sciendo

TATRA MOUNTAINS
Mathematical Publications
DOI: 10.2478/tmmp-2020-0021
Tatra Mt. Math. Publ. **76** (2020), 95–114

ON THE OSCILLATION OF CONFORMABLE IMPULSIVE VECTOR PARTIAL DIFFERENTIAL EQUATIONS

George E. Chatzarakis[1] — Kandhasamy Logaarasi[2] — Thangaraj Raja[2] — Vadivel Sadhasivam[2]

[1]Department of Electrical and Electronic Engineering Educators, School of Pedagogical and Technological Education (ASPETE), 14121, Athens, GREECE

[2]PG and Research Department of Mathematics, Thiruvalluvar Government Arts College, (Affiliated to Periyar University, Salem - 636 011) Rasipuram - 637 401, Namakkal (Dt), Tamil Nadu, INDIA.

ABSTRACT. In this paper, we study the oscillations of a class of conformable impulsive vector partial functional differential equations. For this class, our approach is to reduce the multi-dimensional oscillation problems to that of one dimensional impulsive delay differential inequalities by applying inner product reducing dimension method and an impulsive differential inequality technique. We provide an example to illustrate the effectiveness of our main results.

1. Introduction

The theory of fractional differential equations is considered as an important tool in modelling real life phenomena. It is well-known that fractional differential equations are a more general form of the integer order differential equations, extending those equations to an arbitrary (non-integer) order. Many important mathematical models use fractional order derivatives. But the most frequently used definitions of the fractional derivative are the Riemann-Liouville derivative & the Caputo derivative [6, 7]. However, the fractional derivatives thus defined, have seemed too complex and lack some fundamental properties, like the product and the chain rule. Thus, in 2014, Khalil [15] et. al introduced a new fractional

2010 Mathematics Subject Classification: 35B05, 35L70, 35R10, 35R12.
Keywords: conformable differential equations, neutral partial differential equations, oscillation, impulse, vector.

derivative called the conformable derivative which closely resembles the classical derivative. In the recent years, many researchers have found that fractional differential equations constitute a more accurate description of real world phenomena. Nowadays, they are extensively used in physics, electrochemistry, control theory and electromagnetic fields [16, 26].

The theory of impulsive differential equations has gained importance in mathematical models of processes and phenomena in optimal control, physics, chemical technology, population dynamics, biotechnology, electrical networks and economics. They offer a more natural description of the observed phenomena in these systems. The theory of impulsive differential equations is much richer than the corresponding theory of differential equations without impulse effects and has many real world applications [2, 18, 29–31].

The early work on the oscillation theory of impulsive differential equations appeared in 1989, in [12]. The first paper on impulsive partial differential equations [10] was published in 1991. Several authors worked on the oscillatory behaviour of impulsive partial differential equations with delays [11, 14, 21, 23, 28]. For the essential background on the oscillation theory of differential equations, we refer the reader to the monographs [17, 32, 33] and the references cited therein [3, 5, 9, 22].

In 1970, Domšlak introduced the concept of H-oscillation to study the oscillatory character of vector differential equations, where H is a unit vector in $\mathbb{R}^M$. We refer the reader to [8, 20, 24, 25] for the background in the oscillation of vector differential equations. However, there are only a few papers [4, 19, 27] dealing impulsive vector partial differential equations.

1.1. Formulation of the problems

To the best of our knowledge, there are no known oscillation results, for conformable nonlinear vector partial differential equations with impulse effects. This shortage has been the motivation that has led us to study the model of the form

$$\begin{aligned}
&\frac{\partial^\alpha}{\partial t^\alpha}\left[r(t)\frac{\partial^\alpha}{\partial t^\alpha}\Big(U(x,t)+\lambda(t)U\big(x,\tau(t)\big)\Big)\right]+\int_a^b q(x,t,\xi)U\big(x,\sigma(t,\xi)\big)\,\mathrm{d}\eta(\xi)\\
&= a(t)\Delta U(x,t)+\sum_{i=1}^{n} b_i(t)\Delta U\big(x,\rho_i(t)\big)+F(x,t), \qquad t\neq t_k\\
&U(x,t_k^+)=\alpha_k\big(x,t_k,U(x,t_k)\big),\\
&\frac{\partial^\alpha}{\partial t^\alpha}U(x,t_k^+)=\beta_k\left(x,t_k,\frac{\partial^\alpha}{\partial t^\alpha}U(x,t_k)\right), \quad k=1,2,\dots,(x,t)\in\Omega\times\mathbb{R}_+\equiv G.
\end{aligned} \tag{1}$$

Here Ω is a bounded domain in $\mathbb{R}^M$ with a piecewise smooth boundary $\partial\Omega$, Δ is the Laplacian in the Euclidean N-space $\mathbb{R}^N$, and the integral in (1) is a Stieltjes integral. Moreover, we consider the following boundary condition

$$\frac{\partial U(x,t)}{\partial \gamma} + \mu(x,t)U(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}_+, \tag{2}$$

where γ is the unit exterior normal vector to $\partial\Omega$, $\mu(x,t) \in C(\partial\Omega \times \mathbb{R}_+, \mathbb{R}_+)$ and $\mathbb{R}_+ = [0,+\infty)$ and also $\frac{\partial^\alpha}{\partial t^\alpha}$ denotes the conformable partial derivative of order $\alpha, 0 < \alpha \le 1$.

Next, we define the following set of conditions which we assume to hold, throughout the paper.

(**A₁**) $r(t) \in C^\alpha\big(\mathbb{R}_+, (0,+\infty)\big)$ with $\int\limits_{t_0}^{+\infty} s^{\alpha-1}\frac{1}{r(s)}\,\mathrm{d}s = +\infty$, $q(x,t,\xi) \in C\big(\bar{\Omega} \times \mathbb{R}_+ \times [a,b], \mathbb{R}_+\big)$, $Q(t,\xi) = \min\limits_{x\in\bar{\Omega}} q(x,t,\xi)$, $\sigma(t,\xi) \le t$ for $\xi \in [a,b]$, $\sigma(t,\xi) \in C\big(\mathbb{R}_+ \times [a,b], \mathbb{R}\big)$, $\sigma(t,\xi)$ is non-decreasing with respect to t and ξ respectively, and

$$\liminf_{t\to+\infty,\ \xi\in[a,b]} \sigma(t,\xi) = +\infty.$$

There exists a function $\theta(t) \in C^\alpha(\mathbb{R}_+, \mathbb{R}_+)$ satisfying $\theta(t) \leqslant \sigma(t,a)$, with $T_\alpha\big(\theta(t)\big) > 0$ and $\lim\limits_{t\to+\infty} \theta(t) = +\infty$.

(**A₂**) $a(t), b_i(t) \in PC(\mathbb{R}_+, \mathbb{R}_+)$, $i = 1,2,\dots,n$, where PC denotes the class of functions which are piecewise continuous in t with discontinuities of the first kind only at $t = t_k$, $k = 1,2,\dots$, and left continuous at $t = t_k$, $k = 1,2,\dots$

(**A₃**) $\rho_i(t) \in C(\mathbb{R}_+, \mathbb{R})$, $\lim\limits_{t\to+\infty} \rho_i(t) = +\infty$ for $i = 1,2,\dots,n$, $\eta(\xi) : [a,b] \to \mathbb{R}$ is nondecreasing, $F \in C(\bar{G}, \mathbb{R}^M)$, $f_H(x,t) \in C(\bar{G}, \mathbb{R})$ and $\int\limits_\Omega f_H(x,t)\,\mathrm{d}x \le 0$.

(**A₄**) All the components of $U(x,t)$ and their derivative $\dfrac{\partial^\alpha}{\partial t^\alpha}U(x,t)$ are piecewise continuous in t with discontinuities of the first kind only at $t = t_k$, $k = 1,2,\dots$, and left continuous at $t = t_k$

$$U(x,t_k) = U(x,t_k^-), \quad \frac{\partial^\alpha}{\partial t^\alpha}U(x,t_k) = \frac{\partial^\alpha}{\partial t^\alpha}U(x,t_k^-), \quad k = 1,2,\dots$$

(**A₅**) $\alpha_k,\ \beta_k \in PC(\bar{\Omega} \times \mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+)$ for $k = 1,2,\dots$, and there exist constants c_k, c_k^*, d_k, d_k^* such that for $k = 1,2,\dots$,

$$c_k^* \leqslant \frac{\alpha_k\left(x,t_k,U(x,t_k)\right)}{U(x,t_k)} \leqslant c_k, \qquad d_k^* \leqslant \frac{\beta_k\left(x,t_k,\dfrac{\partial^\alpha U(x,t_k)}{\partial t^\alpha}\right)}{\dfrac{\partial^\alpha U(x,t_k)}{\partial t^\alpha}} \leqslant d_k.$$

Definition 1.1 ([33])**.** By a **solution** of (1) − (2), we mean a function $U(x,t) \in C^{2\alpha}(\overline{\Omega} \times [t_1,+\infty),\mathbb{R}^M) \cap C(\overline{\Omega} \times [\widehat{t_1},+\infty),\mathbb{R}^M)$ which satisfies (1), where

$$t_1 := \min\left\{0, \inf_{t\geq 0} \tau(t), \min_{1\leqslant i\leqslant n}\left\{\inf_{t\geqslant 0} \rho_i(t)\right\}\right\}$$

and

$$\widehat{t_1} := \min\left\{0, \min_{\xi\in[a,b]}\left\{\inf_{t\geqslant 0} \sigma(t,\xi)\right\}\right\}.$$

Now based on this definition of a solution, we can precisely define what we mean by *H-oscillation*.

Definition 1.2 ([33])**.** Let H be a fixed unit vector in $\mathbb{R}^M$. A solution $U(x,t)$ of (1), (2) is said to be H-oscillatory in G if the inner product $\langle U(x,t), H\rangle$ has a zero in

$$\Omega \times [t,+\infty) \quad \text{for } t > 0.$$

Otherwise $U(x,t)$ is said to be H-nonoscillatory.

Definition 1.3 ([15])**.** Given $f : [0,\infty) \to \mathbb{R}$. Then the "conformable derivative" of f of order α is defined by

$$T_\alpha(f)(t) = \lim_{\epsilon\to 0} \frac{f(t+\epsilon t^{1-\alpha}) - f(t)}{\epsilon},$$

for all $t > 0, \alpha \in (0,1]$.

If f is α-differentiable in some $(0,a), a > 0$ and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists, then we define

$$f^{(\alpha)}(0) = \lim_{t\to 0^+} f^{(\alpha)}(t).$$

Definition 1.4. $I_\alpha^a(f)(t) = I_1^a(t^{\alpha-1}f) = \int_a^t \frac{f(x)}{x^{1-\alpha}}\,\mathrm{d}x$, where the integral is the usual Riemann improper integral, and $\alpha \in (0,1)$.

Definition 1.5 ([1])**.** Let f be a function with n variables $x_1, x_2, \ldots, x_n$. Then the conformable partial derivative of f of order $0 < \alpha \leq 1$ in x_i is defined as follows

$$\frac{\partial^\alpha}{\partial x_i^\alpha} f(x_1,x_2,\ldots,x_n) = \lim_{\epsilon\to 0} \frac{f(x_1,x_2,\ldots,x_{i-1},x_i+\epsilon x_i^{1-\alpha},\ldots,x_n) - f(x_1,x_2,\ldots,x_n)}{\epsilon}.$$

Conformable derivatives have the following properties:

Theorem 1.6. *Let $\alpha \in (0,1]$ and f, g be α- differentiable at some point $t > 0$. Then:*

(i) $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$, *for all* $a, b \in \mathbb{R}$.

(ii) $T_\alpha(t^p) = pt^{p-\alpha}$ *for all* $p \in \mathbb{R}$.

(iii) $T_\alpha(\lambda) = 0$ *for all constant functions* $f(t) = \lambda$.

(iv) $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$.

(v) $T_\alpha\left(\frac{f}{g}\right) = \dfrac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$.

(vi) *If f is differentiable, then* $T_\alpha(f)(t) = t^{1-\alpha}\dfrac{\mathrm{d}f}{\mathrm{d}t}(t)$.

Next, we consider the following lemma, which will help us establish our main results.

Lemma 1.7 ([13]). *If X and Y are nonnegative, then*

$$\begin{aligned} X^\delta - \delta XY^{\delta-1} + (\delta - 1)Y^\delta &\geqslant 0, \qquad \text{if} \quad \delta > 1, \\ X^\delta - \delta XY^{\delta-1} - (1 - \delta)Y^\delta &\leqslant 0, \qquad \text{if} \quad 0 < \delta < 1. \end{aligned}$$

In both cases, equality holds if and only if $X = Y$.

For convenience, we use the following notations:

$$\begin{aligned} u_H(x,t) &= \langle U(x,t), H\rangle, & F(t) &= c_0 \int_a^b Q(t,\xi)\,\mathrm{d}\eta(\xi), \\ f_H(x,t) &= \langle F(x,t), H\rangle, & R_H(t) &= \frac{1}{|\Omega|}\int_\Omega u_H(x,t)\,\mathrm{d}x, \end{aligned}$$

where

$$|\Omega| = \int_\Omega \mathrm{d}x.$$

2. Main Results

In this section, we present some sufficient conditions for the H-oscillation of all solutions of the problem (1) – (2).

Lemma 2.1. *Let H be a fixed unit vector in $\mathbb{R}^M$ and let $U(x,t)$ be a solution of* (1).

(i) *If $u_H(x,t)$ is eventually positive, then $u_H(x,t)$ satisfies the scalar impulsive conformable partial differential inequality*

$$\left.\begin{aligned} &\frac{\partial^\alpha}{\partial t^\alpha}\left(r(t)\frac{\partial^\alpha}{\partial t^\alpha}\Big(u_H(x,t)+\lambda(t)u_H\big(x,\tau(t)\big)\Big)\right)+\int_a^b Q(t,\xi)u_H\big(x,\sigma(t,\xi)\big)\,\mathrm{d}\eta(\xi)\\ &\qquad -a(t)\Delta u_H(x,t)-\sum_{i=1}^{n} b_i(t)\Delta u_H\big(x,\rho_i(t)\big)\leqslant f_H(x,t), \qquad t\neq t_k,\\ &c_k^*\leqslant\frac{u_H(x,t_k^+)}{u_H(x,t_k)}\leqslant c_k, \qquad d_k^*\leqslant\frac{\frac{\partial^\alpha}{\partial t^\alpha}u_H(x,t_k^+)}{\frac{\partial^\alpha}{\partial t^\alpha}u_H(x,t_k)}\leqslant d_k, \qquad k=1,2,\dots \end{aligned}\right\} \tag{3}$$

(ii) *If $u_H(x,t)$ is eventually negative, then $u_H(x,t)$ satisfies the scalar impulsive conformable partial differential inequality*

$$\left.\begin{aligned} &\frac{\partial^\alpha}{\partial t^\alpha}\left(r(t)\frac{\partial^\alpha}{\partial t^\alpha}\Big(u_H(x,t)+\lambda(t)u_H\big(x,\tau(t)\big)\Big)\right)+\int_a^b Q(t,\xi)u_H\big(x,\sigma(t,\xi)\big)\,\mathrm{d}\eta(\xi)\\ &\qquad -a(t)\Delta u_H(x,t)-\sum_{i=1}^{n} b_i(t)\Delta u_H\big(x,\rho_i(t)\big)\geqslant f_H(x,t), \qquad t\neq t_k,\\ &c_k^*\geqslant\frac{u_H(x,t_k^+)}{u_H(x,t_k)}\geqslant c_k, \qquad d_k^*\geqslant\frac{\frac{\partial^\alpha}{\partial t^\alpha}u_H(x,t_k^+)}{\frac{\partial^\alpha}{\partial t^\alpha}u_H(x,t_k)}\geqslant d_k, \qquad k=1,2,\dots \end{aligned}\right\} \tag{4}$$

P r o o f. (i) Let $u_H(x,t)$ be eventually positive.
Case(1): $t\neq t_k,\ k=1,2,\dots$ Taking the inner product of (1) and H, we have

$$\begin{aligned} &\frac{\partial^\alpha}{\partial t^\alpha}\left[r(t)\frac{\partial^\alpha}{\partial t^\alpha}\Big(\langle U(x,t),H\rangle+\lambda(t)\big\langle U\big(x,\tau(t)\big),H\big\rangle\Big)\right]\\ &+\int_a^b q(x,t,\xi)\big\langle U\big(x,\sigma(t,\xi)\big),H\big\rangle\,\mathrm{d}\eta(\xi)=a(t)\Delta\langle U(x,t),H\rangle\\ &+\sum_{i=1}^{n} b_i(t)\Delta\big\langle U\big(x,\rho_i(t)\big),H\big\rangle+\langle F(x,t),H\rangle, \qquad t\neq t_k, \end{aligned}$$

that is,

$$\begin{aligned} &\frac{\partial^\alpha}{\partial t^\alpha}\left[r(t)\frac{\partial^\alpha}{\partial t^\alpha}\Big(u_H(x,t)+\lambda(t)u_H\big(x,\tau(t)\big)\Big)\right]+\int_a^b q(x,t,\xi)u_H\big(x,\sigma(t,\xi)\big)\,\mathrm{d}\eta(\xi)\\ &=a(t)\Delta u_H(x,t)+\sum_{i=1}^{m} b_i(t)\Delta u_H\big(x,\rho_i(t)\big)+f_H(x,t), \qquad t\neq t_k. \end{aligned} \tag{5}$$

By condition (A_1), we have

$$\int_a^b q(x,t,\xi)u_H\big(x,\sigma(t,\xi)\big)\,\mathrm{d}\eta(\xi) \geqslant \int_a^b Q(t,\xi)u_H\big(x,\sigma(t,\xi)\big)\,\mathrm{d}\eta(\xi). \tag{6}$$

From (5) and (6), it follows that

$$\left.\begin{aligned} \frac{\partial^\alpha}{\partial t^\alpha}\bigg(r(t)\frac{\partial^\alpha}{\partial t^\alpha}\Big(u_H(x,t)+\lambda(t)u_H\big(x,\tau(t)\big)\Big)\bigg)+\int_a^b Q(t,\xi)u_H\big(x,\sigma(t,\xi)\big)\,\mathrm{d}\eta(\xi)\\ -a(t)\Delta u_H(x,t)-\sum_{i=1}^n b_i(t)\Delta u_H\big(x,\rho_i(t)\big)\leqslant f_H(x,t), \qquad t\neq t_k. \end{aligned}\right\} \tag{7}$$

Case(2): $t = t_k,\ k = 1,2,\dots$ Taking the inner product of (1) and H and using (A_5), we get

$$c_k^* \leqslant \frac{\langle U(x,t_k^+),H\rangle}{\langle U(x,t_k),H\rangle} \leqslant c_k, \qquad d_k^* \leqslant \frac{\left\langle \frac{\partial^\alpha}{\partial t^\alpha}U(x,t_k^+),H\right\rangle}{\left\langle \frac{\partial^\alpha}{\partial t^\alpha}U(x,t_k),H\right\rangle} \leqslant d_k,$$

that is

$$c_k^* \leqslant \frac{u_H(x,t_k^+)}{u_H(x,t_k)} \leqslant c_k, \qquad d_k^* \leqslant \frac{\frac{\partial^\alpha}{\partial t^\alpha}u_H(x,t_k^+)}{\frac{\partial^\alpha}{\partial t^\alpha}u_H(x,t_k)} \leqslant d_k. \tag{8}$$

Therefore, combining (7) and (8) we immediately obtain (3), which shows that $u_H(x,t)$ satisfies the scalar impulsive conformable partial differential inequality (3).

(ii) The proof is similar to case (i) and thus, it is omitted. The proof is complete. □

Let H be a fixed unit vector in $\mathbb{R}^M$. Then the inner product of the boundary condition (2) and H yields the following boundary condition:

$$\frac{\partial u_H(x,t)}{\partial\gamma}+\mu(x,t)u_H(x,t)=0, \quad (x,t)\in\partial\Omega\times\mathbb{R}_+. \tag{9}$$

Lemma 2.2. *Let H be a fixed unit vector in $\mathbb{R}^M$. If the scalar impulsive conformable partial differential inequality* (3) *[(4)] has no eventually positive solutions [negative solutions] and satisfies the boundary condition* (9)*, then every solution $U(x,t)$ of the boundary value problem* (1) − (2) *is H-oscillatory in G.*

Proof. Suppose to the contrary that there is a H-nonoscillatory solution $U(x,t)$ of the boundary value problem (1) – (2) in G, then $u_H(x,t)$ is eventually positive or eventually negative. If $u_H(x,t)$ is eventually positive, by Lemma 2.1, we easily obtain that $u_H(x,t)$ satisfies the scalar impulsive partial differential inequality (3). On the other hand, it is easy to see that $u_H(x,t)$ satisfies the boundary condition (9). This is a contradiction to the hypothesis.

Similarly, if $u_H(x,t)$ is eventually negative, using Lemma 2.1, we easily obtain that $u_H(x,t)$ satisfies the scalar impulsive partial differential inequality (4). It is obvious that $u_H(x,t)$ satisfies the boundary condition (9). This is a contradiction. The proof is complete. □

Theorem 2.3. *Let H be a fixed unit vector in $\mathbb{R}^M$. If the impulsive conformable differential inequality*

$$\left.\begin{aligned} &T_\alpha\left[r(t)T_\alpha\big(Z_H(t)\big)\right] + F(t)Z_H\big(\theta(t)\big) \leqslant 0, \quad t \neq t_k,\\ &c_k^* \leqslant \frac{Z_H(t_k^+)}{Z_H(t_k)} \leqslant c_k, \quad d_k^* \leqslant \frac{T_\alpha(Z_H(t_k^+))}{T_\alpha(Z_H(t_k))} \leqslant d_k, \quad k = 1,2,\dots, \end{aligned}\right\} \tag{10}$$

has no eventually positive solutions and the impulsive conformable differential inequality

$$\left.\begin{aligned} &T_\alpha\left[r(t)T_\alpha\big(Z_H(t)\big)\right] + F(t)Z_H\big(\theta(t)\big) \geqslant 0, \quad t \neq t_k,\\ &c_k^* \geqslant \frac{Z_H(t_k^+)}{Z_H(t_k)} \geqslant c_k, \quad d_k^* \geqslant \frac{T_\alpha(Z_H(t_k^+))}{T_\alpha(Z_H(t_k))} \geqslant d_k, \quad k = 1,2,\dots, \end{aligned}\right\} \tag{11}$$

has no eventually negative solutions satisfying the boundary condition (9), *then every solution $U(x,t)$ of the problem* (1), (2) *is H-oscillatory in G.*

Proof. Suppose that there exists a solution $U(x,t)$ of (1) – (2), which is not H-oscillatory in G. Without loss of generality, we can assume that $u_H(x,t) > 0$ in $\Omega \times [t_0, +\infty)$, for some $t_0 > 0$. Then, from the assumption that there exists a $t_1 > t_0$ such that $\sigma(t,\xi) \geqslant t_0$, for $(t,\xi) \in [t_1,+\infty) \times [a,b]$ and $\tau(t) \geqslant t_0$, $\rho_i(t) \geqslant t_0$, $i = 1,2,\dots,n$ for $t \geqslant t_1$, we have that

$$u_H\big(x,\sigma(t,\xi)\big) > 0, \quad u_H\big(x,\tau(t)\big) > 0 \quad \text{and} \quad u_H\big(x,\rho_i(t)\big) > 0,$$

for $x \in \Omega$, $t \in [t_1,+\infty)$, $\xi \in [a,b]$, $i = 1,2,\dots,n$.

For $t \geqslant t_0$ and $t \neq t_k$ for $k = 1, 2, \dots$, we multiply both sides of inequality (3) by $\dfrac{1}{|\Omega|}$ and integrate with respect to x over the domain Ω to attain

$$\left.\begin{aligned}&t^{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\left[r(t)t^{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{|\Omega|}\int_\Omega u_H(x,t)\,\mathrm{d}x + \lambda(t)\frac{1}{|\Omega|}\int_\Omega u_H\big(x,\tau(t)\big)\,\mathrm{d}x\right)\right]\\&+\frac{1}{|\Omega|}\int_\Omega\int_a^b Q(t,\xi)u_H\big(x,\sigma(t,\xi)\big)\,\mathrm{d}\eta(\xi)\,\mathrm{d}x - a(t)\frac{1}{|\Omega|}\int_\Omega \Delta u_H(x,t)\,\mathrm{d}x\\&-\sum_{i=1}^n b_i(t)\frac{1}{|\Omega|}\int_\Omega \Delta u_H\big(x,\rho_i(t)\big)\,\mathrm{d}x \leqslant \frac{1}{|\Omega|}\int_\Omega f_H(x,t)\,\mathrm{d}x, \quad t \neq t_k.\end{aligned}\right\} \tag{12}$$

Using Green's formula and boundary condition (9), we have that

$$\int_\Omega \Delta u_H(x,t)\,\mathrm{d}x = \int_{\partial\Omega}\frac{\partial u_H(x,t)}{\partial\gamma}\,\mathrm{d}S = -\int_{\partial\Omega}\mu(x,t)u_H(x,t)\,\mathrm{d}S \leqslant 0. \tag{13}$$

For $i = 1, 2, \dots, n$,

$$\begin{aligned}\int_\Omega \Delta u_H(x,\rho_i(t))\,\mathrm{d}x &= \int_{\partial\Omega}\frac{\partial u_H\big(x,\rho_i(t)\big)}{\partial\gamma}\,\mathrm{d}S,\\&= -\int_{\partial\Omega}\mu\big(x,\rho_i(t)\big)u_H\big(x,\rho_i(t)\big)\,\mathrm{d}S \leqslant 0, \quad t \geqslant t_0\end{aligned} \tag{14}$$

where $\mathrm{d}S$ is the surface element on $\partial\Omega$. Moreover, by (A_3), $\int_\Omega f_H(x,t)\,\mathrm{d}x \leqslant 0$. Combining (12)–(14) we get

$$\begin{aligned}&t^{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\left[r(t)t^{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\Big(R_H(t)+\lambda(t)R_H\big(\tau(t)\big)\Big)\right]\\&+\int_a^b Q(t,\xi)R_H\big(\sigma(t,\xi)\big)\,\mathrm{d}\eta(\xi) \leqslant 0, \quad t \geq t_0.\end{aligned}$$

Setting $Z_H(t) = R_H(t) + \lambda(t)R_H\big(\tau(t)\big)$, we have

$$T_\alpha\left[r(t)T_\alpha\big(Z_H(t)\big)\right] + \int_a^b Q(t,\xi)R_H\big(\sigma(t,\xi)\big)\,\mathrm{d}\eta(\xi) \leqslant 0. \tag{15}$$

Clearly, $Z_H(t) > 0$ for $t \geqslant t_1$. Next we prove that $T_\alpha\big(Z_H(t)\big) > 0$ for $t \geqslant t_2$. In fact, assume there exists $K \geqslant t_2$ such that $T_\alpha\big(Z_H(t)\big) \leqslant 0$. Then, we have

$$T_\alpha\left[r(t)T_\alpha\big(Z_H(t)\big)\right] \leqslant -\int_a^b Q(t,\xi)R_H\big(\sigma(t,\xi)\big)\,\mathrm{d}\eta(\xi), \tag{16}$$

from which, we obtain

$$T_\alpha \left[r(t)T_\alpha\big(Z_H(t)\big)\right] \leqslant 0. \tag{17}$$

From (17), we have

$$r(t)T_\alpha\big(Z_H(t)\big) \leq r(K)T_\alpha\big(Z_H(K)\big) \leqslant 0, \quad t \geqslant K.$$

Thus

$$Z_H(t) \leqslant Z_H(K) + r(K)K^{1-\alpha}Z'_H(K)\int_K^t s^{\alpha-1}\frac{\mathrm{d}s}{r(s)}, \qquad \text{for} \qquad t \geqslant K.$$

Also, from (A_1), we have $\lim_{t\to\infty} Z_H(t) = -\infty$, which contradicts the fact that $Z_H(t) > 0$, for $t > 0$. Hence $T_\alpha\big(Z_H(t)\big) > 0$ and since $\tau(t) \leqslant t$ for $t \geqslant t_1$, we have

$$R_H(t) = Z_H(t) - \lambda(t)R_H\big(\tau(t)\big) \geqslant \big(1-\lambda(t)\big)Z_H(t)$$

and

$$R_H\big(\sigma(t,\xi)\big) \geqslant c_0 Z_H\big(\sigma(t,\xi)\big),$$

where $c_0 = 1 - \lambda(t)$ is a positive constant.
Therefore from (15), we have

$$T_\alpha \left[r(t)T_\alpha\big(Z_H(t)\big)\right] + c_0\int_a^b Q(t,\xi)Z_H\big(\sigma(t,\xi)\big)\,\mathrm{d}\eta(\xi) \leqslant 0. \quad t \geq t_0.$$

From (A_1) and $T_\alpha\big(Z_H(t)\big) > 0$, we have

$$Z_H\big(\sigma(t,\xi)\big) \geqslant Z_H\big(\sigma(t,a)\big) > 0, \quad \xi \in [a,b] \quad \text{and} \quad \theta(t) \leqslant \sigma(t,a) \leqslant t.$$

Thus, $Z_H\big(\theta(t)\big) \leqslant Z_H\big(\sigma(t,a)\big)$ and therefore

$$T_\alpha \left[r(t)T_\alpha\big(Z_H(t)\big)\right] + F(t)Z_H\big(\theta(t)\big) \leqslant 0, \quad t \geq t_1. \tag{18}$$

For $t \geqslant t_0$, $t = t_k,\ k = 1, 2, \dots$, multiplying both sides of inequality (3) by $\dfrac{1}{|\Omega|}$ and integrating with respect to x over the domain Ω, we obtain

$$c_k^* \leqslant \frac{R_H(t_k^+)}{R_H(t_k)} \leqslant c_k, \qquad d_k^* \leqslant \frac{T_\alpha(R_H(t_k^+))}{T_\alpha(R_H(t_k))} \leqslant d_k.$$

Since $Z_H(t) = R_H(t) + \lambda(t)R_H\big(\tau(t)\big)$, we have that

$$c_k^* \leqslant \frac{Z_H(t_k^+)}{Z_H(t_k)} \leqslant c_k, \qquad d_k^* \leqslant \frac{T_\alpha(Z_H(t_k^+))}{T_\alpha(Z_H(t_k))} \leqslant d_k. \tag{19}$$

Therefore (18) and (19) show that $Z_H(t) > 0$ is a positive solution of the impulsive differential inequality (10). This is a contradiction.

Suppose now, that $u_H(x,t) < 0$ is a negative solution of the impulsive partial differential inequality (4) satisfying the boundary condition (9), $(x,t) \in \Omega \times [t_0, +\infty)$, $t_0 > 0$. Applying the same procedure as above, we arrive at a contradiction. This completes the proof. □

Theorem 2.4. *If there exists a function $\psi(t) \in C^{\alpha}\big(\mathbb{R}_+, (0,+\infty)\big)$ which is nondecreasing with respect to t, such that*

$$\int_{t_0}^{+\infty} \prod_{t_0 \leqslant t_k < s} \left(\frac{d_k}{c_k^*}\right)^{-1} s^{\alpha-1} \left[\psi(s)F(s) - \frac{E^2(s)}{4G(s)}\right] \mathrm{d}s = +\infty, \tag{20}$$

where

$$E(t) = \frac{T_\alpha(\psi(t))}{\psi(t)} \quad \text{and} \quad G(t) = \frac{1}{r(t)\psi(t)},$$

then every solution of the boundary value problem (1)−(2) *is H-oscillatory in G.*

P r o o f. We show that inequality (10) has no eventually positive solution, if the conditions of Theorem 2.3 hold. Suppose that $Z_H(t)$ is an eventually positive solution of the inequality (10) then there exists a number $t_1 \geqslant t_0$ such that $Z_H\big(\theta(t)\big) > 0$ for $t \geqslant t_1$. Thus we have

$$T_\alpha\left[r(t)T_\alpha\big(Z_H(t)\big)\right] + F(t)Z_H\big(\theta(t)\big) \leqslant 0. \tag{21}$$

Define the Riccati transformation

$$W(t) := \psi(t)\frac{r(t)T_\alpha(Z_H(t))}{Z_H(\theta(t))}.$$

Then

$$W(t) \geqslant 0 \quad \text{and} \quad T_\alpha\big(W(t)\big) \leqslant T_\alpha\big(\psi(t)\big)\frac{W(t)}{\psi(t)} - \psi(t)F(t) - \frac{W^2(t)}{r(t)\psi(t)}.$$

Thus

$$T_\alpha\big(W(t)\big) \leqslant \; E(t)W(t) - F(t)\psi(t) - W^2(t)G(t) \quad \text{and} \quad W(t_k^+) \leqslant \; \frac{d_k}{c_k^*}W(t_k).$$

We define

$$S(t) = \prod_{t_0 \leqslant t_k < t} \left(\frac{d_k}{c_k^*}\right)^{-1} W(t).$$

It is clear that $W(t)$ is continuous in each interval $(t_k, t_{k+1}]$. Since $W(t_k^+) \leq \frac{d_k}{c_k^*}W(t_k)$, it follows that

$$S(t_k^+) = \prod_{t_0 \leqslant t_j \leqslant t_k} \left(\frac{d_k}{c_k^*}\right)^{-1} W(t_k^+) \leqslant \prod_{t_0 \leqslant t_j < t_k} \left(\frac{d_k}{c_k^*}\right)^{-1} W(t_k) = S(t_k)$$

and for all $t \geqslant t_0$,

$$S(t_k^-) = \prod_{t_0 \leqslant t_j \leqslant t_{k-1}} \left(\frac{d_k}{c_k^*}\right)^{-1} W(t_k^-) \leq \prod_{t_0 \leqslant t_j < t_k} \left(\frac{d_k}{c_k^*}\right)^{-1} W(t_k) = S(t_k),$$

which implies that $S(t)$ is continuous on $[t_0, +\infty)$. Also

$$\begin{aligned} & T_\alpha\big(S(t)\big) + \prod_{t_0 \leqslant t_k < t} \left(\frac{d_k}{c_k^*}\right) S^2(t)G(t) + \prod_{t_0 \leqslant t_k < t} \left(\frac{d_k}{c_k^*}\right)^{-1} F(t)\psi(t) - S(t)E(t) \\ &= \prod_{t_0 \leqslant t_k < t} \left(\frac{d_k}{c_k^*}\right)^{-1} \big[T_\alpha\big(W(t)\big) + W^2(t)G(t) - W(t)E(t) + F(t)\psi(t)\big] \\ &\leqslant 0. \end{aligned}$$

Therefore,

$$\begin{aligned} T_\alpha\big(S(t)\big) \leqslant & - \prod_{t_0 \leqslant t_k < t} \left(\frac{d_k}{c_k^*}\right) G(t)S^2(t) + S(t)E(t) \\ & - \prod_{t_0 \leqslant t_k < t} \left(\frac{d_k}{c_k^*}\right)^{-1} F(t)\psi(t). \end{aligned} \tag{22}$$

Taking

$$X(t) = \left(\prod_{t_0 \leqslant t_k < t} \left(\frac{d_k}{c_k^*}\right) G(t)\right)^{\frac{1}{2}} S(t)$$

and

$$Y(t) = \frac{E(t)}{2} \left(\prod_{t_0 \leq t_k < t} \left(\frac{d_k}{c_k^*}\right)^{-1} \frac{1}{G(t)}\right)^{\frac{1}{2}}$$

and using Lemma 1.7, we have

$$E(t)S(t) - \prod_{t_0 \leqslant t_k < t} \left(\frac{d_k}{c_k^*}\right) G(t)S^2(t) \leqslant \frac{E^2(t)}{4G(t)} \prod_{t_0 \leqslant t_k < t} \left(\frac{d_k}{c_k^*}\right)^{-1}.$$

Thus

$$T_\alpha\big(S(t)\big) \leqslant - \prod_{t_0 \leqslant t_k < t} \left(\frac{d_k}{c_k^*}\right)^{-1} \left[F(t)\psi(t) - \frac{E^2(t)}{4G(t)}\right].$$

Integrating both sides from t_0 to t, we have

$$S(t) \leqslant S(t_0) - \int_{t_0}^{t} \prod_{t_0 \leqslant t_k < s} \left(\frac{d_k}{c_k^*}\right)^{-1} s^{\alpha-1} \left[\psi(s)F(s) - \frac{E^2(s)}{4G(s)}\right] \mathrm{d}s.$$

Letting $t \to \infty$ and using (20) we have $\lim_{t\to\infty} S(t) = -\infty$, which leads to a contradiction with $S(t) \geqslant 0$ and completes the proof. □

Theorem 2.5. *Assume that there exist functions ψ and $\phi \in C^\alpha(\mathbb{R}_+, (0, +\infty))$, where ψ is nondecreasing and functions $b, B \in C^\alpha(\mathbb{B}, \mathbb{R})$, where $\mathbb{B} = \{(t,s) \colon t \geqslant s \geqslant t_0 > 0\}$ such that:*

($\mathbf{A_6}$) $B(t,t) = 0$ *and* $B(t,s) > 0$ *for all* $t > s \geqslant t_0$,

($\mathbf{A_7}$) $\dfrac{\partial B(t,s)}{\partial t} \geqslant 0$ *and* $\dfrac{\partial B(t,s)}{\partial s} \leqslant 0$,

($\mathbf{A_8}$) $-\dfrac{\partial B(t,s)}{\partial s} = b(t,s)\sqrt{B(t,s)}$.

If

$$\begin{aligned}\limsup_{t\to+\infty} \frac{1}{B(t,t_0)} \int_{t_0}^{t} \prod_{t_0 \leqslant t_k < r} \left(\frac{d_k}{c_k^*}\right)^{-1} \Bigg(& F(r)\psi(r)B(t,r)\phi(r) \\ & - \frac{1}{4}\Big[r^{1-\alpha}\phi'(r)\sqrt{B(t,r)} - b(t,r)r^{1-\alpha}\phi(r) \\ & + (1-\alpha)r^{-\alpha}\sqrt{B(t,r)}\phi(r) + E(r)\phi(r)\sqrt{B(t,r)}\Big]^2 \cdot \frac{1}{G(r)\phi(r)}\Bigg) \mathrm{d}r \\ = +\infty, &\end{aligned} \tag{23}$$

then every solution of the boundary value problem (1)–(2) *is H-oscillatory in G.*

Proof. Let $Z_H(t)$ be an eventually positive solution of (10). Proceeding as in the proof of Theorem 2.4 we obtain

$$\begin{aligned}T_\alpha(S(t)) \leqslant -\prod_{t_0 \leqslant t_k < t} \left(\frac{d_k}{c_k^*}\right) G(t)S^2(t) + S(t)E(t) \\ - \prod_{t_0 \leqslant t_k < t} \left(\frac{d_k}{c_k^*}\right)^{-1} F(t)\psi(t).\end{aligned}$$

Multiplying the above inequality by $B(t,s)\phi(s)$, for $t \geqslant s \geqslant K$ and integrating from K to t, we have

$$\begin{aligned}
&\int_K^t r^{1-\alpha} S'(r) B(t,r)\phi(r)\,\mathrm{d}r \\
&\leqslant -\int_K^t \prod_{t_0 \leqslant t_k < r} \left(\frac{d_k}{c_k^*}\right) G(r) S^2(r) B(t,r)\phi(r)\,\mathrm{d}r \\
&\quad + \int_K^t S(r)E(r)B(t,r)\phi(r)\,\mathrm{d}r \\
&\quad - \int_K^t \prod_{t_0 \leqslant t_k < r} \left(\frac{d_k}{c_k^*}\right)^{-1} F(r)\psi(r)B(t,r)\phi(r)\,\mathrm{d}r.
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\int_K^t \prod_{t_0 \leqslant t_k < r} \left(\frac{d_k}{c_k^*}\right)^{-1} F(r)\psi(r)B(t,r)\phi(r)\,\mathrm{d}r \\
&\leqslant S(K)B(t,K)K^{1-\alpha}\phi(K) \\
&\quad + \int_K^t \left[\frac{\partial B(t,r)}{\partial r} r^{1-\alpha}\phi(r) + B(t,r) r^{1-\alpha}\phi'(r) + (1-\alpha) r^{-\alpha} B(t,r)\phi(r)\right] S(r)\,\mathrm{d}r \\
&\quad + \int_K^t E(r)B(t,r)\phi(r)S(r)\,\mathrm{d}r - \int_K^t \prod_{t_0 \leqslant t_k < r} \left(\frac{d_k}{c_k^*}\right) G(r)S^2(r)B(t,r)\phi(r)\,\mathrm{d}r.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_K^t \prod_{t_0 \leqslant t_k < r} \left(\frac{d_k}{c_k^*}\right)^{-1} F(r)\psi(r)B(t,r)\phi(r)\,\mathrm{d}r \\
&\quad - \frac{1}{4}\int_K^t \prod_{t_0 \leqslant t_k < r} \left(\frac{d_k}{c_k^*}\right)^{-1} \Big[r^{1-\alpha}\phi'(r)\sqrt{B(t,r)} - b(t,r) r^{1-\alpha}\phi(r) \qquad (24) \\
&\quad + (1-\alpha) r^{-\alpha}\sqrt{B(t,r)}\phi(r) + E(r)\phi(r)\sqrt{B(t,r)} \Big]^2 \cdot \frac{1}{G(r)\phi(r)}\,\mathrm{d}r \\
&\leqslant S(K)B(t,K)K^{1-\alpha}\phi(K).
\end{aligned}$$

From (24), for $t \geqslant K \geqslant t_0$, we have

$$\begin{aligned}
&\frac{1}{B(t,t_0)} \int_{t_0}^{t} \prod_{t_0 \leqslant t_k < r} \left(\frac{d_k}{c_k^*}\right)^{-1} \\
&\quad \times \Bigg[F(r)\psi(r)B(t,r)\phi(r) - \frac{1}{4}\Big[r^{1-\alpha}\phi'(r)\sqrt{B(t,r)} - b(t,r)r^{1-\alpha}\phi(r) \\
&\quad +(1-\alpha)r^{-\alpha}\sqrt{B(t,r)}\phi(r) + E(r)\phi(r)\sqrt{B(t,r)}\Big]^2 \cdot \frac{1}{G(r)\phi(r)} \Bigg] \mathrm{d}r \\
&= \frac{1}{B(t,t_0)} \left[\int_{t_0}^{K} + \int_{K}^{t} \right] \Bigg\{ \prod_{t_0 \leq t_k < r} \left(\frac{d_k}{c_k^*}\right)^{-1} \Bigg(F(r)\psi(r)B(t,r)\phi(r) \\
&\quad - \frac{1}{4}\Big[r^{1-\alpha}\phi'(r)\sqrt{B(t,r)} - b(t,r)r^{1-\alpha}\phi(r) \\
&\quad +(1-\alpha)r^{-\alpha}\sqrt{B(t,r)}\phi(r) + E(r)\phi(r)\sqrt{B(t,r)}\Big]^2 \cdot \frac{1}{G(r)\phi(r)} \Bigg) \Bigg\} \mathrm{d}r \\
&\leqslant \int_{t_0}^{K} \prod_{t_0 \leqslant t_k < r} \left(\frac{d_k}{c_k^*}\right)^{-1} F(r)\psi(r)\phi(r)\,\mathrm{d}r + \phi(K)K^{1-\alpha}S(K).
\end{aligned}$$

Letting $t \to +\infty$, we have

$$\begin{aligned}
&\limsup_{t\to+\infty} \frac{1}{B(t,t_0)} \int_{t_0}^{t} \prod_{t_0 \leqslant t_k < r} \left(\frac{d_k}{c_k^*}\right)^{-1} \Bigg(F(r)\psi(r)B(t,r)\phi(r) \\
&\quad - \frac{1}{4}\Big[r^{1-\alpha}\phi'(r)\sqrt{B(t,r)} - b(t,r)r^{1-\alpha}\phi(r) \\
&\quad +(1-\alpha)r^{-\alpha}\sqrt{B(t,r)}\phi(r) + E(r)\phi(r)\sqrt{B(t,r)}\Big]^2 \cdot \frac{1}{G(r)\phi(r)} \Bigg) \mathrm{d}r \\
&\leqslant \int_{t_0}^{K} \prod_{t_0 \leqslant t_k < r} \left(\frac{d_k}{c_k^*}\right)^{-1} F(r)\psi(r)\phi(r)\,\mathrm{d}r + \phi(K)K^{1-\alpha}S(K) \\
&< +\infty,
\end{aligned}$$

which contradicts (23). The proof of the theorem is complete. □

Choosing $\phi(r) = \psi(r) \equiv 1$, in Theorem 2.5, we obtain the following result.

Corollary 2.6. *Assume that the conditions of Theorem 2.5 hold and*

$$\limsup_{t\to+\infty}\frac{1}{B(t,t_0)}\int\limits_{t_0}^{t}\prod_{t_0\leqslant t_k<s}\left(\frac{d_k}{c_k^*}\right)^{-1}\Bigg(F(r)B(t,r)$$
$$-\frac{1}{4}\Big[(1-\alpha)r^{-\alpha}\sqrt{B(t,r)}-b(t,r)r^{1-\alpha}+E(r)\sqrt{B(t,r)}\Big]^2\cdot\frac{1}{G(r)}\Bigg)\,\mathrm{d}r=+\infty.$$

Then every solution of the boundary value problem (1) − (2) *is H-oscillatory in G.*

From Theorem 2.5 and Corollary 2.6, we can obtain several oscillatory criteria, depending on the choice of the weighted function $B(t,s)$. For example, choosing $B(t,r)=(t-r)^{\nu-1},\ t\geqslant r\geqslant t_0$, in which $\nu>2$ is an integer, then

$$b(t,r)=(\nu-1)(t-r)^{(\nu-3)/2},\quad t\geqslant r\geqslant t_0.$$

Corollary 2.6 leads to the following result.

Corollary 2.7. *If $\nu>2$ is an integer such that*

$$\limsup_{t\to+\infty}\frac{1}{(t-t_0)^{\nu-1}}\int\limits_{t_0}^{t}\prod_{t_0\leqslant t_k<r}\left(\frac{d_k}{c_k^*}\right)^{-1}(t-r)^{\nu-1}$$
$$\times\left(F(r)-\frac{1}{4G(r)}\Big[\frac{-(\nu-1)r^{1-\alpha}}{t-r}+(1-\alpha)r^{-\alpha}+E(r)\Big]^2\right)\mathrm{d}r=+\infty.$$

Then every solution of the boundary value problem (1)–(2) *is H-oscillatory in G.*

3. Example

In this section, we provide an example to illustrate our results.

Example 1. Consider the following impulsive partial differential equations

$$\begin{cases}\dfrac{\partial^{\frac12}}{\partial t^{\frac12}}\left(2\dfrac{\partial^{\frac12}}{\partial t^{\frac12}}\big(U(x,t)+\tfrac12U(x,t-\pi)\big)\right)+\dfrac{3}{4}\int\limits_{\pi/2}^{\pi}U(x,t-\xi)\,\mathrm{d}\xi=\Delta U(x,t)\\ \qquad+\dfrac{5}{4}\Delta U(x,t-\tfrac{\pi}{2})+F(x,t),\quad t\neq t_k,\quad k=1,2,\dots,\\ U(x,t_k^+)=\dfrac{k}{k+1}U(x,t_k),\\ \dfrac{\partial^\alpha}{\partial t^\alpha}U(x,t_k^+)=\dfrac{\partial^\alpha}{\partial t^\alpha}U(x,t_k),\qquad k=1,2,\dots,\end{cases}\tag{25}$$

for $(x,t) \in (0,2\pi)\times\mathbb{R}_+$, with the boundary condition

$$\frac{\partial}{\partial x}U(0,t) = \frac{\partial}{\partial x}U(2\pi,t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad t \geq 0. \tag{26}$$

Here

$$\begin{array}{rrrr} \Omega = (0,2\pi), & \mu(x,t) = 1, & N = 1, & M = 2, \\ n = 1, & \alpha = \dfrac{1}{2}, & c_k = c_k^* = \dfrac{k}{k+1}, & d_k = d_k^* = 1, \\ r(t) = 2, & \lambda(t) = \dfrac{1}{2}, & \tau(t) = t-\pi, & \sigma(t,\xi) = t-\xi, \\ Q(t,\xi) = \dfrac{3}{4}, & a(t) = 1, & b_1(t) = \dfrac{5}{4}, & \rho_1(t) = t - \frac{\pi}{2}, \end{array}$$

$$[a,b] = [\pi/2, \pi]$$

and

$$F(x,t) = \begin{pmatrix} -\cos x\,(3/2\cos t + (t-1/4)\sin t) \\ \cos x e^{-t}\left(2t + (t+1/4)e^{\pi} + 1/2e^{\pi/2}\right) \end{pmatrix}.$$

Let $H = e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then we have

$$f_H(x,t) = f_{e_1}(x,t) = -\cos x\left(\frac{3}{2}\cos t + \left(t - \frac{1}{4}\right)\sin t\right)$$

and

$$\int_\Omega f_{e_1}(x,t)\,\mathrm{d}x = -\int_\Omega \cos x\left(\frac{3}{2}\cos t + \left(t-\frac{1}{4}\right)\sin t\right)\mathrm{d}x \leq 0.$$

Take $\theta(t) = t/2$, $\psi(t) = t$. Since

$$t_0 = 1, \quad t_k = 2^k, \quad E(t) = t^{-1/2}, \quad G(t) = \frac{1}{2t}, \quad F(t) = \frac{3\pi}{16}.$$

Then hypotheses $(A_1)-(A_5)$ hold, and moreover

$$\begin{aligned} \lim_{t\to+\infty}\int_{t_0}^{t}\prod_{t_0\leqslant t_k<s}\Big(\frac{d_k^*}{c_k}\Big)^{-1}\mathrm{d}s &= \int_1^{+\infty}\prod_{1<t_k<s}\frac{k}{k+1}\,\mathrm{d}s \\ &= \int_1^{t_1}\prod_{1<t_k<s}\frac{k}{k+1}\,\mathrm{d}s + \int_{t_1^+}^{t_2}\prod_{1<t_k<s}\frac{k}{k+1}\,\mathrm{d}s + \cdots \\ &= 1 + \frac{1}{2}\times 2 + \frac{1}{2}\times\frac{2}{3}\times 2^2 + \cdots \\ &= \sum_{n=0}^{\infty}\frac{2^n}{n+1} = +\infty. \end{aligned}$$

Thus

$$\int_{1}^{+\infty} \prod_{1<t_k<s} \frac{k+1}{k} s^{-1/2} \left[\frac{3\pi s}{16} - \frac{1}{2}\right] \mathrm{d}s = +\infty.$$

Therefore all the conditions of Theorem 2.4 are satisfied and hence every solution $U(x,t)$ of the problem (25)-(26) is e_1-oscillatory in G. One such solution is

$$U(x,t) = \begin{pmatrix} \cos x \sin t \\ \cos x \ e^{-t} \end{pmatrix}.$$

We should note that above solution $U(x,t)$ is not e_2-oscillatory in G, where

$$e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

REFERENCES

[1] ATANGANA, A.—BALEANU, D.—ALSAEDI, A.: *New properties of conformable derivative*, Open Math. **13** (2015), 889–898.

[2] BENCHOHRA, M.—HENDERSON, J.—NTOUYAS, S.: *Impulsive Differential Equations and Inclusions*. Hindawi Publishing Corporation, New York, 2006.

[3] CHATZARAKIS, G. E.—LOGAARASI, K.—RAJA, T.—SADHASIVAM, V.: *Interval oscillation criteria for conformable fractional differential equations with impulses*, Appl. Math. E-Notes **19** (2019), 354–369.

[4] CHATZARAKIS, G. E.—SADHASIVAM, V.—RAJA, T.: *On the oscillation of impulsive vector partial differential equations with distributed deviating arguments*, Analysis (Berlin) **38** (2018), no. 2, 101–114.

[5] CHATZARAKIS, G.-E.—SADHASIVAM, V.—RAJA, T. —KALAIMANI, T.: *Oscillation of certain nonlinear impulsive neutral partial differential equations with continuous distributed deviating arguments and a damping term*, Dynam. Contin. Discrete Impuls. Systems **25** (2018), 329–345.

[6] CHEN, D.-X.: *Oscillation criteria of fractional differential equations*, Adv. Difference Equ. **2012** (2012), no. 33, 10 pp.

[7] GRACE, S.-R.—AGARWAL, R.-P.—WONG, P.-J.-Y.—ZAFER, A.: *On the oscillation of fractional differential equations*, Frac. Calc. Appl. Anal. **15** (2012), 222–231.

[8] DOMŠLAK, JU.-I.: *On the oscillation of solutions of vector differential equations*, Soviet Math. Dokl. **11** (1970), 839–841.

[9] CHIU, K.-S.—LI, T.: *Oscillatory and periodic solutions of differential equations with piecewise constant generalized mixed arguments*, Math. Nachr. **292** (2019), no. 10, 2153–2164.

[10] ERBE, L.-H.—FREEDMAN, H.-I.—LIU, X.-Z.—WU, J.-H.: *Comparison principles for impulsive parabolic equations with application to models of single species growth*, J. Austral. Math. Soc. Ser (B). **32** (1991), no. 4, 382–400.

[11] GAO, Z.—TENG, Z.: *Oscillation criteria of impulsive neutral parabolic equations with nonlinear diffusion coefficient*, Int. J. Nonlinear Sci., **11** (2011), no. 2, 168–172.

[12] GOPALSAMY, K.—ZHANG, B.-G.: *On delay differential equations with impulses*, J. Math. Anal. Appl. **139** (1989), 110–122.

[13] HARDY, G. H.—LITTLEWOOD, J. E.—PÓLYA, G.: *Inequalities*. Reprint of the 1952 edition. Cambridge Mathematical Library. Cambridge University Press, Cambridge, UK, 1988.

[14] KALAIMANI, T.—RAJA, T.—SADHASIVAM, V.—SAKER, S. H.: *Oscillation of impulsive neutral partial differential equations with distributed deviating arguments*, Bul. Math. Soc. Sci. Math. Roumanie (N.S.) **61** (2018), no. 109, 51–68.

[15] KHALIL, R. R.—AL HORANI, M.—YOUSEF, A.—SABABHEH, M.: *A new definition of fractional derivative*, J. Comput. Appl. Math., **264** (2014), 65–70.

[16] KILBAS, A. A.—SRIVASTAVA, H. M.—TRUJILLO, J. J.: *Theory and Applications of Fractional Differential Equations*. Elsevier, Amsterdam, 2006.

[17] LADDE, G. S.—LAKSHMIKANTHAM, V.—ZHANG, B. G.: *Oscillation Theory of Differential Equations with Deviating Arguments*. In: *Monographs and Textbooks in Pure and Applied Mathematics Vol. 110*. Marcel Dekker, Inc., New York, 1987.

[18] LAKSHMIKANTHAM, V.—BAINOV, D. D.—SIMEONOV, P. S.: *Theory of Impulsive Differential Equations*. In: *Series in Modern Applied Mathematics Vol. 6*. World Scientific Publishing Co., Inc., Teaneck, NJ, 1989.

[19] LI, W. N.—HAN, M.: *Oscillation of solutions for certain impulsive vector parabolic differential equations with delays*, J. Math. Anal. Appl., **326(1)** (2007), 363–371.

[20] LI, W. N.—HAN, M.—MENG, F. W.: *H-oscillation of solutions of certain vector hyperbolic differential equations with deviating arguments*, Appl. Math. Comput., **158** (2004), 637–653.

[21] LIU, G. J.—WANG, C. Y. : *Forced oscillation of neutral impulsive parabolic partial differential equations with continuous distributed deviating arguments*, Open Access Library Journal **1** (2014), no. 9, 1–8.

[22] MA, Q. X.—LIU, A. P.: *Oscillation criteria of neutral type impulsive hyperbolic equations*, Acta Math. Sci. **34 B** (2014), no. 6, 1845–1853.

[23] MIL'MAN, V. D.—MYŠKIS, A. D.: *On the stability of motion in the presence of impulse*, Sibirsk. Math. Ž., **1** (1960), no. 2, 233–237.

[24] MINCHEV, E.—YOSHIDA, N.: *Oscillation of vector differential equations of hyperbolic type with functional arguments*, Math. J. Toyama Univ. **26** (2003), 75–84.

[25] MINCHEV, E.—YOSHIDA, N.: *Oscillation of solutions of vector differential equations of parabolic type with functional arguments*, J. Comput. Appl. Math., **151(1)** (2003), no. 1, 107–117.

[26] PODLUBNY, I.: *Fractional Differential Equations*. Academic Press, San Diego, 1999.

[27] PRAKASH, P.—HARIKRISHNAN, S.: *Oscillation of solutions of impulsive vector hyperbolic differential equations with delays*, Appl. Anal. **91** (2012), no. 3, 459–473.

[28] SADHASIVAM, V.—LOGAARASI, K.—KALAIMANI, T.: *Oscillation of even order impulsive partial differential equations with deviating arguments*, Int. J. Pure Appl. Math.,**115(9)** (2017), 83–91.

[29] SAMOILENKO, A. M.—PERSESTYUK, N. A.: *Impulsive Differential Equations*. World Scientific, Singapore, 1995.

[30] TANG, S.—ZADA, A.—FAISAL, S.—EL SHEIKH, M. M. A.—LI, T.: *Stability of higher-order nonlinear impulsive differential equations*, J. Nonlinear Sci. Appl., **9** (2016), no. 6, 4713–4721.

[31] WANG, P.—LI, C.—ZHANG, J.—LI, T.: *Quasilinearization method for first-order impulsive integro differential equations*, Electron. J. Differential Equations, **2019** (2019), 1–14.

[32] WU, J. H.: *Theory and Applications of Partial Functional Differential Equations*. In: *Applied Mathematical Sciences Vol. 119*. Springer-Verlag, New York, 1996.

[33] YOSHIDA, N.: *Oscillation Theory of Partial Differential Equations*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2008.

Received February 13, 2019

Department of Electrical and Electronic Engineering Educators, School of Pedagogical and Technological Education (ASPETE)
14121, Athens
GREECE

E-mail: geaxatz@otenet.gr

PG and Research Department of Mathematics, Thiruvalluvar Government Arts College
(Affiliated to Periyar University, Salem - 636 011)
Rasipuram - 637 401, Namakkal (Dt), Tamil Nadu, INDIA

E-mail: logajoni@gmail.com
trmaths19@gmail.com
ovsadha@gmail.com