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Some Subclasses of Meromorphically Functions Associated with the Convolution Structure

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Abstract. In this present paper we introduce and investigate each of the following new subclasses $\mathcal{F}^{m}_{p,\lambda,\ell,k}(\alpha;\varphi)$, $\hat{G}^{m}_{p,\lambda,\ell}(\alpha;\varphi)$ and $\aleph^{m}_{p,\lambda,\ell}(\alpha;\varphi)$ as well as $\Im^{m}_{p,\lambda,\ell,k}(\alpha;\varphi)$, $\hat{C}^{m}_{p,\lambda,\ell}(\alpha;\varphi)$ and $\acute{R}^{m}_{p,\lambda,\ell}(\alpha;\varphi)$ of meromorphic functions, which is defined by means of a certain meromorphically p-modified version of the convolution structure. Such results as inclusion relationships, integral representations and convolution properties for these function classes are proved.

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1 Introduction

Let Σ_p denote the class of all meromorphic functions f of the form

$$f(z) = z^{-p} + \sum_{n=1-p}^{\infty} a_n z^n \quad (p \in N = \{1, 2, ...\}) \tag{1.1}$$

which are analytic in the punctured disc $U^* = \{z : z \in C \text{ and } 0 < |z| < 1\} = U \backslash \{0\}$. For simplicity, we write $\Sigma_1 = \Sigma$. If f and g are analytic in U, we say that f is subordinate to g written symbolically as follows:

$$f \prec g \text{ or } f \prec g,$$

if there exists a Schwarz function w, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$, such that $f = g(w(z))$ $(z \in U)$. In particular, if the function g is univalent in U, then we have the following equivalence (cf., e.g., [4]; see also [12], [13])

$$f \prec g \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

For functions $f \in \Sigma_p$, given by (1.1), and $g \in \Sigma_p$ defined by

$$g(z) = z^{-p} + \sum_{n=1-p}^{\infty} b_n z^n \qquad (p \in N), \tag{1.2}$$

then the Hadamard product (or convolution) of f and g is given by

$$(f * g)(z) = z^{-p} + \sum_{n=1-p}^{\infty} a_n b_n z^n = (g * f)(z). \tag{1.3}$$

Now, we defined a linear operator For $f, g \in \Sigma_p$, $\lambda \geq 0$, $\ell > 0$, $p \in \mathbb{N}$, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, we define the linear operator $D^m_{\lambda,\ell,p}(f * g) : \Sigma_p \to \Sigma_p$ by:

$$\begin{aligned}
D^0_{\lambda,l,p}(f * g)(z) &= (f * g)(z) = z^{-p} + \sum_{n=1-p}^{\infty} a_n b_n z^n. \\
D^1_{\lambda,l,p}(f * g)(z) &= (1-\lambda)(f * g)(z) + \frac{\lambda}{\ell z^{p+\ell-1}} \left(z^{p+\ell}(f * g)(z)\right)' \\
&= (1-\lambda)\left[z^{-p} + \sum_{n=1-p}^{\infty} a_n b_n z^n\right] + \frac{\lambda}{\ell z^{p+\ell-1}} \left[z^{\ell} + \sum_{n=1-p}^{\infty} a_n b_n z^{n+p+\ell}\right]' \\
&= z^{-p} + \sum_{n=1-p}^{\infty} \left[\frac{\ell + \lambda(n+p)}{\ell}\right] a_n b_n z^n. \\
D^2_{\lambda,\ell,p}(f * g)(z) &= (1-\lambda) D^1_{\lambda,\ell,p}(f * g)(z) + \frac{\lambda}{\ell z^{p+\ell-1}} \left(z^{p+\ell} D^1_{\lambda,\ell,p}(f * g)(z)\right)' \\
&= z^{-p} + \sum_{n=1-p}^{\infty} \left[\frac{\ell + \lambda(n+p)}{\ell}\right]^2 a_n b_n z^n
\end{aligned} \tag{1.4}$$

and (in general)

$$\begin{aligned} D^{m}_{\lambda,\ell,p}(f*g)(z) &= (1-\lambda)\,D^{m-1}_{\lambda,\ell,p}(f*g)(z) + \frac{\lambda}{\ell z^{p+\ell-1}}\left(z^{p+\ell}D^{m-1}_{\lambda,\ell,p}(f*g)(z)\right)' \\ &= z^{-p} + \sum_{n=1-p}^{\infty}\left[\frac{\ell+\lambda\,(n+p)}{\ell}\right]^{m} a_n b_n z^n. \end{aligned} \tag{1.5}$$

From (1.5) it is easy to verify that

$$\lambda z\left(D^{m}_{\lambda,\ell,p}(f*g)\right)'(z) = \ell D^{m+1}_{\lambda,\ell,p}(f*g)(z) - (\ell+\lambda p)\,D^{m}_{\lambda,\ell,p}(f*g)(z) \tag{1.6}$$

We observe that the linear operator $D^{m}_{\lambda,\ell,p}(f*g)$ reduces to several interesting operators for different choices of n, λ, ℓ, p and the function g:

(i) For $g = \frac{z^{-p}}{1-z}$ (or $b_n = 1$), $D^{m}_{\lambda,\ell,p}(f*g) = I^{m}_{p}(\lambda,\ell)$, was introduced and studied by El-Ashwah [9], the operator $I^{m}_{p}(\lambda,\ell)$, contains as special cases (see [2], [5] and [17]);

(ii) For $m = 0$ and

$$g = z^{-p} + \sum_{n=1-p}^{\infty}\frac{(\alpha_1)_n...(\alpha_q)_n}{(\beta_1)_n...(\beta_s)_n}.\frac{z^n}{n!} \tag{1.7}$$

$$\left(\alpha_i\in\mathbb{C}; i=1,...,q; \beta_j\in\mathbb{C}\backslash\mathbb{Z}_0^- = \{0,-1,-2,...\}; j=1,...,s; \right.$$
$$\left. q\le s+1; q,s\in\mathbb{N}_0, p\in\mathbb{N}; z\in U\right),$$

and

$$(\theta)_\nu = \frac{\Gamma(\theta+\nu)}{\Gamma(\theta)} = \begin{cases} 1 & if \quad \nu=0; \theta\in\mathbb{C}^* = \mathbb{C}\backslash\{0\}, \\ \theta(\theta-1)...(\theta+\nu-1) & if \quad \nu\in\mathbb{N}; \theta\in\mathbb{C}. \end{cases}$$

We have $D^{0}_{\lambda,p}(f*g)(z) = (f*g)(z) = H^{q,s}_{p}(\alpha_1)f$, where $H^{q,s}_{p}(\alpha_1)$ is a meromorphically $p-$ modified version of familiar Dziok-Srivastava linear operator [6, 7] .

Recently, Liu and Srivastava [11], Raina and Srivastava [15], and Aouf [1] obtained many interesting results involving the linear operator $H^{q,s}_{p}(\alpha_1)$, and was further studied in a subsequent investigation by wang et al [18]. In particular, for

$q = 2, \qquad s = 1, \qquad \alpha_1 = a \qquad \beta_1 = c \quad and \quad \alpha_2 = 1$

we obtain the following linear operator

$$\mathcal{L}_p(a,c)f = H_p(\alpha_1, 1; \beta_1)f \qquad (z\in U^*)$$

which was introduced and investigated earlier by Liu and Srivastava [10], and was further studied in a subsequent investigation by Srivastava et al [16].

Let P denote the class of functions of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \ ,$$

which are analytic in U and satisfy the following condition

$$\operatorname{Re} p(z) > 0 \quad (z \in U) \ .$$

Throughout this paper, we assume that $p, k \in N$, $\epsilon_k = \exp\left(\frac{2\pi i}{k}\right)$,

$$F^m_{p,\lambda,\ell,k}(f*g)(z) = \frac{1}{k}\sum_{j=0}^{k-1} \epsilon_k^{ip} D^m_{\lambda,\ell,p}(f*g)(z)\left(\epsilon_k^j z\right) = z^{-p} + ...(f, g \in \Sigma_p), \tag{1.8}$$

$$G^m_{p,\lambda,\ell}(f*g)(z) = \frac{1}{2}\left[D^m_{\lambda,\ell,p}(f*g)(z) + \overline{D^m_{\lambda,\ell,p}(f*g)(\bar{z})}\right] = z^{-p} + ...(f, g \in \Sigma_p) \ , \tag{1.9}$$

and

$$H^m_{p,\lambda,\ell}(f*g)(z) = \frac{1}{2}\left[D^m_{\lambda,\ell,p}(f*g)(z) - \overline{D^m_{\lambda,\ell,p}(f*g)(-\bar{z})}\right] = z^{-p} + ... \ (f, g \in \Sigma_p). \tag{1.10}$$

Clearly, for $k = 1$, we have

$$F^m_{p,\lambda,\ell,1}(f*g)(z) = D^m_{\lambda,\ell,p}(f*g)(z) \ .$$

Making use of the integral operator $D^m_{\lambda,\ell,p}(f*g)$ and the above mentioned principle of subordination between analytic functions, we now interoduce and investigate the following subclasses of the class Σ_p of meromorphic functions.

Definition 1. *Let $g \in \Sigma_p$ be defined by (1.2). A function $f \in \Sigma_p$ is said to be in the class $\mathcal{F}^m_{p,\lambda,\ell,k}(\alpha;\varphi)$ if it satisfies the following subordination condition:*

$$-\frac{z\left[(1+\alpha)\left(D^m_{\lambda,\ell,p}(f*g)\right)'(z) + \alpha\left(D^{m+1}_{\lambda,\ell,p}(f*g)\right)'(z)\right]}{p\left[(1+\alpha)F^m_{p,\lambda,\ell,k}(f*g)(z) + \alpha F^{m+1}_{p,\lambda,\ell,k}(f*g)(z)\right]} \prec \varphi(z), \tag{1.11}$$

for some α $(\alpha \geq 0)$, where $\varphi \in P$, $F^m_{p,\lambda,\ell,k}(f*g)$ is defined by (1.8) and $F^{m+1}_{p,\lambda,\ell,k}(f*g)(z) \neq 0$ $(z \in U^*)$.

For simplicity, we write

$$\mathcal{F}^m_{p,\lambda,\ell,k}(0;\varphi) = \mathcal{F}^m_{p,\lambda,\ell,k}(\varphi) \ .$$

Remark 1. *In [20], Zou and Wu introduced and investigated a subclass $MS_s^*(\alpha)$ of Σ consisting of functions which are meromorphically α-starlike with respect to symmetric points and satisfy the following inequality:*

$$\operatorname{Re}\left\{-\frac{z\left[(1+\alpha)(f*g)'(z)+\alpha(z(f*g)'(z))'\right]}{(1+\alpha)T_s(f*g)(z)+\alpha z(T_s(f*g))'(z)}\right\}>0 \quad (z\in U)\ ,$$

where

$$T_s(f*g)(z)=\frac{1}{2}\left[(f*g)(z)-(f*g)(-z)\right]. \tag{1.12}$$

Remark 2. *For $\alpha=0$ and $\lambda=\ell=1$, we have the class $\mathcal{F}^m_{p,1,1k}(0;\varphi)=\mathcal{F}^m_{p,k}(\varphi)$, where the class $\mathcal{F}^m_{p,k}(\varphi)$ consisting of functions $f,g\in\Sigma_p$ which satisfy the following subordination condition:*

$$-\frac{z\left(D_p^m(f*g)\right)'(z)}{pF^m_{p,k}(f*g)(z)}\prec\varphi(z),$$

where $\varphi\in P$ and

$$F^m_{p,k}(f*g)(z)=\frac{1}{k}\sum_{j=0}^{k-1}\epsilon_k^{jp}(D_p^m(f*g))(\epsilon_k^j z)\neq 0 \quad (z\in U^*)\ .$$

Definition 2. *Let $g\in\Sigma_p$ be defined by (1.2). A function $f\in\Sigma_p$ is said to be in the class $\hat{G}^m_{p,\lambda,\ell}(\alpha;\varphi)$ if it satisfies the following subordination condition:*

$$-\frac{\left[(1+\alpha)\left(D^m_{\lambda,\ell,p}(f*g)\right)'(z)+\alpha\left(D^{m+1}_{\lambda,\ell,p}(f*g)\right)'(z)\right]}{p\left[(1+\alpha)G^m_{p,\lambda,\ell}(f*g)(z)+\alpha G^{m+1}_{p,\lambda,\ell}(f*g)(z)\right]}\prec\varphi(z) \quad (\ \alpha\geq 0).$$

Definition 3. *Let $g\in\Sigma_p$ be defined by (1.2). A function $f\in\Sigma_p$ is said to be in the class $\aleph^m_{p,\lambda,\ell}(\alpha;\varphi)$ if it satisfies the following subordination condition:*

$$-\frac{z\left[(1+\alpha)\left(D^m_{\lambda,\ell,p}(f*g)(z)\right)'+\alpha\left(D^{m+1}_{\lambda,\ell,p}(f*g)(z)\right)'\right]}{p\left[(1+\alpha)H^m_{p,\lambda,\ell}(f*g)(z)+\alpha H^{m+1}_{p,\lambda,\ell}(f*g)(z)\right]}\prec\varphi(z) \quad (\alpha\geq 0).$$

Remark 3. *In [19], Zou and Wu introduced and investigated a subclass $MS_{sc}^*(\alpha)$ of Σ consisting of functions which are meromorphically α-starlike with respect to symmetric conjugate points and satisfy the following inequality:*

$$\operatorname{Re}\left\{-\frac{z\left[(1+\alpha)(f*g)'(z)+\alpha(z(f*g)'(z))'\right]}{(1+\alpha)T_{sc}(f*g)(z)+\alpha z(T_{sc}(f*g)(z))'}\right\}>0 \quad (z\in U)\ ,$$

where

$$T_{sc}(f*g)(z)=\frac{1}{2}\left[\left((f*g)(z)-\overline{(f*g)(-\overline{z})}\right)\right] . \tag{1.13}$$

Definition 4. *Let $g\in\Sigma_p$ be defined by (1.2). A function $f\in\Sigma_p$ is said to be in the class $\Im^m_{p,\lambda,\ell,k}(\alpha;\varphi)$ if it satisfies the following subordination condition:*

$$-\frac{z\left[(1+\alpha)\left(D^m_{\lambda,\ell,p}(f*g)\right)'(z)+\alpha(D^{m+1}_{\lambda,\ell,p}(f*g))'(z)\right]}{p\left[(1+\alpha)\pounds^m_{p,\lambda,\ell,k}(f*g)(z)+\alpha\pounds^{m+1}_{p,\lambda,\ell,k}(f*g)(z)\right]}\prec\varphi(z)$$

$$(\alpha\geq 0;\pounds\in\mathfrak{F}^m_{p,\lambda,\ell,k}(\alpha;\varphi).$$

Definition 5. *Let $g\in\Sigma_p$ be defined by (1.2). A function $f\in\Sigma_p$ is said to be in the class $\hat{C}^m_{p,\lambda,\ell}(\alpha;\varphi)$ if it satisfies the following subordination condition:*

$$-\frac{z\left[(1+\alpha)\left(D^m_{\lambda,\ell,p}(f*g)\right)'(z)+\alpha(D^{m+1}_{\lambda,\ell,p}(f*g))'(z)\right]}{p\left[(1+\alpha)\chi^m_{p,\lambda,\ell}(f*g)(z)+\alpha\chi^{m+1}_{p,\lambda,\ell}(f*g)(z)\right]}\prec\varphi(z)$$

$$(\alpha\geq 0;\chi\in\hat{G}^m_{p,\lambda,\ell}(\alpha;\varphi).$$

Definition 6. *Let $g\in\Sigma_p$ be defined by (1.2). A function $f\in\Sigma_p$ is said to be in the class $\acute{R}^m_{p,\lambda,\ell}(\alpha;\varphi)$ if it satisfies the following subordination condition:*

$$-\frac{z\left[(1+\alpha)\left(D^m_{\lambda,\ell,p}(f*g)\right)'(z)+\alpha(D^{m+1}_{\lambda,\ell,p}(f*g))'(z)\right]}{p\left[(1+\alpha)\eta^m_{p,\lambda,\ell}(f*g)(z)+\alpha\eta^{m+1}_{p,\lambda,\ell}(f*g)(z)\right]}\prec\varphi(z)$$

$$(\alpha\geq 0;\eta\in\aleph^m_{p,\lambda,\ell}(\alpha;\varphi).$$

In order to establish our main results we shall make use the following lemmas.

Lemma 1 ([8], [12]). *Let $\beta,\gamma\in C$. Suppose also that ϕ is convex and univalent in U with*

$$\phi(0)=1 \text{ and } \mathrm{Re}(\beta\phi(z)+\gamma)>0 \quad (z\in U) .$$

If p is analytic in U with $p(0)=1$, then the following subordination:

$$p(z)+\frac{zp'(z)}{\beta p(z)+\gamma}\prec\phi(z)$$

implies that

$$p(z)\prec\phi(z) .$$

Lemma 2 [14]. *Let $\beta, \gamma \in C$. Suppose also that ϕ is convex and univalent in U with*

$$\phi(0) = 1 \quad \text{and} \quad \mathrm{Re}(\beta\phi(z) + \gamma) > 0 \ .$$

Also let

$$q(z) \prec \phi(z).$$

If $p \in P$ and satisfies the following subordination:

$$p(z) + \frac{zp^{'}(z)}{\beta p(z) + \gamma} \prec \phi(z) \ ,$$

then

$$p(z) \prec \phi(z) \ .$$

Lemma 3. *Let $f \in \mathfrak{F}^m_{p,\lambda,\ell,k}(\alpha; \varphi)$. Then*

$$-\frac{z\left[(1+\alpha)\left(F^m_{p,\lambda,\ell,k}(f*g)\right)^{'}(z) + \alpha\left(F^{m+1}_{p,\lambda,\ell,k}(f*g)\right)^{'}(z)\right]}{p\left[(1+\alpha)F^m_{p,\lambda,\ell,k}(f*g)(z) + \alpha F^{m+1}_{p,\lambda,\ell,k}(f*g)(z)\right]} \prec \varphi(z). \tag{1.14}$$

Furthermore, if $\varphi \in P$ with

$$\mathrm{Re}\left(\frac{\ell}{\alpha\lambda} + 2\frac{\ell}{\lambda} + p - p\varphi(z)\right) > 0 \quad (\alpha > 0; \lambda > 0; z \in U) \ ,$$

then

$$-\frac{z\left(F^m_{p,\lambda,\ell,k}(f*g)\right)^{'}(z)}{pF^m_{p,\lambda,\ell,k}(f*g)(z)} \prec \varphi(z).$$

Proof. Making use of (1.8), we have

$$\begin{aligned}
F^m_{p,\lambda,\ell,k}(f*g)(\epsilon^j_k z) &= \frac{1}{k}\sum_{n=0}^{k-1}\epsilon^{np}_k D^m_{\lambda,\ell,p}(f*g)\left(\epsilon^{n+j}_k z\right) \\
&= \epsilon^{-jp}_k . \frac{1}{k}\sum_{n=0}^{k-1}\epsilon^{(n+j)p}_k D^m_{\lambda,\ell,p}(f*g)\left(\epsilon^{n+j}_k z\right) \\
&= \epsilon^{-jp}_k F^m_{p,\lambda,\ell,k}(f*g)(z) \ (j \in \{0, 1, ..., k-1\})
\end{aligned} \tag{1.15}$$

and

$$\left(F^m_{p,\lambda,\ell,k}(f*g)\right)^{'}(z) = \frac{1}{k}\sum_{n=0}^{k-1}\epsilon^{j(p+1)}_k \left(D^m_{\lambda,\ell,p}(f*g)\right)^{'}\left(\epsilon^j_k z\right) \ . \tag{1.16}$$

Replacing m by $m+1$ in (1.15) and (1.16), respectively, we obtain

$$F_{p,\lambda,\ell,k}^{m+1}(f*g)(\epsilon_k^j z) = \epsilon_k^{-jp} F_{p,\lambda,\ell,k}^{m+1}(f*g)(z) \ (j \in \{0,1,...,k-1\}) \tag{1.17}$$

and

$$\left(F_{p,\lambda,\ell,k}^{m+1}(f*g)\right)^{'}(z) = \frac{1}{k}\sum_{n=0}^{k-1} \epsilon_k^{j(p+1)} \left(D_{\lambda,\ell,p}^{m+1}(f*g)\right)^{'}\left(\epsilon_k^j z\right) \ . \tag{1.18}$$

From (1.15) and (1.18), we obtain

$$\begin{aligned} & -\frac{z\left[(1+\alpha)\left(F_{p,\lambda,\ell,k}^{m}(f*g)\right)^{'}(z) + \alpha\left(F_{p,\lambda,\ell,k}^{m+1}(f*g)\right)^{'}(z)\right]}{p\left[(1+\alpha)F_{p,\lambda,\ell,k}^{m}(f*g)(z) + \alpha F_{p,\lambda,\ell,k}^{m+1}(f*g)(z)\right]} \\ = \ & -\frac{1}{k}\sum_{j=0}^{k-1} \frac{\epsilon_k^j z\left[(1+\alpha)\left(D_{\lambda,\ell,p}^{m}(f*g)\right)^{'}(\epsilon_k^j z) + \alpha\left(D_{\lambda,\ell,p}^{m+1}(f*g)\right)^{'}(\epsilon_k^j z)\right]}{p\left[(1+\alpha)F_{p,\lambda,\ell,k}^{m}(f*g)(\epsilon_k^j z) + \alpha F_{p,\lambda,\ell,k}^{m+1}(f*g)(\epsilon_k^j z)\right]} \quad (z \in U). \end{aligned} \tag{1.19}$$

Moreover, since $f \in \mathcal{F}_{p,\lambda,\ell,k}^{m}(\alpha;\varphi)$, it follows that

$$-\frac{\epsilon_k^j z\left[(1+\alpha)\left(D_{\lambda,\ell,p}^{m}(f*g)\right)^{'}(\epsilon_k^j z) + \alpha\left(D_{\lambda,\ell,p}^{m+1}(f*g)\right)^{'}(\epsilon_k^j z)\right]}{p\left[(1+\alpha)F_{p,\lambda,\ell,k}^{m}(f*g)(\epsilon_k^j z) + \alpha F_{p,\lambda,\ell,k}^{m+1}(f*g)(\epsilon_k^j z)\right]} \prec \varphi(z)$$

$$(j \in \{0,1,...,k-1\}) \ . \tag{1.20}$$

By noting that φ is convex and univalent in U, we conclude from (1.19) and (1.20) that the assertion (1.14) of Lemma 3 holds true.

Next, making use of the relationships (1.6) and (1.8), we have

$$\begin{aligned} z\left(F_{p,\lambda,\ell,k}^{m}(f*g)\right)^{'}(z) + \left(p+\frac{\ell}{\lambda}\right)F_{p,\lambda,\ell,k}^{m}(f*g)(z) &= \frac{\ell}{\lambda k}\sum_{j=0}^{k-1}\epsilon_k^{jp}\left(D_{\lambda,\ell,p}^{m+1}(f*g)\right)\left(\epsilon_k^j z\right) \\ &= \frac{\ell}{\lambda}F_{p,\lambda,\ell,k}^{m+1}(f*g)(z) \quad (f \in \Sigma_p)\,. \end{aligned} \tag{1.21}$$

Let $f \in \mathcal{F}_{p,\lambda,\ell,k}^{m}(\alpha;\varphi)$ and suppose that

$$\psi(z) = -\frac{z\left(F_{p,\lambda,\ell,k}^{m}(f*g)\right)^{'}(z)}{pF_{p,\lambda,\ell,k}^{m}(f*g)(z)} \quad (z \in U) \ . \tag{1.22}$$

Then ψ is analytic in U and $\psi(0) = 1$. It follows from (1.21) and (1.22) that

$$\frac{\ell}{\lambda} + p - p\psi(z) = \frac{\ell}{\lambda}\frac{F_{p,\lambda,\ell,k}^{m+1}(f*g)(z)}{F_{p,\lambda,\ell,k}^{m}(f*g)(z)} \ . \tag{1.23}$$

From (1.22) and (1.23), we obtain

$$z\left(F^{m+1}_{p,\lambda,\ell,k}(f*g)\right)'(z)=\frac{-p\lambda}{\ell}\left\{z\psi'(z)+\left[\frac{\ell}{\lambda}+p-p\psi(z)\right]\psi(z)\right\}F^{m}_{p,\lambda,\ell,k}(f*g)(z)\ (z\in U^*)\,. \tag{1.24}$$

It now follows from (1.14) and (1.22)-(1.24) that

$$\begin{aligned}
&-\frac{z\left[(1+\alpha)\left(F^{m}_{p,\lambda,\ell,k}(f*g)\right)'(z)+\alpha\left(F^{m+1}_{p,\lambda,\ell,k}(f*g)\right)'(z)\right]}{p\left[(1+\alpha)F^{m}_{p,\lambda,\ell,k}(f*g)(z)+\alpha F^{m+1}_{p,\lambda,\ell,k}(f*g)(z)\right]}\\
&=\frac{\frac{\alpha\lambda}{\ell}z\psi'(z)+\left\{(1+\alpha)+\frac{\alpha\lambda}{\ell}\left[\frac{\ell}{\lambda}+p-p\psi(z)\right]\right\}\psi(z)}{(1+\alpha)+\frac{\alpha\lambda}{\ell}\left[\frac{\ell}{\lambda}+p-p\psi(z)\right]}\\
&=\psi(z)+\frac{z\psi'(z)}{\frac{\ell}{\lambda\alpha}+2\frac{\ell}{\lambda}+p-p\psi(z)}\prec\varphi(z)\,.
\end{aligned} \tag{1.25}$$

Thus, since

$$\mathrm{Re}\left(\frac{\ell}{\lambda\alpha}+2\frac{\ell}{\lambda}+p-p\psi(z)\right)>0\quad(\alpha>0;\lambda>0;z\in U)\,,$$

by means of (1.25) and Lemma 1, we find that

$$\psi(z)=-\frac{z\left(F^{m}_{p,\lambda,\ell,k}(f*g)\right)'(z)}{pF^{m}_{p,\lambda,\ell,k}(f*g)(z)}\prec\varphi(z).$$

This completes the proof of Lemma 3.

By similarly applying the method of proof of Lemma 3, we can easily get the following results for the classes $\hat{G}^{m}_{p,\lambda,\ell}(\alpha;\varphi)$ and $\aleph^{m}_{p,\lambda,\ell}(\alpha;\varphi)$.

Lemma 4. *Let $f\in\hat{G}^{m}_{p,\lambda,\ell}(\alpha;\varphi)$. Then*

$$-\frac{z\left[(1+\alpha)\left(G^{m}_{p,\lambda,\ell}(f*g)\right)'(z)+\alpha\left(G^{m+1}_{p,\lambda,\ell}(f*g)\right)'(z)\right]}{p\left[(1+\alpha)G^{m}_{p,\lambda,\ell}(f*g)(z)+\alpha G^{m+1}_{p,\lambda,\ell}(f*g)(z)\right]}\prec\varphi(z)\qquad.$$

Furthermore, if $\varphi\in P$ with

$$\mathrm{Re}\left(\frac{\ell}{\lambda\alpha}+2\frac{\ell}{\lambda}+p-p\varphi(z)\right)>0\quad(\alpha>0;\lambda>0;z\in U)\,,$$

then

$$-\frac{z\left(G^{m}_{p,\lambda,\ell}(f*g)(z)\right)'}{pG^{m}_{p,\lambda,\ell}(f*g)(z)}\prec\varphi(z).$$

Lemma 5. *Let* $f \in \aleph^m_{p,\lambda,\ell}(\alpha;\varphi)$. *Then*

$$-\frac{z\left[(1+\alpha)\left(H^m_{p,\lambda,\ell}(f*g)\right)'(z)+\alpha\left(H^{m+1}_{p,\lambda,\ell}(f*g)\right)'(z)\right]}{p\left[(1+\alpha)H^m_{p,\lambda,\ell}(f*g)(z)+\alpha H^m_{p,\lambda,\ell}(f*g)(z)\right]} \prec \varphi(z).$$

Furthermore, if $\varphi \in P$ with

$$\mathrm{Re}\left(\frac{\ell}{\lambda\alpha}+2\frac{\ell}{\lambda}+p-p\varphi(z)\right)>0 \quad (\alpha>0;\lambda>0;z\in U)\ ,$$

then

$$-\frac{z\left(H^m_{p,\lambda,\ell}(f*g)\right)'(z)}{pH^m_{p,\lambda,\ell}(f*g)(z)} \prec \varphi(z).$$

In this paper, we obtain inclusion relationships integral representation, and convolution properties for each of the following function classes which we have introduced here: $\mathfrak{F}^m_{p,\lambda,\ell,k}(\alpha;\varphi), \hat{G}^m_{p,\lambda,\ell}(\alpha;\varphi)$ and $\aleph^m_{p,\lambda,\ell}(\alpha;\varphi)$ as well as $\Im^m_{p,\lambda,\ell,k}(\alpha;\varphi)$, $\hat{C}^m_{p,\lambda,\ell}(\alpha;\varphi)$ and $\acute{R}^m_{p,\lambda,\ell}(\alpha;\varphi)$. The methods used here to obtain our main results are similar to those of Wang et al. [18], Srivastava et al. [16], and Zou et al.([19],[20]).

2 A set of inclusion relationships

We first provide some inclusion relationships for the following function classes $\mathfrak{F}^m_{p,\lambda,\ell,k}(\alpha;\varphi), \hat{G}^m_{p,\lambda,\ell}(\alpha;\varphi)$ and $\aleph^m_{p,\lambda,\ell}(\alpha;\varphi)$ as well as $\Im^m_{p,\lambda,\ell,k}(\alpha;\varphi)$, $\hat{C}^m_{p,\lambda,\ell}(\alpha;\varphi)$ and $\acute{R}^m_{p,\lambda,\ell}(\alpha;\varphi)$.

Theorem 1. *Let* $\varphi \in P$ *with*

$$\mathrm{Re}\left(\frac{\ell}{\lambda\alpha}+2\frac{\ell}{\lambda}+p-p\varphi(z)\right)>0 \quad (\alpha>0;\lambda>0;z\in U)\ .$$

Then

$$\mathfrak{F}^m_{p,\lambda,\ell,k}(\alpha;\varphi) \subset \mathfrak{F}^m_{p,\lambda,\ell,k}(\varphi)\ .$$

Proof. Let $f \in \mathfrak{F}^m_{p,\lambda,\ell,k}(\alpha;\varphi)$ and suppose that

$$q(z)=-\frac{z\left(D^m_{\lambda,\ell,p}(f*g)\right)'(z)}{pF^m_{p,\lambda,\ell,k}(f*g)(z)} \quad (z\in U)\ . \tag{2.1}$$

Then q is analytic in U and $q(0) = 1$. It follows from (1.6) and (2.1) that

$$q(z)F^m_{p,\lambda,\ell,k}(f*g)(z) = \frac{-\ell}{\lambda p}D^{m+1}_{\lambda,\ell,p}(f*g)(z) + \frac{\frac{\ell}{\lambda}+p}{p}D^m_{\lambda,\ell,p}(f*g)(z) \ . \qquad (2.2)$$

Differentiating both sides of (2.2) with respect to z and using (2.1), we obtain

$$\begin{aligned} & zq^{'}(z) + \left(\frac{\ell}{\lambda} + p + \frac{z(F^m_{p,\lambda,\ell,k}(f*g))'(z)}{F^m_{p,\lambda,\ell,k}(f*g)(z)}\right)q(z) \\ = \ & \frac{-\ell}{\lambda p}\frac{z\left(D^{m+1}_{\lambda,\ell,p}(f*g)\right)^{'}(z)}{F^m_{p,\lambda,\ell,k}(f*g)(z)} \ . \end{aligned} \qquad (2.3)$$

It now follows from (1.11), (1.22), (1.23), (2.1) and (2.3) that

$$\begin{aligned} & -\frac{z\left[(1+\alpha)\left(D^m_{\lambda,\ell,p}(f*g)\right)^{'}(z) + \alpha\left(D^{m+1}_{\lambda,\ell,p}(f*g)\right)^{'}(z)\right]}{p\left[(1+\alpha)F^m_{p,\lambda,\ell,k}(f*g)(z) + \alpha F^{m+1}_{p,\lambda,\ell,k}(f*g)(z)\right]} \\ = \ & \frac{\frac{\alpha\lambda}{\ell}zq^{'}(z) + \left\{(1+\alpha) + \frac{\alpha\lambda}{\ell}\left[\frac{\ell}{\lambda} + p - p\psi(z)\right]\right\}q(z)}{(1+\alpha) + \frac{\alpha\lambda}{\ell}\left[\frac{\ell}{\lambda} + p - p\psi(z)\right]} \\ = \ & q(z) + \frac{zq^{'}(z)}{\frac{\ell}{\lambda\alpha} + 2\frac{\ell}{\lambda} + p - p\psi(z)} \prec \varphi(z) \ . \end{aligned} \qquad (2.4)$$

Moreover, since

$$\mathrm{Re}\left(\frac{\ell}{\lambda\alpha} + 2\frac{\ell}{\lambda} + p - p\varphi(z)\right) > 0 \quad (\alpha > 0; \lambda > 0; z \in U) \ ,$$

by Lemma 3, we have

$$\psi(z) = -\frac{z\left(F^m_{p,\lambda,\ell,k}(f*g)\right)^{'}(z)}{pF^m_{p,\lambda,\ell,k}(f*g)(z)} \prec \varphi(z).$$

Thus, by (2.4) and Lemma 2, we find that

$$q(z) \prec \varphi(z) \ ,$$

that is, that $f \in \mathcal{F}^m_{p,\lambda,\ell,k}(\varphi)$. This implies that

$$\mathcal{F}^m_{p,\lambda,\ell,k}(\alpha;\varphi) \subset \mathcal{F}^m_{p,\lambda,\ell,k}(\varphi) \ .$$

The proof of Theorem 1 is evidently completed.

In view of Lemmas 4 and 5, and by similarly applying the method of proof of Theorem 1, we can easily obtain the inclusion relationships $\hat{G}^m_{p,\lambda,\ell}(\alpha;\varphi) \subset \hat{G}^m_{p,\lambda,\ell}(\varphi)$ and $\aleph^m_{p,\lambda,\ell}(\alpha;\varphi) \subset \aleph^m_{p,\lambda,\ell}(\varphi)$.

Theorem 2. *Let $\varphi \in P$ with*

$$\mathrm{Re}\left(\frac{\ell}{\lambda\alpha} + 2\frac{\ell}{\lambda} + p - p\varphi(z)\right) > 0 \quad (\alpha > 0; \lambda > 0; z \in U)\ .$$

Then

$$\Im^m_{p,\lambda,\ell,k}(\alpha;\varphi) \subset \Im^m_{p,\lambda,\ell,k}(\varphi).$$

Proof. Let $f \in \Im^m_{p,\lambda,\ell,k}(\alpha;\varphi)$ and suppose that

$$p(z) = -\frac{z\left(D^m_{\lambda,\ell,p}(f*g)\right)'(z)}{p\pounds^m_{p,\lambda,\ell,k}(f*g)(z)} \quad (z \in U)\ . \tag{2.5}$$

Then p is analytic in U and $p(0) = 1$. It follows from (1.6) and (2.5) that

$$p(z)\pounds^m_{p,\lambda,\ell,k}(f*g)(z) = -\frac{\ell}{\lambda p}D^{m+1}_{\lambda,\ell,p}(f*g)(z) + \frac{\frac{\ell}{\lambda}+p}{p}D^m_{\lambda,\ell,p}(f*g)(z)\ . \tag{2.6}$$

Differentiating both sides of (2.6) with respect to z and using (2.5), we have

$$\begin{aligned} & zp'(z) + \left(\frac{\ell}{\lambda} + p + \frac{z\left(\pounds^m_{p,\lambda,\ell,k}(f*g)\right)'(z)}{\pounds^m_{p,\lambda,\ell,k}(f*g)(z)}\right)p(z) \\ = \ & -\frac{\ell}{\lambda p}\frac{z\left(D^{m+1}_{\lambda,\ell,p}(f*g)\right)'(z)}{\pounds^m_{p,\lambda,\ell,k}(f*g)(z)}\ . \end{aligned}$$

Furthermore, we suppose that

$$\varphi(z) = -\frac{z\left(\pounds^m_{p,\lambda,\ell,k}(f*g)\right)'(z)}{p\pounds^m_{p,\lambda,\ell,k}(f*g)(z)} \quad (z \in U).$$

The remainder of the proof of Theorem 2 is similar to that of Theorem 1. We, therefore, choose to omit the analogous details involved. We thus find that

$$p(z) \prec \varphi(z),$$

which implies that $f \in \Im^m_{p,\lambda,\ell,k}(\varphi)$. The proof of Theorem 2 is thus completed.

In view of Lemmas 4 and 5, and by similarly applying the method of proof of Theorem 2, we can easily obtain the inclusion relationships $\hat{C}^m_{p,\lambda,\ell}(\alpha;\varphi) \subset \hat{C}^m_{p,\lambda,\ell}(\varphi)$ and $\acute{R}^m_{p,\lambda,\ell}(\alpha;\varphi) \subset \acute{R}^m_{p,\lambda,\ell}(\varphi)$.

In view of Lemmas 3 to 5, and by similarly applying the method of proofs of Theorems 1 and 2 obtained by Srivastava et al. [16], we can easily obtain the following inclusion relationships.

Corollary 1. *Let* $\varphi \in P$ *with*

$$\operatorname{Re}\left(\frac{\ell}{\lambda} + p - p\varphi(z)\right) > 0 \quad (\lambda > 0; z \in U) .$$

Then

$$\mathcal{F}^{m+1}_{p,\lambda,\ell,k}(\varphi) \subset \mathcal{F}^{m}_{p,\lambda,\ell,k}(\varphi) .$$

The result of Corollary1 also holds true for the classes $\hat{G}^{m+1}_{p,\lambda,\ell}(\varphi)$ and $\aleph^{m+1}_{p,\lambda,\ell}(\varphi)$.

Corollary 2. *Let* $\varphi \in P$ *with*

$$\operatorname{Re}\left(\frac{\ell}{\lambda} + p - p\varphi(z)\right) > 0 \quad (\lambda > 0; z \in U) .$$

Then

$$\Im^{m+1}_{p,\lambda,\ell,k}(\varphi) \subset \Im^{m}_{p,\lambda,\ell,k}(\varphi) .$$

The result of Corollary2 also holds true for the classes $\hat{C}^{m+1}_{p,\lambda,\ell}(\varphi)$ and $\acute{R}^{m+1}_{p,\lambda,\ell}(\varphi)$.

Remark 3. (i) *Putting* $m = 0, \frac{\ell}{\lambda} = \alpha_1$ *and* g *is* given by (1.7), in Theorem 1, we obtain the result obtained by Wang et al [18];

(ii) *Putting* $g = \frac{z^{-p}}{1-z}$ (or $b_n = 1$), in Theorem 1, we obtain the result obtained by Aouf et al [3] .

3 Integral representation

In this section, we prove a number of integral representations associated with the function classes $\mathcal{F}^{m}_{p,\lambda,\ell,k}(\varphi), \hat{G}^{m}_{p,\lambda,\ell}(\varphi)$ and $\aleph^{m}_{p,\lambda,\ell}(\varphi)$.

Theorem 3. *Let* $f \in \mathcal{F}^{m}_{p,\lambda,\ell,k}(\varphi)$. *Then*

$$F^{m}_{p,\lambda,\ell,k}(f * g)(z) = z^{-p}.\exp\left(-\frac{p}{k}\sum_{j=0}^{k-1}\int_{0}^{z}\frac{\varphi(w(\epsilon_k^j\xi)) - 1}{\xi}d\xi\right) , \tag{3.1}$$

where $F^{m}_{p,\lambda,\ell,k}(f * g)$ is defined by (1.8) and w is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$.

Proof. Suppose that $f \in \mathcal{F}^m_{p,\lambda,\ell,k}(\varphi)$. We observe that the condition (1.11) (with $\alpha = 0$) can be written as follows:

$$-\frac{z\left(D^{m+1}_{\lambda,\ell,p}(f*g)\right)'(z)}{pF^m_{p,\lambda,\ell,k}(f*g)(z)} = \varphi(w(z)) \quad (z \in U)\ , \tag{3.2}$$

where w is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$.
Replacing z by $\epsilon_k^j z$ $(j = 0, 1, ..., k-1)$ in (3.2), we find that (3.2) also holds true, that is, that

$$-\frac{\epsilon_k^j z\left(D^{m+1}_{\lambda,\ell,p}(f*g)\right)'(\epsilon_k^j z)}{pF^m_{p,\lambda,\ell,k}(f*g)(\epsilon_k^j z)} = \varphi\left(w(\epsilon_k^j z)\right) \quad (z \in U)\ . \tag{3.3}$$

We note that

$$F^m_{p,\lambda,\ell,k}(f*g)(\epsilon_k^j z) = \epsilon_k^{-jp} F^m_{p,\lambda,\ell,k}(f*g)(z) \quad (z \in U)\ .$$

Thus, by letting $j = 0, 1, ..., k-1$ in (3.3), successively, and summing the resulting equations, we get

$$-\frac{z\left(F^m_{p,\lambda,\ell,k}(f*g)\right)'(z)}{pF^m_{p,\lambda,\ell,k}(f*g)(z)} = \frac{1}{k}\sum_{j=0}^{k-1}\varphi\left(w(\epsilon_k^j z)\right) \quad (z \in U)\ . \tag{3.4}$$

We next find from (3.4) that

$$\frac{\left(F^m_{p,\lambda,\ell,k}(f*g)\right)'(z)}{F^m_{p,\lambda,\ell,k}(f*g)(z)} + \frac{p}{z} = \frac{-p}{k}\sum_{j=0}^{k-1}\frac{\varphi\left(w(\epsilon_k^j z)\right) - 1}{z} \quad (z \in U^*)\ , \tag{3.5}$$

which, upon integration, yields

$$\log\left(z^p F^m_{p,\lambda,\ell,k}(f*g)(z)\right) = \frac{-p}{k}\sum_{j=0}^{k-1}\int_0^z \frac{\varphi\left(w(\epsilon_k^j \xi)\right) - 1}{\xi} d\xi\ . \tag{3.6}$$

The assertion (3.1) of Theorem 3 can now easily be derived from (3.6).
Theorem 4. *Let* $f \in \mathcal{F}^m_{p,\lambda,\ell,k}(\varphi)$. *Then*

$$D^m_{\lambda,\ell,p}(f*g)(z) = -p\int_0^z \zeta^{-p-1}\varphi(w(\zeta)).\exp\left(\frac{-p}{k}\sum_{j=0}^{k-1}\int_0^\zeta \frac{\varphi(w(\epsilon_k^j \xi)) - 1}{\xi} d\xi\right) d\zeta, \tag{3.7}$$

where w is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$.

Proof. Suppose that $f \in \mathcal{F}^m_{p,\lambda,\ell,k}(\varphi)$. Then, in light of (3.1) and (3.2), we have

$$\left(D^m_{\lambda,\ell,p}(f*g)\right)'(z) = -\frac{pF^m_{p,\lambda,\ell,k}(f*g)(z)}{z}.\varphi(w(z))$$

$$= -pz^{-p-1}\varphi(w(z)).\exp\left(\frac{-p}{k}\sum_{j=0}^{k-1}\int_0^z \frac{\varphi(w(\epsilon_k^j\xi))-1}{\xi}d\xi\right), \qquad (3.8)$$

which, upon integration, leads us easily to the assertion (3.7) of Theorem 4.

In view of Lemma 3, we can obtain another integral representation for the function class $\mathcal{F}^m_{p,\lambda,\ell,k}(\varphi)$.

Theorem 5. *Let* $f \in \mathcal{F}^m_{p,\lambda,\ell,k}(\varphi)$. *Then*

$$D^m_{\lambda,\ell,p}(f*g)(z) = -p\int_0^z \zeta^{-p-1}\varphi(w_2(\zeta)).\exp\left(-p\int_0^z \frac{\varphi(w_1(\xi))-1}{\xi}d\xi\right)d\,\zeta, \qquad (3.9)$$

where the function w_j $(j=1,2)$ are analytic in U with $w_j(0)=0$ and $|w_j(z)|<1$ $(z\in U; j=1,2)$.

Proof. Suppose that $f \in \mathcal{F}^m_{p,\lambda,\ell,k}(\varphi)$. We then find from (1.14) (with $\alpha=0$) that

$$-\frac{z\left(F^m_{p,\lambda,\ell,k}(f*g)\right)'(z)}{pF^m_{p,\lambda,\ell,k}(f*g)(z)} = \varphi(w_1(z)) \quad (z\in U)\ ,$$

where w_1 is analytic in U and $w_1(0)=0$. Thus, by similarly applying the method of proof of Theorem 3, we find that

$$F^m_{p,\lambda,\ell,k}(f*g)(z) = z^{-p}.\exp\left(-p\int_0^z \frac{\varphi(w_1(\xi))-1}{\xi}d\xi\right)\ . \qquad (3.11)$$

From (3.2) and (3.11), we have

$$\left(D^m_{\lambda,\ell,p}(f*g)\right)'(z) = -\frac{pF^m_{p,\lambda,\ell,k}(f*g)(z)}{z}\ .\ \varphi(w_2(z))$$

$$= -pz^{-p-1}\varphi(w_2(z)).\exp\left(-p\int_0^z \frac{\varphi(w_1(\xi))-1}{\xi}d\xi\right)\ , \qquad (3.12)$$

where the functions $w_j(z)$ $(j=1,2)$ are analytic in U with $w_j(0)=0$ and $|w_j(z)|<1$ $(z\in U; j=1,2)$. Upon integrating both sides of (3.12), we readily arrive at the assertion (3.9) of Theorem 5.

Remark 4. The result of Theorem 5 also holds true for the classes $\hat{G}^{m}_{p,\lambda,\ell}(\varphi)$ and $\aleph^{m}_{p,\lambda,\ell}(\varphi)$. So we omit the details involved.

In view of Lemmas 4 and 5, and by similarly applying the methods of proof of Theorems 3 and 4, we can easily obtain the results for the function classes $\hat{G}^{m}_{p,\lambda,\ell}(\varphi)$ and $\aleph^{m}_{p,\lambda,\ell}(\varphi)$.

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References

[1] **M. K. Aouf**, Certain subclasses of meromorphically multivalent functions associated with generalized hypergeometric function, *Comput. Math. Appl.*, **55**, (2008), 494-509.

[2] **M. K. Aouf and H. M. Hossen**, New criteria for meromorphic p-valent starlike functions, *Tsukuba J. Math.*, **17**, (1993), 481-486.

[3] **M. K. Aouf, A. Shamandy, R. M. El-Ashwah, and E. E. Ali**, Some subclasses of meromorphically multivalent functions associated with the generalized multiplier transformations, *Asian Europe. J. Math.*, **5**, (2012), 1-20.

[4] **T. Bulboaca**, *Differential subordinations and superordinations, Recent Results*, House of Scientific Book Publ., Cluj-Napoca, 2005.

[5] **N. E. Cho, O. S. Kwon, and H. M Srivastava**, Inclusion and argument properties for certain subclasses of meromorphic functions associated with a family of multiplier transformations, *J. Math. Anal. Appl.*, **300**, (2004), 505-520.

[6] **J. Dziok and H. M. Srivastava**, Classes of analytic functions associated with the generalized hypergeometric function, *Appl. Math. Comput.*, **103**, (1999), 1-13.

[7] **J. Dziok and H. M. Srivastava**, Certain subclasses of analytic functions associated with the generalized hypergeometric function, *Integral Transforms Spec. Funct.*, **14**, (2003), 7-18.

[8] **P. Eenigenburg, S. S. Miller, P. T. Mocanu, and M. O. Reade**, *On a Briot-Bouquet differential subordination*, in: General Mathematics 3, International Series of Numerical Mathemtics, Vol. 64, Birkhauser. Verlag, Basel, 1983, 339-348, see also Rev. Roumaine Math. Pures Appl. 29(1984), 567-573.

[9] **R. M. El-Ashwah**, A note on certain meromorphic p-valent functions, *Appl. Math. Lett.*, **22**, (2009), 1756-1759.

[10] **J. L. Liu and H. M. Srivastava**, A linear operator and associated families of meromorphically multivalent functions, *J. Math. Anal. Appl.*, **259**, (2001), 566-581.

[11] **J. L. Liu and H. M. Srivastava**, Classes of meromorphically multivalent functions associated with the generalized hypergeometric function, *Math. Comput. Model.*, **39**, (2004), 21-34.

[12] **S. S. Miller and P. T. Mocanu**, On some classes of first order differential subordinations, *Michigan Math. J.*, **32**, (1985), 185-195.

[13] **S. S. Miller and P. T. Mocanu**, Differential Subordinations: Theory and Applications, *Series on Monographs and Texbooks in Pure and Applied Mathematics*, **225**, (2000.)

[14] **K. S. Padmanabhan and R. Parvatham**, Some applications of differential subordination, *Bull. Aust. Math. Soc.*, **32**, (1985), 321-330.

[15] **R. K. Raina and H. M. Srivastava**, A new class of meromorphically multivalent functions with applications to generalized hypergeometric functions, *Math. Comput. Model.*, **43**, (2006), 350-356.

[16] **H. M. Srivastava, D. -G. Yang, and N. E. Xu**, Some subclasses of meromorphically multivalent functions associated with a linear operator, *Appl. Math. Comput.*, **195**, (2008), 11-23.

[17] **B. A. Uralegaddi and C. Somanatha**, New Criteria for meromorphic starlike functions, *Bull. Aust. Math. Soc.*, **43**, (1991), 137-140.

[18] **Z. G. Wang, Y. P. Jiang, and H. M. Srivastava**, Some subclasse of meromorphically multivalent functions associated with the generalized hypergeometric function, *Comput. Math. Appl.*, **57**, (2009), 571-586.

[19] **Z. -Z. Zou and Z. R. Wu**, On meromorphically starlike functions and functions meromorphically starlike with respect to symmetric conjuagate points, *J. Math. Anal. Appl.*, **261**, (2001), 17-27.

[20] **Z. -Z. Zou and Z. R. Wu**, On functions meromorphically starlike with respect to symmetric points, *J. Math. Res. Expos.*, **23**, (2003), 71-76.

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