quadratic fields. To work or determine the structure of real quadratic fields is more difficult than the imaginary one.
The Dirichlet class number formula is defined as a special case of a more general class number formula satisfying any types of number field. It includes regulator, $\mathscr{L}$-function, Dedekind zeta function and discriminant for the field. The Dirichlet's class number $h(d)$ formula in real quadratic fields claims that we have

$$h(d).log\varepsilon_d = \sqrt{\Delta}.\mathscr{L}(1,\ \chi_d)$$

for positive $d > 0$ and the fundamental unit ε_d of $\mathbb{Q}\left(\sqrt{d}\right)$. It is seen that discriminant, $\mathscr{L}$-function and fundamental unit ε_d are significant and necessary tools for determining the structure of real quadratic fields.
The focus of this paper is to determine structure of some special real quadratic fields for $d > 0$ and $d \equiv 2,3\ (mod4)$. In this paper, we provide a handy technique so as to calculate particular continued fraction expansion of integral basis element w_d, fundamental unit ε_d, and so on for such real quadratic number fields. In this paper, we get fascinating results in the development of real quadratic fields.

Keywords: Quadratic Fields, Continued Fraction Expansion, Fundamental Unit, Special Integer Sequences.
MSC codes: 11R11, 11A55, 11R27, 11k31.

1 Introduction and Preliminaries Section

Structure of $\mathbb{Q}\left(\sqrt{d}\right)$ real quadratic number fields depend on the $d > 0$ positive non-square integer. It means that we have two different structures whether $d \equiv 2,3\,(mod4)$ or $d \equiv 1\,(mod4)$.

In this brief paper, we focus on certain types of $\mathbb{Q}\left(\sqrt{d}\right)$ real quadratic fields for $d \equiv 2,3\,(mod4)$ positive non-square integers. We define an integer sequence and determine such fields from parameterization of positive

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non-square integers d by using defined integer sequence. These types of real quadratic number fields contain the special written continued fraction expansion of the integral basis element w_d such as $\left[\gamma_0; \overline{11,\ 11,\ \ldots, 11,\ 2\gamma_0}\right]$, where period length is represented by $l = l(d)$ of w_d.

We also demonstrate fundamental units of such fields in the case of $d \equiv 2,3\,(mod 4)$. All the results help us to get a practical and handy method to calculate continued fraction expansions and fundamental units.

We refer all references [1-21] to readers for more information on the structure of quadratic fields.

In this section, we give following basic notations to use in our Main Results section.

Definition 1.1. Let $\{\aleph_i\}$ be an integer sequence. It is defined by the recurrence relation

$$\aleph_i = 11\aleph_{i-1} + \aleph_{i-2}$$

with the seed values $\aleph_0 = 0$ and $\aleph_1 = 1$ for $i \geq 2$.

Lemma 1.1. Let d be a square-free positive integer such that $d \equiv 2,3\,(mod 4)$. If we put $w_d = \sqrt{d}$ and $\gamma_0 = \sqrt{d}$ into $w_R = \gamma_0 + w_d$, then we get $w_d \notin R(d)$ but $w_R \in R\ (d)$.

Furthermore, for the period $l = l(d)$ of w_R, we have continued fraction expansion of $w_R = \left[\overline{2\gamma_0, \gamma_1, \gamma_2, \ldots,\ \gamma_{l(d)-1}}\right]$ as pur-periodic and continued fraction expansion of w_d as periodic $w_d = \left[\gamma_0;\ \overline{\gamma_1, \gamma_2, \ldots,\ \gamma_{l(d)-1},\ 2\gamma_0}\right]$. Besides, let

$$w_R = \frac{w_R P_l + P_{l-1}}{w_R Q_l + Q_{l-1}} = \left[\ 2\gamma_0, \gamma_1, \gamma_2, \ldots,\ \gamma_{l(d)-1}, w_R\right]$$

be a modular automorphism of w_R. Then, the fundamental unit ε_d of $\left(\sqrt{d}\right)$ real quadratic number field is given by the following formula:

$$\varepsilon_d = \frac{t_d + u_d\sqrt{d}}{2} = \left(\gamma_0 + \sqrt{d}\right) Q_{l(d)} + Q_{l(d)-1}$$

$$t_d = 2\gamma_0 Q_{l(d)} + 2Q_{l(d)-1} \text{ and } u_d = 2Q_{l(d)}$$

where Q_i is determined by $Q_0 = 0$, $Q_1 = 1$ and Q_{i+1}=$\gamma_i Q_i + Q_{i-1}$ for $i \geq 1$.

Note. $I(d)$ is a set of all quadratic irrational numbers in $\left(\sqrt{d}\right)$. α in $I(d)$ is reduced if $\alpha > 1$ and $-1 < \alpha' < 0$ (α' is the conjugate of α). $R(d)$ is also a set of all reduced quadratic irrational numbers in $I(d)$. Besides, for any number α, $R(d)$ is purely periodic in the continued fraction expansion.

2 Main Results

In this section, we give two theorems and two corollaries, which carry out the main aim of the brief paper.

Theorem 2.1. Let d be a square-free positive integer and $\ell \geq 2$ be a positive integer such that it is not divided by three. Supposing that the parameterization of d is

$$d = \left(\frac{11 + (2J+1)\aleph_\ell}{2}\right)^2 + (2J+1)\aleph_{\ell-1} + 1$$

where $J \geq 0$ is a positive integer.

If $\ell \equiv 2,\ 4,\ 5\ (mod 6)$ and $J \geq 0$ is a even positive integer, then $d \equiv 2,3\ (mod 4)$. Besides, we obtain

$$w_d = \left[\frac{(2J+1)\aleph_\ell + 11}{2}; \overline{\underbrace{11, 11, \ldots, 11}_{\ell-1}, (2J+1)\aleph_\ell + 11}\right]$$

with $\ell = \ell(d)$ for $d \equiv 2,3\,(mod 4)$. Furthermore, we have the fundamental unit ε_d as follows:

$$\varepsilon_d = \left(\frac{11 + (2J+1)\aleph_\ell}{2}\aleph_\ell + \aleph_{\ell-1}\right) + \aleph_\ell\sqrt{d}.$$

Theorem 2.2. Assume that d is a square-free positive integer, period length is $\ell \geq 2$ and $3 \nmid \ell$. The parameterization of d is given by

$$d = \left(\frac{11 + (2J+1)\,\aleph_\ell}{2} \right)^2 + (2J+1)\,\aleph_{\ell-1} + 1$$

where $J \geq 0$ is a positive integer. If $\ell \equiv 1 \ (mod 6)$ and J is odd positive integer, then $d \equiv 2\,(mod 4)$. Also, we have same form of the w_d defined in Theorem 2.1, while $\ell = \ell(d)$ and $d \equiv 2\,(mod 4)$. In addition, we obtain the coefficients of fundamental unit t_d and u_d as follows:

$$t_d = (2+1)\,\aleph_\ell^2 + 11\,\aleph_\ell + 2\,\aleph_{\ell-1} \text{ and } u_d = 2\,\aleph_\ell.$$

Corollary 2.1. Let d be square-free positive integer and $\ell \geq 2$ be a positive integer satisfying the conditions of Theorem 2.1. Let the parameterization of d be defined as $d = \left(\frac{11+\aleph_\ell}{2}\right)^2 + \aleph_{\ell-1} + 1$. Then, we obtain $d \equiv 2, 3\,(mod 4)$, and the continued fraction expansion of w_d is given by

$$w_d = \left[\frac{11 + \aleph_\ell}{2}; \overline{\underbrace{11,\ 11,\ \ldots,\ 11}_{\ell-1}, 11 + \aleph_\ell} \right]$$

for $\ell = \ell(d)$. Also, we get the fundamental unit as $\varepsilon_d = \left(\frac{11+\aleph_\ell}{2} + \sqrt{d}\right)\aleph_\ell + \aleph_{\ell-1}$.

Proof. It is obtained by Theorem 2.1, if we chose $J = 0$. By the way, we prepare the following table, which includes some of infinite numerical examples considering condition $\ell \equiv 2,\ 4,\ 5 \ (mod 6)$ for Corollary 2.1.

Table 1 Numerical Results for Corollary 2.1.

d	$\ell(d)$	w_d	ε_d
123	2	$[11; \overline{11,\ 22}]$	$122 + 11\sqrt{123}$
465247	4	$[682; \overline{11, 11,\ 11,\ 1364}]$	$922868 + 1353\sqrt{465247}$
56371418	5	$[7508; \overline{11, 11,\ 11, 11,\ 15016}]$	$112658893 + 15005\sqrt{56371418}$
104722907208735	8	$[10233421; \overline{11, 11,\ \ldots, 11,\ 20466842}]$	$209445700004344 + 20466801\sqrt{104722907208735}$

Corollary 2.2. Let d be a square-free positive integer and $\ell > 1$ and $\ell \equiv 5\,(mod 6)$. Assume that the parameterization of d is given by

$$d = \left(\frac{11 + 3\,\aleph_\ell}{2} \right)^2 + 3\,\aleph_{\ell-1} + 1$$

Then, we obtain $d \equiv 2\,(mod 4)$ and

$$w_d = \left[\frac{11 + 3\,\aleph_\ell}{2}; \overline{\underbrace{11, 11, \ldots, 11}_{\ell-1}, 11 + 3\,\aleph_\ell} \right]$$

for $\ell = \ell(d)$. Also, we obtain following equation for the coefficient of fundamental units t_d andu_d.

$$t_d = 3\,\aleph_\ell^2 + 11\,\aleph_\ell + 2\,\aleph_{\ell-1} \text{ and } u_d = 2\,\aleph_\ell$$

Proof. We get this corollary from Theorem 2.2 if J is chosen as $J = 1$. Besides, we prepare the following table, which contains several of infinite numeric illustrations under the conditions of $\ell > 1$ and $\ell \equiv 1 \ (mod 6)$ for Corollary 2.2. We cannot write $\ell = 7$ in the table since $d = 763180879250$ has a square factor.

Remark. Readers can see that the real quadratic fields depend on two different parameters such as period length $\ell = \ell(d)$ and $J \geq 0$ integer. So, we can determine infinitely many real quadratic number fields with their structures if we change the values of these parameters.

Table 2 Numerical Results for Corollary 2.2.

d	$\ell(d)$	w_d	ε_d
26525631841181822843479178	13	$[5150304053275;\overline{11, 11, \ldots, 11, 10300608106550}]$	$17683754560768687845706259+$ $3433536035513\sqrt{26525631841181822843479178}$

3 Conclusion

In the topic of real quadratic fields, there are some tools such as fundamental unit and continued fraction expansion That are useful for determining structures of such fields. The main aim of this paper was to provide a practical method to calculate fundamental unit rapidly and simply for such real quadratic number fields. We are sure that this paper will be useful for readers.

References

[1] Robert B. Ash, Algebraic number theory, (Dover Books on Mathematics) , Dover Publications, 2010.
[2] David M. Burton, Elementary number theory, McGraw-Hill, 2002.
[3] K. Chakraborty, S. Kanemitsu and T. Kuzumaki, "On the class number formula of certain real quadratic fields", Hardy Ramanujan Journal., 2013, 36, 1-7.
[4] L. E. Clemens, K. D. Merill and D. W. Roeder, "Continues fractions and series", J. Number Theory., 1995, 54, 309-317.
[5] N. Elezovic, "A note on continued fractions of quadratic irrationals", Math. Commun., 1997, 2, 27-33.
[6] C. Friesen, "On continued fraction of given period", Proc. Amer. Math. Soc., 1988, 10, 9 – 14.
[7] F. Kawamoto and K. Tomita, "Continued fraction and certain real quadratic fields of minimal Type", J. Math. Soc., 2008, 60, 865 - 903.
[8] J. M. Kim and J. Ryu, "On the class number and fundamental unit of the quadratic field $k = \left(\sqrt{p.q}\right)$", Bull. Aus. Math. Soc, 2012, 85(3), 359-370.
[9] S. Louboutin, "Continued fraction and real quadratic fields", J. Number Theory., 1988, 30, 167- 176.
[10] R. A. Mollin, "Quadratics", 3^{rd} Edn, CRC Press, Boca Rato, FL, 1996.
[11] Richard A. Mollin - Algebraic Number Theory-CRC Press , 2011.
[12] M. Ram Murty, J. Esmonde, Problems in Algebraic Number Theory, (Graduate Texts in Mathematics 190), Springer-Verlag New York (2005)
[13] Ö. Özer, "On real quadratic number fields related with specific type of continued Fractions", J. Anal. Number Theory, 2016, 4(2), 85-90.
[14] Ö. Özer, "Notes on especial continued fraction expansions and real quadratic number fields, Kirklareli University Journal of Engineering and Science, 2016, 2(1), 74-89.
[15] Ö. Özer, "Fibonacci sequence and continued fraction expansion in real quadratic number fields, Malaysian Journal of Mathematical Science., 2016, 11(1),97-118.
[16] Ö. Özer, A.B.M. Salem, "A Computational Technique For Determining Fundamental Unit in Explicit Type of Real Quadratic Number Fields, International Journal of Advanced and Applied Sciences, 2017, Vol.4, Issue.2, 22-27.
[17] Ö. Özer, "A Study On The Fundamental Unit of Certain Real Quadratic Number Fields", Turkish Journal of Analysis and Number Theory (TJANT), Turkish Journal of Analysis and Number Theory, 2018, 6(1), 1-8.
[18] K. Tomita, "Explicit representation of fundamental units of some quadratic fields", Proc. Japan Acad., 1995, 71(2), Series A, 41-43.
[19] K. Tomita and K. Yamamuro, "Lower bounds for fundamental units of real quadratic Fields", Nagoya Math. J., 2002, 166, 29-37.
[20] H. Yokoi, "The fundamental unit and class number one problem of real quadratic fields with prime discriminant", Nagoya Math. J., 1990, 120, 51-59.
[21] H. Yokoi, "The fundamental unit and bounds for class numbers of real quadratic fields", Nagoya Math. J., 1991, 124, 181-197.