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DOES MACAULAY DURATION PROVIDE THE MOST COST-EFFECTIVE IMMUNIZATION METHOD – A THEORETICAL APPROACH

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Abstract: In the following, we offer a theoretical approach that attempts to explain (Comments 1-3) why and when the Macaulay duration concept happens to be a good approximation of a bond's price sensitivity. We are concerned with the basic immunization problem with a single liability to be discharged at a future time q. Our idea is to divide the class K of all shifts a(t) of a term structure of interest rates s(t) into many classes and then to find a sufficient and necessary condition a given bond portfolio, dependent on a class of shifts, must satisfy to secure immunization at time q against all shifts a(t) from that class. For this purpose, we introduce the notions of dedicated duration and dedicated convexity. For each class of shifts, we show how to choose from a bond market under consideration a portfolio with maximal dedicated convexity among all immunizing portfolios. We demonstrate that the portfolio yields the maximal unanticipated rate of return and appears to be uniquely determined as a barbell strategy (portfolio) built up with 2 zero-coupon bearing bonds with maximal and respective minimal dedicated durations. Finally, an open problem addressed to researchers performing empirical studies is formulated.

Keywords: barbell strategy, convexity, dedicated duration, Macaulay duration, unanticipated rate of return.

1 Introduction

Consider an investor who possessing C dollars today must achieve an investment goal of L dollars (L > C) q years from now by means of a purchase of appropriately selected bond portfolio (BP). If not successful, he/she will incur a penalty, while achieving more than L dollars will result in practically no rewards. Such investors are called bond immunizers. It is natural to assume that C is the present value of L dollars.

By the term structure of interest rates, one understands a schedule of spot interest rates s(t) which is estimated from the yields of all coupon-bearing bonds available on a given debt market M under consideration. The basic immunization problem (BIP) relies on a construction of such a bond portfolio BP with the present value of C dollars that the single liability to pay L dollars (L is the future value of C) q years from now will be discharged by means of the inflows c(t) generated by portfolio BP, no matter what shocks/shifts a(t) of s(t) will occur.

The new term structure is always of the form:

$$s^*(t) = s(t) + a(t) \tag{1}$$

with a(t) standing for a shift / shock of our term structure s(t), which satisfies Assumption 1 only. The function s(t) can exhibit various behaviors, for example, it can be flat, rising, declining, humped, or twisted. The classical results refer to flat shifts a(t) and flat term structures s(t), and they go back as far as to the pioneering work of Macaulay (1938), Redington (1952), and Fisher (1971).

In this paper, we approach the BIP by dividing the set K of all possibly shits a(t) into infinitely many classes K_v and then solve BIP for each of these classes separately, with durations D_v accordingly tailored to the specifics of the class K_v. Similar to Zheng in (2002) and (2007), we are not interested in building or borrowing from the literature a more or less accurate stochastic or deterministic model of the term structure s(t), as is the case in some publications, for example, Bansal and Zhou (2002) and Litterman (1991), simply because the specifics of term structure models will not play any role in our studies, as is stated in our Assumption 1.

Assumption 1

All shifts a(t), as well as all realizations of the governing term structure s(t), are continuous functions.

As of today, however, no one was successful in building up a bond portfolio BP, with the present value of C dollars, whose value at a future time q would never be less than the future value of C dollars (L dollars) at time q, no matter what shocks a(t) of the term structure s(t) will occur in the future.

As a matter of fact, it was demonstrated (Corollary 2) in a recent paper by Zaremba and Rządkowski (2016) that given an arbitrary bond portfolio BP, the set of all continuous shocks a(t) of any continuous term structure s(t), against which BP is immunized, is an (m − 1) dimensional linear subspace in the m-dimensional linear space K of all continuous shifts a(t), with m standing for the number of instances when BP promises to pay cash (coupons or par values)

Our goal in this paper is fourfold:

(i) to divide set K into infinitely many classes K_v;

(ii) for each class K_v (one of them will comprise all parallel shifts), to find a sufficient and necessary condition, a given bond portfolio BP must satisfy to secure immunization at time q against all shifts a(t) from class K_v; from now on, the set of these immunizing portfolios BP_v against shocks a(t) from class K_v will be denoted by Φ_v;

(iii) to identify a bond portfolio $BP_v * \in \Phi_v$ with the highest convexity; and

(iv) finally, to find a bond portfolio with maximal unanticipated rate of return among all $BP_v \in \Phi_v$.

We will shortly see that set Φ_v of all immunizing bond portfolios against shifts a(t) from class K_v consists of portfolios such as $BP_v = (w_1, w_2, \ldots, w_m)$ whose weights w_i satisfy Equation (2) and (2a).

The opposite (in some sense) problem, although closely related, was tackled in two recent papers by Rzadkowski and Zaremba (2010; 2016) where the main task was to characterize all shifts a(t) that satisfy Equation (2) for a given specific bond portfolio

$$BP = (w_1, w_2, w_3, \ldots, w_m).$$

For this purpose, in 2010, a Hilbert space approach was used with orthonormal polynomials playing the major role, while in 2015, the key role was played by the so-called triangular functions.

As far as goals (iii) and (iv) are concerned, similar problems were already investigated in Zaremba (1998) for proportional shifts explored earlier by Elton and Gruber (1995) and next in Zaremba and Smolenski (2000) for the so-called generalized proportional shifts. However, in these two papers, rates of return were compounded in a discrete manner, while the continuous compounding framework is used in this paper.

2 Initial considerations

The inspiration as how to divide set K of all shifts a(t) of the term structure s(t) into infinitely many classes K_v will come soon from Equation (2).

The resulting division into classes K_v is presented in Section 3.

Before we arrive at Equation (2), let us remind ourselves that all bond portfolios BP_v that we will be dealing with are constructed from debt instruments available on a given financial market M. Let t_0 stands for the very moment when an investor bought BP_v, while

$$t_1, t_2, t_3, \ldots, t_m = T$$

comprise all instances from interval $[t_0; T]$ representing the life span (expressed in years) of portfolio BP_v, when either BP_v generates payments c_i at t_i, $1 \le i \le m$ (in the form of coupons or par values) or the owner of BP_v expects to be required to pay his/her liabilities at some time q. In other words, we assume that $q = t_n$ for some n, $1 \le n \le m$.

We start by invoking Theorem 1 from Rzadkowski and Zaremba (2000), which can be formulated as the following.

Fact 1

Let q denotes a future date when a single liability of L dollars has to be discharged by means of the cumulated value of the inflows generated by some bond portfolio BP whose present value equals the present value of L. In addition, let Assumption 1 holds.

Then the payment of L dollars at time q will be guaranteed (immunization will be secured), provided the following necessary and sufficient condition, having nothing to do with the kind of dynamics of the continuous term structure $s(t)$ of interest rates, but referring solely to its continuous shifts $a(t)$ holds:

$$a(q)q = \sum_{i=1}^{i=m} t_i \cdot w_i \cdot a(t_i) \tag{2}$$

with

$$w_k = \frac{c_k \cdot \exp(-s(t_k) \cdot t_k)}{\sum_{i=1}^{i=m} c_i \cdot \exp(-(t_i) \cdot t_i)} \tag{2a}$$

standing for the weight of the cash c_k, $1 \le k \le m$, payable by BP at time t_k.

Looking at (2), one sees that what really matters in Equation (2) are the values of $a(t)$ at instances t_1, t_2, t_3, …, t_m only, that is, $a(t_i)$, including $a(q)$. Let us separately consider 2 cases:

(a) $a(q) = 0$

(b) $a(q) \neq 0$ (3)

In case of scenario (a), when the new interest rate s*(q) at time q remains the same as it was before the shift a(t) has occurred, that is

s*(q) = s(q) + a(q) = s(q),

the right-hand side of (2) must be equal to 0.

Fact 2a

Assume a shift a*(t) satisfies two conditions:

(i) $a^*(q) = 0$ and

(ii) for all instances t_1, t_2, t_3, …, t_m, the values $a^*(t_i)$ are nonnegative numbers or all $a^*(t_i)$ are nonpositive numbers except for $q = t_n$.

Then BP = ($w_1, w_2, \ldots, w_m$) is an immunizing bond portfolio against shift a*(t) if and only if the equalities (4) hold, where

$$w_i \cdot a^*(t_i) = 0, \; 1 \le i \le m \tag{4}$$

Proof. Observe that all t_i are positive numbers, w_k are nonnegative, and consequently, the right-hand side of Equation (2) must be either

- nonnegative (whenever $a^*(t_i)$ are nonnegative) or
- nonpositive (whenever $a^*(t_i)$ are nonpositive).

As the right-hand side of Equation (2) equals 0, the condition (4) follows.

Fact 2b

Assume that a shift a*(t) satisfies two conditions:

(i) $a^*(q) = 0$ and

(ii) for all instances t_1, t_2, t_3, …, t_m, the values $a^*(t_i)$ are negative numbers or all $a^*(t_i)$ are positive numbers except for $q = t_n$.

Then the only immunizing bond portfolio against shift a*(t) is the one (if it exists on the debt market M), say B*, that matures at time q and reduces to a zero-coupon bond. If, however, such B* is not tradable on bond market M, then there is no immunizing bond portfolio against a*(t).

Proof. Observe that by virtue of Fact 2a, $w_i \cdot a^*(t_i) = 0$.

If a zero-coupon bond B* maturing at time $q = t_n$ is tradable on market M, then condition (4) is satisfied with $w_n = 1, w_i = 0$, $i \neq n$. But, if there is no such bond B* on market M, then each other immunizing bond B, if existed, would generate at least one payment at some instance $t_i \neq q$. This, however, would mean $w_i \neq 0$, and by virtue of (ii) would lead to the inequality $w_i \cdot a^*(t_i) \neq 0$ that cannot hold because we have already demonstrated that $w_i \cdot a^*(t_i) = 0$.

The proof is completed.

3 Dedicated for class K_v duration D_v (when $a(q) \neq 0$)

In what follows we will be dealing with the most general scenario (b) when $a(q) \neq 0$. In such a case, we can rewrite Equation (2) into an equivalent form

$$q = \sum_{i=1}^{i=m} t_i \cdot w_i \cdot \frac{a(t_i)}{a(q)} \tag{5}$$

For each vector

$$v = (v_1, v_2, ..., v_m) \in R^m$$

we will define the class K_v of those shifts a(t) of the term structure s(t) for which

$$v_i = \frac{a(t_i)}{a(q)} \tag{6}$$

For example, when $v = (1, 1, \ldots, 1)$, then the corresponding class K_v comprises all parallel shifts because each parallel shift a(t), being a constant function, satisfies the condition

$$1 = v_i = \frac{a(t_i)}{a(q)} \tag{7}$$

Clearly, in case (7), our necessary and sufficient immunization condition (5) reduces to the very well-known fact stating that

$$q = D, \text{ with } D = \sum_{i=1}^{i=m} t_i \cdot w_i$$

being the classical duration. (8)

For each vector

$$v = (v_1, v_2, ..., v_m) \in R^m$$

the natural question arises how to solve Equation (5) with unknown variables w_i that depend on decision variables c_i shown in Equation (2) or, equivalently, how to solve the equation

$$q = \sum_{i=1}^{i=m} t_i \cdot w_i \cdot v_i \tag{9a}$$

with the known parameters t_i, v_i. Clearly, condition (9a) can be equivalently rewritten as q = D_v with:

$$D_v = \sum_{i=1}^{i=m} t_i \cdot w_i \cdot v_i \tag{9b}$$

$$v = (v_1, v_2, ..., v_m) \in R^m, \tag{9b}$$

so that Fact 1 can be reformulated as follows.

Fact 3

Let q denotes a future date when a single liability of L dollars has to be discharged by means of the cumulated value of the inflows generated by bond portfolio BP. In addition, let Assumption 1 holds.

Then, assuming that all admissible shifts a(t) belong to class K_v, the payment of L dollars at time q will be secured provided the following necessary and sufficient condition holds:

$$D_v(BP) = \sum_{i=1}^{i=m} t_i \cdot w_i \cdot v_i = q \text{ with } v_i = \frac{a(t_i)}{a(q)},$$

$$v = (v_1, v_2, ..., v_m), \tag{10}$$

and w_k standing the weight of the cash c_k, $1 \leq k \leq m$, payable by BP at time t_k.

Definition 1

Let

$$BP = (w_1, w_2, w_3, \ldots, w_m)$$

be the bond portfolio with w_i standing for the weight of the cash c_i payable by BP at time t_i, and let:

$$v = (v_1, v_2, ..., v_m) \in R^m$$

be an arbitrary m-dimensional vector. Then the number D_v defined by Equation (9b) is said to be the duration dedicated for class K_v (K_v consists of all shifts a(t) satisfying

$$v_i = \frac{a(t_i)}{a(q)}$$).

It immediately follows from this definition that a zero-coupon bearing bond maturing at time t_i has dedicated duration $D_v = t_i \cdot v_i$ depending on class

$$K_v, \ v = (v_1, v_2, ..., v_m) \in R^m.$$

Formula (9b) occurring in the definition of dedicated duration can thus be rephrased as follows.

If we divide bond portfolio BP into real or imaginary zero-coupon bearing bonds

B_i, $1 \le i \le m$

with par values of B_i being equal to the cash c_i payable by BP at instant t_i, then the dedicated duration of BP, D_v(BP), is a weighted average of dedicated durations of B_i, denoted by $D_v(B_i)$, that is:

$$D_v(BP) = \sum_{i=1}^{i=m} t_i \cdot w_i \cdot v_i = \sum_{i=1}^{i=m} w_i \; D_v(B_i) \qquad (11a)$$

Remark 1

To avoid in future the possible misunderstandings, let us clarify what we shall mean in this paper by writing "a bond portfolio BP consists of n different bonds B_k". By writing this, we are trying to say that BP is built up with n (say three) types of bonds, for example, BP consists of 210 bonds of type B_1, 440 bonds of type B_2, and 170 bonds of type B_3. In other words, we treat all copies of the same type of bond as just 1 bond (1 cash flow) and denote it by B_k depending on the type of a bond.

Fact 4 tells us that Formula (11a) can be further generalized.

Fact 4

Let a bond portfolio BP consists of n different (not necessarily zero-coupon) bonds B_k with dedicated durations

$D_v(B_k)$, $1 \le k \le n$.

Then the dedicated for class K_v duration of BP, D_v(BP), is given by the formula

$$D_v(BP) = \sum_{k=1}^{k=n} w_k \cdot D_v(B_k) \qquad (11b)$$

where each w_k, $1 \le k \le n$, represents the proportion of money spent on the purchase of bond B_k, expressed as a fraction of the total cost incurred for the purchase of BP.

Proof. Suppose, for the sake of simplicity of presentation, that a bond portfolio BP is built of just two bonds, say B^1 and B^2, whose dedicated for class K_v durations $D_v{}^1$ and $D_v{}^2$, respectively, satisfy the relationships, by Definition 1,

$$D_v{}^1 = \sum_{i=1}^{m} t_i \cdot \frac{c_i{}^1 \cdot \exp(-s(t_i)\cdot t_i)}{PV(B^1)} \cdot v_i\,,$$

$$D_v{}^2 = \sum_{i=1}^{m} t_i \cdot \frac{c_i{}^2 \cdot \exp(-s(t_i)\cdot t_i)}{PV(B^2)} \cdot v_i \qquad (12)$$

with $c_i{}^1 + c_i{}^2 = c_i$, where c_i is the cash paid by BP at time t_i.

Therefore,

$$D_v(BP) =$$
$$\sum_{i=1}^{m} t_i \cdot \frac{(c_i{}^1 + c_i{}^2)\exp - s(t_i)\cdot t_i}{PV(BP)} \cdot v_i =$$
$$\frac{PV(B^1)}{PV(PB)} \cdot \sum_{i=1}^{m} t_i \cdot \frac{c_i{}^1 \exp - s(t_i)\cdot t_i}{PV(B^1)} \cdot v_i +$$
$$\frac{PV(B^2)}{PV(PB)} \cdot \sum_{i=1}^{m} t_i \cdot \frac{c_i{}^2 \exp - s(t_i)\cdot t_i}{PV(B^2)} \cdot v_i =$$
$$w_1 \cdot D_v{}^1 + w_2 \cdot D_v{}^2,\ w_i = \frac{PV(B^k)}{PV(PB)},\ 1 \le k \le 2.$$

Let us now comment a fairly widespread opinion/hypothesis among the researchers that "the simplest Macaulay duration provides the most cost-effective immunization method."

Comment 1

Looking at the definition of dedicated duration D_v for class K_v, cf. Equation (9b), one sees that the classical Macaulay duration measure, being identical to our dedicated duration D_v, with

$v = (1, 1, \ldots, 1)$

is a natural, possible, and the most likely candidate for a one-number sensitivity indicator of a bond's price applicable to all nonparallel shifts of term structure s(t).

To formulate such an opinion/hypothesis, one may not need to do any empirical research of the debt market under consideration.

However, it seems reasonable and desirable to perform empirical research referring to the computation of vectors

$$v = (v_1, v_2, ..., v_m) \in R^m \text{ with } v_i = \frac{a(t_i)}{a(q)}$$

based on historical data. As a result of such research, it might, for example, happen that on some real debt markets, for example, in Japan, Kazachstan, and Poland, the Macaulay duration would occur not to be "the most cost-effective immunization method."

However, we will prove in Theorem 1 that for those bond immunizers who maximize the unanticipating rate of return, the knowledge of all coordinates of the vector $v = (v_1, v_2, ..., v_m)$ is not really needed (see, Comment 2). To prove Theorem 1 in Section 5, we will need the following assumption.

Assumption 2

All zero-coupon bearing bonds B_i maturing at time t_i, $1 \le i \le m$, have mutually different dedicated durations, that is, $D_v(B_i) \ne D_v(B_j)$ if and only if $i \ne j$ which is the same as to say that

$$t_i \cdot v_i \ne t_j \cdot v_j \leftrightarrow i \ne j.$$

4 Dedicated convexity

When the term structure of interest rates is a function s(t), then the present value of an arbitrary bond portfolio BP generating cash flow c_t is given by the formula

$$PV[s(\cdot)] = \sum_t c_t \cdot \exp[-s(t) \cdot t] \tag{13}$$

with index *t* varying over all instances t_1, t_2, t_3, …, t_m when portfolio BP is paying cash. Suppose that immediately after the acquisition of BP, the term structure s(t) shifted to its new level s*(t) = s(t) + a(t) with a continuous shift a(t) according to Assumption 1.

Then, the present value of BP will also shift to its new value, namely,

$$PV[s(\cdot) + a(\cdot)] = \sum_t c_t \cdot \exp[-s(t) - a(t)] \cdot t \tag{14}$$

where, as previously, t varies over all instances

t_1, t_2, t_3, …, t_m.

In order to explain formula (16), let us note that if

$h(s) = \exp[-s \cdot t]$,

then:

$$h'(s) = -t \cdot \exp[-s \cdot t]$$

$$h''(s) = t^2 \cdot \exp[-s \cdot t] \tag{15a}$$

and, consequently, by the Taylor approximation formula, one has

$$h(s+a) - h(s) \approx -t \cdot \exp[-s \cdot t] \cdot a + \frac{1}{2} t^2 \exp[-s \cdot t] \cdot a^2 + O(a) \cdot a^2, \tag{15b}$$

with

$$\lim O(a) = 0 \text{ when } a \to 0. \tag{15c}$$

Generalizing this line of reasoning, we can estimate the unanticipated change in the value of the bond portfolio BP as follows (the term $\sum_{k=1}^{m} O[a(t_k)] \cdot a(t_k)^2$ was omitted):

$$PV[s(\cdot) + a(\cdot)] - PV[s(\cdot)] \approx \sum_{k=1}^{m} \{-t_k c_k [\exp - s(t_k) t_k] \cdot a(t_k) + \frac{1}{2}[t_k{}^2 c_k \exp[-s(t_k) t_k] \cdot a^2(t_k)\} \tag{16}$$

Dividing both sides by the present value of BP, that is, $PV[s(\cdot)]$, we obtain the following relationship for the unanticipating rate of return (occurring on the left-hand side of the following equality):

$$\frac{PV[s(\cdot) + a(\cdot)] - PV(s(\cdot)]}{PV(s(\cdot)]} = \sum_{k=1}^{m} [-t_k w_k a(t_k)/a(q)] a(q\quad) + \frac{1}{2} \sum_{k=1}^{m} \{t^2 w_k a^2(t_k) + O[a(t_k)] a(t_k)^2\} \tag{17a}$$

Making use of the notion of dedicated duration D_v for class K_v of those shifts a(t) of the term structure of interest rates s(t) for which

$$v_i = \frac{a(t_i)}{a(q)},\ v = (v_1, v_2, ..., v_m),$$

one arrives at

$$\frac{PV[s(\cdot)+a(\cdot)] - PV(s(\cdot)]}{PV(s(\cdot)]} = -D_v(BP)\cdot a(q) +$$

$$\tfrac{1}{2}\sum_{k=1}^{m}\{t^2 w_k a^2(t_k) + O[a(t_k)]a(t_k)^2\}. \qquad (17b)$$

Definition 2

Let

$$BP = (w_1, w_2, \dots, w_m)$$

be a bond portfolio BP with weights w_i, and let

$$v = (v_1, v_2, ..., v_m)$$

be an arbitrary m-dimensional vector.

The number

$$C_v(BP) = \tfrac{1}{2}\sum_{k=1}^{m} t_k^{\,2}\cdot w_k \cdot v_k^{\,2} \qquad (18)$$

is said to be dedicated for class K_v convexity of portfolio BP, having in mind that K_v is a class of shifts a(t) satisfying $v_i = \frac{a(t_i)}{a(q)}$. It immediately follows from Definition 2 that the dedicated for class K_v convexity C_v of a zero-coupon bond maturing at time t_i is given by the formula $C_v = \tfrac{1}{2}\cdot t_i^{\,2}\cdot v_i^{\,2}$.

Formula (18) occurring in the definition of convexity can thus be read as follows: If we divide bond portfolio BP into zero-coupon bearing bonds B_i, $1 \le i \le m$, with par values of B_i being equal to the cash c_i paid by BP at instant t_i, then the dedicated convexity of BP, C_v(BP), is a weighted average of dedicated convexities of B_i, denoted by $C_v(B_i)$, that is

$$C_v(BP) = \sum_{k=1}^{k=m} w_k \cdot C_v(B_k). \qquad (19)$$

Fact 5 tells us that formula (19) can be further generalized (see Remark 1).

Fact 5

Let a bond portfolio BP consists (is built up) of n different (not necessarily zero-coupon) bonds B_k, $1 \le k \le n$, with convexities $C_v(B_k)$. Then the convexity of BP is a weighted average of convexities $C_v(B_k)$ given by formula(19), with w_k representing the proportion of money spent on the purchase of bond B_k, expressed as a fraction of the total cost incurred for the purchase of BP.

Proof. Suppose, for the sake of simplicity of presentation, that a bond portfolio BP is built of bonds B^1 and B^2, whose convexities $C_v^{\,1}$ and $C_v^{\,2}$ satisfy by Definition 2 the relationships

$$C_v^{\,1} = \sum_{i=1}^{m} t_i^{\,2}\cdot\frac{c_i^{\,1}\exp(-s(t_i)\cdot t_i)}{PV(B^1)}\cdot v_i^{\,2},$$

$$C_v^{\,2} = \sum_{i=1}^{m} t_i^{\,2}\cdot\frac{c_i^{\,2}\exp(-s(t_i)\cdot t_i)}{PV(B^2)}\cdot v_i^{\,2}, \qquad (20)$$

with $c_i^{\,1} + c_i^{\,2} = c_i$, where c_i is the cash paid by BP at time t_i. Therefore,

$$C_v(BP) = \sum_{i=1}^{m} t_i^{\,2}\cdot\frac{(c_i^{\,1}+c_i^{\,2})\exp(-s(t_i)\cdot t_i)}{PV(BP)}\cdot v_i^{\,2} =$$

$$\frac{PV(B^1)}{PV(PB)}\cdot\sum_{i=1}^{m} t_i^{\,2}\cdot\frac{c_i^{\,1}\exp(-s(t_i)\cdot t_i)}{PV(B^1)}\cdot v_i^{\,2} +$$

$$\frac{PV(B^2)}{PV(PB)}\cdot\sum_{i=1}^{m} t_i^{\,2}\cdot\frac{c_i^{\,2}\exp(-s(t_i)\cdot t_i)}{PV(B^2)}\cdot v_i^{\,2} =$$

$$w_1\cdot C_v^{\,1} + w_2\cdot C_v^{\,2},$$

$$w_k = \frac{PV(B^k)}{PV(PB)},\ 1 \le i \le 2.$$

We can now rephrase formula (17b) into a more convenient (handy) manner as follows.

Fact 6

Suppose a bond portfolio BP had been purchased and then the term structure s(t) of interest rates switched to its new level s*(t) = s(t) + a(t).

Then the unanticipated rate of return resulting from the purchase of BP satisfies the relationship

$$\frac{PV[s(\cdot)+a(\cdot)]-PV(s(\cdot)]}{PV(s(\cdot)]}$$
$$=-D_v(BP)\cdot a(q)+C_v(BP)\cdot a^2(q)+$$
$$\sum_{k=1}^{m} O[a(t_k)]\cdot a(t_k)^2 \qquad (21)$$

with O(a) satisfying (15c). It's worth to notice that $a^2(t_k)$ are rather small factors because the absolute values of shifts a(t) are often smaller than 1% = 0.01, and pretty often they are even close to 0.001.

If we apply Equation (21) to a zero-coupon bearing bond maturing at time t_i, say B_i, we will obtain that the unanticipated rate of return on

$$B_i=-t_i v_i a(q)+\tfrac{1}{2}\cdot t_i^2 v_i^2 a^2(q)+O[a(t_i)]\cdot a(t_i)^2$$

provided instantly after the acquisition of bond B_i, the term structure s(t) shifted to its new level

$$s^*(t) = s(t)+a(t), \text{ with } v_i=\frac{a(t_i)}{a(q)}.$$

5 Maximizing the unanticipating rate of return

In this section, we approach the problems formulated in (iii) and (iv), namely, we want to characterize/identify the bond portfolios $\overline{BP}\in\Phi_v$ that yield the maximal unanticipating rate of return among all portfolios $BP\in\Phi_v$ that, by definition, are immunized against shifts a(t) from class K_v.

As we have already derived a formula for the unanticipating rate of return in the form of relationship (21) and we know that the dedicated duration $D_v(BP)$ = q, all we need to do to solve (iv) is to solve (iii), that is, we should maximize the dedicated convexity (18).

The latter can be done by optimally choosing weights w_k given in Equation (2a) through the appropriate choice of the cash flow $\{\bar{c}_k\}$, $1\le k\le m$, representing our decision variables.

Following Elton and Gruber (1995, p.552), a bond portfolio BP is said to be a barbell strategy (barbell portfolio) if it is built up of just two bonds, say B^1 and B^2, with significantly different dedicated durations $D_v^{\,1}$ and $D_v^{\,2}$, respectively.

On the other hand, BP is said to be a focused strategy (focused portfolio) if it consists of several bonds whose dedicated durations $D_v^{\,j}$ are centered around duration of the liability (q, in this context).

Theorem 1

Suppose Assumptions 1 and 2 hold.

- Let $BP_v^*\in\Phi_v$ be a bond portfolio with the highest dedicated convexity (18) in the class Φ_v of all bond portfolios immunized against shifts a(t) from set K_v.
- Let B^k, $1\le k\le n$, denote those bonds tradable on the debt market M under consideration from which BP_v^* is built up.

Then BP_v^* is uniquely determined as the barbell strategy built up of two specific zero-coupon bearing bonds, namely, B_s with the minimal and B_l (maximal) dedicated durations, respectively.

The weights $\bar{x}_s$ and $\bar{x}_l$ associated with the payments of BP_v^* at instances t_s and t_l are given by formula (27); they represent the proportions of money spent on the purchase of bonds B_s and B_l, expressed as a fraction of the total cost incurred for the purchase of $\overline{BP}$.

Proof. Although BP_v^* has been created by purchasing n bonds B^k, $1\le k\le n$, tradable on the debt market M, we will look at BP_v^* from a different prospective. Namely, we will say that BP_v^* consists of m zero-coupon bonds B_i, each of them maturing at time t_i and generating the same payment of $\overline{c}_i$ dollars as BP_v* is promising to pay ($1 \le i \le m$).

By virtue of formulas (9b) and (18), the maximization of dedicated convexity $C_v(BP)$ leads to the following optimization problem, with decision variables x_i, $1 \le i \le m$, representing the proportion of money spent on the purchase of bond B_i, expressed as a fraction of the total cost incurred for the purchase of BP:

$$\max \tfrac{1}{2}\sum_{i=1}^{m} t_i^{\,2} \cdot x_i \cdot v_i^{\,2} : \sum_{i=1}^{m} x_i = 1; \sum_{i=1}^{m} t_i \cdot x_i \cdot v_i = q$$

$$x_i \ge 0 \tag{22}$$

As all functions occurring in Equation (22) are linear in x_i, the Kuhn–Tucker conditions

$$0 = \frac{\partial L}{\partial x_i}(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_m) =$$

$$\tfrac{1}{2} t_i^{\,2} \cdot v_i^{\,2} + \mu_1 + \mu_2 \cdot t_i \cdot v_i + \lambda_i$$

$$1 \le i \le m \tag{23a}$$

$$\overline{x}_i \cdot \lambda_i = 0, \lambda_i \ge 0, \overline{x}_i \ge 0, 1 \le i \le m \tag{23b}$$

with $(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_m)$ representing the weights of BP_v* are necessary and sufficient for optimality of BP_v*, with the Lagrangian

$$L(x_1, x_2, \ldots, x_m) =$$

$$\tfrac{1}{2}\sum_{i=1}^{m}(t_i^{\,2} \cdot x_i \cdot v_i^{\,2} + \mu_1 x_i + \mu_2 \cdot t_i \cdot x_i \cdot v_i + \lambda_i \cdot x_i)$$

$$1 \le i \le m. \tag{24}$$

We start by proving that all but two weights $\overline{x}_i$ are equal to 0. Suppose on the contrary that

$$\overline{x}_i \ne 0,\ \overline{x}_j \ne 0,\ \overline{x}_s \ne 0. \tag{25}$$

Then by virtue of Equation (23b), we would then infer that

$$\lambda_i = 0,\ \lambda_j = 0,\ \lambda_s = 0,$$

and then, by invoking Equation (23a), we would get that the quadratic function

$$f(t \cdot v) = \tfrac{1}{2}(t \cdot v)^2 + \mu_1 + \mu_2 \cdot (t \cdot v)$$

would be equal to 0 for three distinct (by virtue of Assumption 2) roots $t_i v_i$, $t_j v_j$, and $t_s v_s$, what is impossible because each polynomial of degree 2 has no more than 2 roots. Thus let $\overline{x}_l \ne 0$ and $\overline{x}_s \ne 0$ be the two only weights that are different (greater than 0), meaning that BP_v* generates cash in just two instances, t_l and t_s. From Equation (22), we know that

$$t_l \cdot \overline{x}_l \cdot v_l + t_s \cdot \overline{x}_s \cdot v_s = q;$$

$$\overline{x}_l + \overline{x}_s = 1; \quad \overline{x}_l > 0; \quad \overline{x}_s > 0. \tag{26}$$

This is a system of two equations with two unknown variables $\overline{x}_l$ and $\overline{x}_s$. Solving it, we obtain the formulas (27) for $\overline{x}_l$ and $\overline{x}_s$ (without loss of generality one may assume that

$t_l \cdot v_l > q > t_s \cdot v_s$)

$$\overline{x}_s = \frac{t_l v_l - q}{t_l v_l - t_s v_s},\ \overline{x}_l = \frac{q - t_s v_s}{t_l v_l - t_s v_s},$$

with the remaining variables

$$\overline{x}_i = 0. \tag{27}$$

We will demonstrate that bonds B_s and B_l have the shortest and resp. longest dedicated durations and that BP_v* is uniquely determined. In fact, it follows from Equation (23a) that:

$$0 = \frac{\partial L}{\partial x_s}(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_m) \text{ and}$$

$$0 = \frac{\partial L}{\partial x_l}(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_m)\text{, that is,}$$

$$0 = \tfrac{1}{2} t_s^{\,2} \cdot v_s^{\,2} + \mu_1 + \mu_2 \cdot t_s \cdot v_s =$$

$$\tfrac{1}{2} t_l^{\,2} \cdot v_l^{\,2} + \mu_1 + \mu_2 \cdot t_l \cdot v_i \tag{28}$$

because

$$\lambda_s = \lambda_l = 0$$

by virtue of

$$\overline{x}_s \cdot \lambda_s = 0 = \overline{x}_l \cdot \lambda_l.$$

From Equation (28), we immediately derive two formulas for μ_1, namely,

$$\mu_1 = -\mu_2 \cdot t_s \cdot v_s - \tfrac{1}{2} t_s^{\,2} \cdot v_s^{\,2} \text{ and}$$

$$\mu_1 = -\mu_2 \cdot t_l \cdot v_l - \tfrac{1}{2} t_l^{\,2} \cdot v_l^{\,2}, \tag{29}$$

the latter resulting in

$$\mu_2 = -\tfrac{1}{2} \cdot (t_l \cdot v_l + t_s \cdot v_s). \tag{30}$$

Having proved Equation (30), we infer from Equation (29) that

$$\mu_1 = \tfrac{1}{2} t_l \cdot v_l \cdot t_s \cdot v_s, \tag{31}$$

and then substitute Equations (30) and (31) into Equation (23a) to obtain the relationship

$$0 = \tfrac{1}{2} t_i^{\,2} \cdot v_i^{\,2} + (\tfrac{1}{2} t_l \cdot v_l \cdot t_s \cdot v_s) -$$
$$\tfrac{1}{2}(t_l \cdot v_l + t_s \cdot v_s) \cdot t_i \cdot v_i + \lambda_i \tag{32}$$

from which the following formula

$$\lambda_i = \tfrac{1}{2}(t_l \cdot v_l - t_i \cdot v_i) \cdot (t_i \cdot v_i - t_s \cdot v_s) \tag{33}$$

can be deduced by performing elementary transformations.

In order to have the necessary condition $\lambda_i \geq 0$ satisfied, we must choose $t_l \cdot v_l$ to be the largest number and $t_s \cdot v_s$ to be the smallest number among all $t_i \cdot v_i$.

Such a choice is the same as picking from the market M two zero-coupon bonds, one B_l with the maximal dedicated duration $t_l \cdot v_l$, and the other bond B_s with the smallest dedicated duration $t_s \cdot v_s$

With such a choice of dedicated durations, all three necessary and sufficient conditions, namely,

$$\overline{x}_i \cdot \lambda_i = 0,\ \lambda_i \geq 0,\ \overline{x}_i \geq 0,\ 1 \leq i \leq m,$$

are now satisfied by the barbell bond portfolio $BP_v{*}$, what is clearly seen from Equations (27) and (33).

In this way, we have uniquely identified via formula (27) the weights of the bond portfolio BP with the maximal dedicated convexity.

In order to claim the uniqueness of $BP_v{*}$, it is enough to invoke the fact that the present value of $BP_v{*}$ equals the present value of liability L to be paid off at the specified time

$$q,\ t_1 \leq q \leq t_m.$$

6 Concluding remarks

Comment 2

It follows from Theorem 1 that what really matters, from the point of view of bond immunizers, are not all coordinates of the vector

$$v = (v_1, v_2, \ldots, v_m) \in R^m$$

determine the class K_v of shifts a(t), but two of them, namely, v_s and v_l for which $t_s \cdot v_s$ and $t_i \cdot v_l$ stand for the minimal and resp. maximal dedicated durations, respectively, among all durations

$$D_v = t_i \cdot v_i,\ 1 \leq i \leq m,$$

if bond immunizers are choosing portfolio $BP_v{*}$ specified in Theorem 1.

However, it seems likely to this author that many bond managers do not immunize their liabilities by means of portfolio $BP_v{*}$, but making use of other immunizing barbell portfolios BP_v. Therefore, there is no point that they will be much interested in estimating all coordinates of the vector

$$v = (v_1, v_2, \ldots, v_m)$$

if only two of them really matter.

Now we begin to realize that the answer to the question raised in the title of this paper depends on the choices made by bond immunizers. If they immunize their liabilities by choosing barbell portfolios paying cash at instances t_i and t_j such that

$$\text{(a) } v_i < 1,\ v_j < 1 \text{ or (b) } v_j > 1,\ v_j > 1 \tag{34}$$

then such bond immunizers diminish the chances that Macaulay duration

$$D = \sum_{i=1}^{i=m} t_i w_i$$

reducing in case of barbell portfolios to the formula

$$D = t_i w_i + t_j w_j$$

accurately approximates the dedicated duration

$$D_v = t_i w_i v_i + t_j w_j v_j .$$

But if they select such barbell portfolios BP_v for which $v_i \cdot v_j = 1$ or the product $v_i \cdot v_j$ is close to 1 ($v_i \cdot v_j \approx 1$), then the dedicated duration and Macaulay duration become closer to each other.

Comment 3

The natural question arises how much (to what extend) the weights given by Equation (27), with $\bar{x}_s$, $\bar{x}_l$ meaning the same as w_s and w_l above, differ from the weights

$$x_s = \frac{t_l - q}{t_l - t_s} \quad \text{and} \quad x_l = \frac{q - t_s}{t_l - t_s}$$

derived from the Macaulay duration when v = (1, 1, …, 1). Various approaches can be used to answer this question. Some of them may involve knowledge of historical data, while the other may rely solely on simulation techniques independent on the history of shifts a(t) of the term structure s(t).

Open Problem: Is the Macaulay duration the best choice among all dedicated durations D_v? In what sense the best? Call this the question Q. To give answer to question Q, one may attempt to approach it by answering the following subquestions:

(a) Is the relationship

(i) $v_s \cdot v_l = 1$ or

(ii) $v_s \cdot v_l \approx 1$

sufficient for the positive answer to question Q ?

(b) Suppose that the weights (27) of the barbell portfolio BP_v * with the highest convexity differ "very little" from the weights

$$x_s = \frac{t_l - q}{t_l - t_s} \text{ and } x_l = \frac{q - t_s}{t_l - t_s}.$$

Does this guarantee the positive answer to question Q ?

(c) Suppose that the weights

$$x_1 = \frac{t_j v_j - q}{t_j v_j - t_i v_i} \text{ and } x_2 = \frac{q - t_i v_i}{t_j v_j - t_i v_i}$$

of some (or many) other than BP_v * barbell portfolios immunized against shifts a(t) from class K_v differ "very little" from the weights

$$x_1 = \frac{t_j - q}{t_j - t_i} \quad \text{and} \quad x_2 = \frac{q - t_i}{t_j - t_i}.$$

Does this guarantee the positive answer to question Q ?

(d) When the answer to question Q will be negative?

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