

# SOME REMARK ON OSCILLATION OF SECOND ORDER IMPULSIVE DELAY DYNAMIC EQUATIONS ON TIME SCALES

GOKULA NANDA CHHATRIA

Sambalpur University, Sambalpur, INDIA

**ABSTRACT.** This article deals with the oscillation criteria for a very extensively studied second order impulsive delay dynamic equations on time scale by using the Riccati transformation technique. Some examples are given to show the effect of impulse and to illustrate our main results.

## 1. Introduction

Oscillation theory of impulsive differential/difference equation has brought the attention of many researchers, as it provides a more adequate mathematical model for numerous process and phenomena studied in physics, biology, engineering and to mention a few. In the literature, most of the results obtained for difference equations is the continuous analogues of differential equations and vice versa. Hence, it was an immediate question to find a way for which one can unify the qualitative properties of both equations. In 1988 Stefen Hilger introduced the concept of time scales calculus, which unify the continuous and discrete calculus in his Ph.D. thesis [15]. The study of impulsive dynamic equations on time scales has been initiated by Benchora et al. [5].

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In [18], Huang has considered the second order impulsive dynamic equation of the form

$$\begin{cases} [r(t)(y^\Delta(t))^\gamma]^\Delta + f(t, y^\sigma(t)) = 0, & t \in \mathbb{J}_\mathbb{T} := [0, \infty) \cap \mathbb{T}, \quad t \neq t_k, \quad t \geq t_0, \\ y(t_k^+) = g_k(y(t_k^-)), & y^\Delta(t_k^+) = h_k(y^\Delta(t_k^-)), \quad k \in \mathbb{N}, \\ y(t_0^+) = y_0, & y^\Delta(t_0^+) = y_0^\Delta \end{cases}$$

and improved the results of [16] and [17].

In [2], Agwa et al. have studied the oscillation properties of the solution of second order impulsive dynamic equations of the form

$$\begin{cases} [r(t)g(y^\Delta(t))]^\Delta + f(t, y^\sigma(t)) = g(t, y^\sigma(t)), & t \in \mathbb{J}_\mathbb{T} := [0, \infty) \cap \mathbb{T}, \quad t \neq t_k, \quad t \geq t_0, \\ y(t_k^+) = I_k(y(t_k^-)), & y^\Delta(t_k^+) = J_k(y^\Delta(t_k^-)), \quad k \in \mathbb{N}, \\ y(t_0^+) = y_0, & y^\Delta(t_0^+) = y_0^\Delta \end{cases}$$

and improved the results of [16, 17] and [18].

In [19], Huang and Wen have considered the second order forced impulsive dynamic equation of the form

$$\begin{cases} y^{\Delta\Delta}(t) + p(t)f(y^\sigma(t)) = e(t), & t \in \mathbb{J}_\mathbb{T} := [0, \infty) \cap \mathbb{T}, \quad t \neq t_k, \quad t \geq t_0, \\ y(t_k^+) = a_k y(t_k^-), & y^\Delta(t_k^+) = b_k y^\Delta(t_k^-), \quad k \in \mathbb{N}, \\ y(t_0^+) = y_0, & y^\Delta(t_0^+) = y_0^\Delta \end{cases}$$

and improved the results of [20].

Motivated by the above mention work, our objective is to study the second order impulsive nonlinear dynamic equations of the form

$$[r(t)|u^\Delta(t)|^{\gamma-1}u^\Delta(t)]^\Delta + f(t, u(t), u(t-\delta)) = 0, \quad t \in \mathbb{J}_\mathbb{T} := [0, \infty) \cap \mathbb{T}, \quad t \neq \theta_k \quad (1a)$$

$$r(\theta_k^+)|u^\Delta(\theta_k^+)|^{\gamma-1}u^\Delta(\theta_k^+) = I_k(r(\theta_k)|u^\Delta(\theta_k)|^{\gamma-1}u^\Delta(\theta_k)), \quad k \in \mathbb{N}, \quad (1b)$$

$$u(t) = \phi(t), \quad t_0 - \delta \leq t \leq t_0 \quad (1c)$$

where  $\gamma > 0$ ,  $\mathbb{T}$  is an unbounded above time scale with  $0 \in \mathbb{T}$  and  $\theta_k \in \mathbb{T}$ ,  $k \in \mathbb{N}$  are the fixed moment of impulse satisfying the properties:

$$0 \leq t_0 < \theta_1 < \theta_2 < \cdots < \theta_k, \quad \lim_{k \rightarrow \infty} \theta_k = \infty.$$

$$u(\theta_k^+) = \lim_{h \rightarrow 0^+} u(\theta_k + h), \quad u^\Delta(\theta_k^+) = \lim_{h \rightarrow 0^+} u^\Delta(\theta_k + h),$$

represent the right limit of  $u(t)$  at  $t = \theta_k$  in the sense of time scale, if  $\theta_k$  is right scattered, then

$$u(\theta_k^+) = u(\theta_k), \quad u^\Delta(\theta_k^+) = u^\Delta(\theta_k).$$

Similarly, we can define

$$u(\theta_k^-), \quad u^\Delta(\theta_k^-).$$

Throughout this paper, we assume that the following hypotheses hold:

- (H<sub>1</sub>)  $r(t) > 0$ ,  $\delta \in \mathbb{R}_+$ ,  $t - \delta \in \mathbb{T}$ ,  $\theta_k - \theta_{k-1} > \delta$ ;
- (H<sub>2</sub>)  $f \in C_{rd}(\mathbb{T} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ ,  $xf(t, x, y) > 0$  for  $x, y \geq 0$  and  $\frac{f(t, x, y)}{\varphi(y)} \geq q(t)(y \neq 0)$ , where  $q(t) \in C_{rd}(\mathbb{T}, [t_0, \infty)_{\mathbb{T}})$ ,  $\varphi \in C(\mathbb{R}, \mathbb{R})$  and  $y\varphi(y) > 0(y \neq 0)$ ,  $\varphi'(y) \geq 0$ ;
- (H<sub>3</sub>)  $I_k : \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function,  $I_k(0) = 0$  and there exist positive numbers  $b_k, b_k^*$  such that  $b_k \leq \frac{I_k(x)}{x} \leq b_k^*$ ,  $u \neq 0$ ,  $k \in \mathbb{N}$ .

To the best of the authors knowledge, this equation has not been considered before. In this direction, we refer the reader to some works [2]–[4], [7]–[13] and the references cited therein. About the time scale concept and fundamentals of time scale calculus we refer the monographs [7] and [8].

**DEFINITION 1.1** ([10]). A function  $f : \mathbb{T} \rightarrow \mathbb{R}$  is said to be absolutely continuous on  $\mathbb{T}$  if for every  $\epsilon > 0$  there exists a  $\delta > 0$  such that if  $\{[c_k, d_k) \cap \mathbb{T}\}_{k=1}^n$ , with  $c_k, d_k \in \mathbb{T}$ , is a finite pairwise disjoint family of subintervals of  $\mathbb{T}$  satisfying

$$\sum_{k=1}^n (d_k - c_k) < \delta, \quad \text{then} \quad \sum_{k=1}^n |f(d_k) - f(c_k)| < \epsilon.$$

$AC^i = \{u : \mathbb{J}_{\mathbb{T}} \rightarrow \mathbb{R} \text{ is } i\text{-times } \Delta\text{-differentiable, whose } i\text{th delta derivative } u^{\Delta(i)} \text{ is absolutely continuous}\}.$

$PC = \{u : \mathbb{J}_{\mathbb{T}} \rightarrow \mathbb{R} \text{ is rd-continuous at the points } \theta_k, k \in \mathbb{N} \text{ for which } u(\theta_k^-), u(\theta_k^+), u^{\Delta}(\theta_k^-) \text{ and } u^{\Delta}(\theta_k^+) \text{ exist with } u(\theta_k^-) = u(\theta_k), u^{\Delta}(\theta_k^-) = u^{\Delta}(\theta_k)\}.$

**DEFINITION 1.2.** A solution of  $u(t)$  of (1) is said to be regular if it is defined on some half line  $[t_x, \infty)_{\mathbb{T}} \subset [t_0, \infty)_{\mathbb{T}}$  and  $\sup\{|u(t)| : t \geq t_x\} > 0$ . A regular solution  $u(t)$  of (1) is said to be eventually positive (eventually negative), if there exists  $t_1 > 0$  such that  $u(t) > 0$  ( $u(t) < 0$ ), for  $t \geq t_1$ .

**DEFINITION 1.3.** A function  $u(t) \in PC \cap AC^2(\mathbb{J}_{\mathbb{T}} \setminus \{\theta_1, \theta_2, \dots\}, \mathbb{R})$  is called a solution of (1) if:

- (I) it satisfies (1a) a.e on  $\mathbb{J}_{\mathbb{T}} \setminus \{\theta_k\}, k \in \mathbb{N}$ ;
- (II) for  $t = \theta_k, k \in \mathbb{N}$ ,  $u(t)$  satisfies (1b);
- (III) and satisfies the initial condition (1c).

**DEFINITION 1.4.** A nontrivial solution  $u(t)$  of (1) is said to be nonoscillatory, if there exists a point  $t_0 \geq 0$  such that  $u(t)$  has a constant sign for  $t \geq t_0$ . Otherwise, the solution  $u(t)$  is said to be oscillatory.

## 2. Basic Lemmas

**LEMMA 2.1.** *Let  $u(t)$  be a solution of (1). Furthermore, assume that there exists  $T \geq t_0$  such that  $u(t) > 0$  for  $t \geq T$  and*

$$(H_4) \quad \int_{\theta_j}^{\infty} \frac{1}{r^{\frac{1}{\gamma}}(s)} \prod_{\theta_j < \theta_k < s} b_k^{\frac{1}{\gamma}} \Delta s = \infty.$$

*Then  $u^\Delta(\theta_k^+) \geq 0$  and  $u^\Delta(t) \geq 0$  for  $t \in (\theta_k, \theta_{k+1}]_{\mathbb{T}}$ , where  $\theta_k \geq T$ .*

**Proof.** Let  $u(t)$  be an eventually positive solution of (1) for  $t \geq t_0$ . Without loss of generality we assume that  $u(t) > 0$  and  $u(t - \delta) > 0$  for  $t \geq t_1 > t_0 + \delta$ . Set

$$z(t) = r(t)|u^\Delta(t)|^{\gamma-1}u^\Delta(t).$$

Therefore, from (1) we get

$$z^\Delta(t) = -f(t, u(t), u(t - \delta)) \leq -q(t)\varphi(u(t - \delta)) \leq 0.$$

Therefore,  $z^\Delta(t)$  is monotonically decreasing on  $[t_2, \infty)_{\mathbb{T}}$ ,  $t_2 > t_1 + \delta$ . Assume that  $\theta_k > t_2$  for  $k \in \mathbb{N}$ . Consider the interval  $(\theta_k, \theta_{k+1}]_{\mathbb{T}}$ ,  $k \in \mathbb{N}$ . We assert that  $u^\Delta(\theta_k) \geq 0$ . If not, there exists  $\theta_j \geq t_2$  such that  $u^\Delta(\theta_j) < 0$  and hence

$$\begin{aligned} r(\theta_j^+)|u^\Delta(\theta_j^+)|^{\gamma-1}u^\Delta(\theta_j^+) &= I_k(r(\theta_j)|u^\Delta(\theta_j)|^{\gamma-1}u^\Delta(\theta_j)) \\ &\leq b_j r(\theta_j)|u^\Delta(\theta_j)|^{\gamma-1}u^\Delta(\theta_j) < 0. \end{aligned}$$

Let

$$r(\theta_j)|u^\Delta(\theta_j)|^{\gamma-1}u^\Delta(\theta_j) = -\alpha^\gamma, \quad \alpha > 0.$$

Now for  $t \in (\theta_j, \theta_{j+1}]_{\mathbb{T}}$ , we have

$$r(\theta_{j+1})|u^\Delta(\theta_{j+1})|^{\gamma-1}u^\Delta(\theta_{j+1}) \leq r(\theta_j^+)|u^\Delta(\theta_j^+)|^{\gamma-1}u^\Delta(\theta_j^+),$$

that is,

$$r(\theta_{j+1})|u^\Delta(\theta_{j+1})|^{\gamma-1}u^\Delta(\theta_{j+1}) \leq I_j(r(\theta_j)|u^\Delta(\theta_j)|^{\gamma-1}u^\Delta(\theta_j)) = -b_j\alpha^\gamma < 0.$$

If  $t \in (\theta_{j+1}, \theta_{j+2}]_{\mathbb{T}}$ , then

$$\begin{aligned} r(\theta_{j+2})|u^\Delta(\theta_{j+2})|^{\gamma-1}u^\Delta(\theta_{j+2}) &\leq r(\theta_{j+1}^+)|u^\Delta(\theta_{j+1}^+)|^{\gamma-1}u^\Delta(\theta_{j+1}^+) \\ &= I_{j+1}(r(\theta_{j+1})|u^\Delta(\theta_{j+1})|^{\gamma-1}u^\Delta(\theta_{j+1})) \\ &\leq b_{j+1}r(\theta_{j+1})|u^\Delta(\theta_{j+1})|^{\gamma-1}u^\Delta(\theta_{j+1}), \end{aligned}$$

that is,

$$u^\Delta(\theta_{j+2}) \leq -b_j b_{j+1} \alpha^\gamma < 0.$$

Hence, by the method of induction,

$$\begin{aligned} r(t)|u^\Delta(t)|^{\gamma-1}u^\Delta(t) &\leq -b_j b_{j+1} b_{j+2} \cdots b_{j+n} \alpha^\gamma \\ &= -\alpha^\gamma \prod_{\theta_j \leq \theta_k < t} b_k < 0, \quad \text{for } t \in (\theta_{j+n-1}, \theta_{j+n}]_{\mathbb{T}}. \end{aligned}$$

Therefore,

$$u^\Delta(t) \leq \frac{-\alpha \prod_{\theta_j \leq \theta_k < t} b_k^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(t)}. \quad (2)$$

Integrating (2) from  $\theta_j$  to  $t$ , we get

$$\begin{aligned} u(t) &\leq u(\theta_j^+) - \alpha \int_{\theta_j}^t \left( \frac{1}{r(s)} \right)^{\frac{1}{\gamma}} \prod_{\theta_j < \theta_k < s} b_k^{\frac{1}{\gamma}} \Delta s \\ &\rightarrow -\infty \text{ as } t \rightarrow \infty \end{aligned}$$

due to  $(H_4)$ , a contradiction to the fact that  $u(t) > 0$  eventually. Hence our assertion holds, that is,  $u^\Delta(\theta_k) \geq 0$  for  $\theta_k \geq T$  and hence  $u^\Delta(t) > u^\Delta(\theta_k^+)$ . Since  $z^\Delta(t) \leq 0$  for any  $t \in (\theta_k, \theta_{k+1}]_{\mathbb{T}}$ ,  $\theta_k \geq T$ , then

$$r(t)|u^\Delta(t)|^{\gamma-1}u^\Delta(t) \geq r(\theta_{k+1})|u^\Delta(\theta_{k+1})|^{\gamma-1}u^\Delta(\theta_{k+1}) \geq 0, \quad t \in (\theta_k, \theta_{k+1}]_{\mathbb{T}}.$$

Therefore,

$$u^\Delta(\theta_k^+) \geq 0 \quad \text{and} \quad u^\Delta(t) \geq 0 \quad \text{for } t \in (\theta_k, \theta_{k+1}]_{\mathbb{T}}, \quad t \geq t_2.$$

Therefore, the lemma is proved.  $\square$

**Remark 1.** If  $u(t)$  is an eventually negative solution of (1). Then using  $(H_4)$ , it is easy to prove that

$$u^\Delta(\theta_k^+) \leq 0 \quad \text{and} \quad u^\Delta(t) \leq 0$$

for  $t \in (\theta_k, \theta_{k+1}]_{\mathbb{T}}$  and  $\theta_k \geq T \geq t_0$ .

We need the time scale version of the following well-known results for our use in the sequel.

**LEMMA 2.2** ([1]). *Let  $y, f \in C_{rd}$  and  $p \in \mathcal{R}$ . Then*

$$y^\Delta(t) \leq p(t)y(t) + f(t)$$

*implies that for all  $t \in \mathbb{T}$*

$$y(t) \leq y(t_0)e_p(t, t_0) + \int_{t_0}^t e_p(t, \sigma(s))f(s)\Delta s.$$

**LEMMA 2.3** ([17]). *Assume that*

- (i)  $m \in PC \cap AC^1(\mathbb{J}_{\mathbb{T}} \setminus \{\theta_k\}, \mathbb{R})$ ;
- (ii)  $k \in \mathbb{N}$  and  $t \geq t_0$ , we have

$$m^\Delta(t) \leq p(t)m(t) + v(t), \quad t \in \mathbb{J}_{\mathbb{T}} = [0, \infty) \cap \mathbb{T}, \quad t \neq \theta_k,$$

$$m(\theta_k^+) \leq d_k m(\theta_k) + e_k.$$

Then the following inequality holds

$$\begin{aligned}
 m(t) \leq m(t_0) \prod_{t_0 < \theta_k < t} d_k e_p(t_0, t) + \int_{t_0}^t \prod_{s < \theta_k < t} d_k e_p(t, \sigma(s)) v(s) \Delta s \\
 + \sum_{t_0 < \theta_k < t} \left( \prod_{\theta_k < \theta_j < t} d_j e_p(t, \theta_k) \right) e_k, \quad t \geq t_0.
 \end{aligned}$$

### 3. Main Results

**THEOREM 3.1.** *Let all conditions of Lemma 2.1 hold. Furthermore, assume that*

$$(H_5) \quad \int_{t_0}^{\infty} \prod_{t_0 < \theta_k < t} \frac{1}{b_k^*} q(s) \Delta s = \infty.$$

*Then every solution of (1) oscillates.*

**Proof.** Suppose, on the contrary, that  $u(t)$  is a nonoscillatory solution of (1). Without loss of generality, we assume that  $u(t) > 0$ ,  $u(t - \delta) > 0$  for  $t \geq t_1$ . Hence by Lemma 2.1, there exists  $t_2 > t_1$  such that  $u^\Delta(t) > 0$  for  $t \in (\theta_k, \theta_{k+1}]_{\mathbb{T}}$ ,  $k \in \mathbb{N}$  and  $\theta_k \geq t_2$ . Let

$$w(t) = \frac{r(t)|u^\Delta(t)|^{\gamma-1}u^\Delta(t)}{\varphi(u(t-\delta))}, \quad (3)$$

then  $w(\theta_k^+) \geq 0$  and  $w(t) \geq 0$  for  $\theta_k \geq t_3$ . From (3), for  $t \neq \theta_k$  we have

$$\begin{aligned}
 w^\Delta(t) &= \frac{[r(t)|u^\Delta(t)|^{\gamma-1}u^\Delta(t)]^\Delta}{\varphi(u(\sigma(t)-\delta))} - \frac{r(\sigma(t))|u^\Delta(\sigma(t))|^{\gamma-1}u^\Delta(\sigma(t))\varphi^\Delta(u(t-\delta))}{\varphi(u(t-\delta))\varphi(u(\sigma(t)-\delta))} \\
 &\leq \frac{[r(t)|u^\Delta(t)|^{\gamma-1}u^\Delta(t)]^\Delta}{\varphi(u(\sigma(t)-\delta))} \\
 &\leq \frac{-f(t, u(t), u(t-\delta))}{\varphi(u(t-\delta))},
 \end{aligned}$$

where we have used the fact that  $u^\Delta(t) > 0$ . Therefore, due to  $(H_2)$  we get

$$w^\Delta(t) \leq -q(t), \quad t \neq \theta_k. \quad (4)$$

We note that

$$w(t_k^+) = \frac{r(\theta_k^+)|u^\Delta(\theta_k^+)|^{\gamma-1}u^\Delta(\theta_k^+)}{\varphi(u(\theta_k^+-\delta))} \leq \frac{b_k^* r(\theta_k)|u^\Delta(\theta_k)|^{\gamma-1}u^\Delta(\theta_k)}{\varphi(u(\theta_k-\delta))} = b_k^* w(\theta_k).$$

Now, we have the following impulsive dynamics inequalities

$$\begin{aligned} w^\Delta(t) &\leq -q(t), & t \neq \theta_k, \\ w(\theta_k^+) &\leq b_k^* w(\theta_k), & k \in \mathbb{N} \end{aligned}$$

and by Lemma 2.3 it follows that

$$\begin{aligned} w(t) &\leq w(t_3) \prod_{t_3 < \theta_k < t} b_k^* - \int_{t_3}^t \prod_{t_3 < \theta_k < s} b_k^* q(s) \Delta s \\ &\leq \prod_{t_3 < \theta_k < t} b_k^* \left[ w(t_3) - \int_{t_3}^t \prod_{t_3 < \theta_k < s} \frac{1}{b_k^*} q(s) \right] \Delta s \\ &\rightarrow -\infty \text{ as } t \rightarrow \infty \end{aligned}$$

due to (H<sub>5</sub>), a contradiction to the fact that  $w(t) > 0$  for  $t \in (\theta_k, \theta_{k+1}]_{\mathbb{T}}$ ,  $k \in \mathbb{N}$ . This completes the proof of the theorem.  $\square$

**COROLLARY 3.2.** *Let all conditions of Lemma 2.1 hold. Assume that there exists a positive integer  $k_0$  such that  $b_k^* \leq 1$  for  $k \geq k_0$ . Furthermore, assume that*

(H<sub>6</sub>)  $\int_{t_0}^\infty q(s) \Delta s = \infty$  *hold, then every solution of (1) oscillates.*

**Proof.** Without loss of generality, we assume that  $k_0 = 1$ . Since  $b_k^* \leq 1$ , then  $\frac{1}{b_k^*} \geq 1$ . Therefore,

$$\int_{t_0}^t \prod_{t_0 \leq \theta_k < s} \frac{1}{b_k^*} q(s) \Delta s \geq \int_{t_0}^t q(s) \Delta s.$$

Letting  $t \rightarrow \infty$  and in view of Theorem 3.1, we get that every solution of (1) is oscillatory. This completes the proof.  $\square$

**COROLLARY 3.3.** *Let all conditions of Lemma 2.1 hold. Assume that there exists a positive integer  $k_0$  and a positive constant  $\alpha$  such that  $\frac{1}{b_k^*} \geq \left(\frac{\theta_{k+1}}{\theta_k}\right)^\alpha$  for  $k \geq k_0$ . Furthermore, assume that*

(H<sub>7</sub>)  $\int_{t_0}^\infty s^\alpha q(s) \Delta s = \infty$  *hold, then every solution of (1) oscillates.*

Proof. Without loss of generality, we assume that  $k_0 = 1$ . Now

$$\begin{aligned}
 \int_{t_0}^t \prod_{t_0 < \theta_k < s} \frac{1}{b_k^*} q(s) \Delta s &= \sum_{i=1}^n \prod_{t_0 < \theta_k < \theta_{i+1}} \frac{1}{b_k^*} \int_{\theta_i}^{\theta_{i+1}} q(s) \Delta s \\
 &\geq \frac{1}{\theta_1^\alpha} \sum_{i=1}^n \theta_{i+1}^\alpha \int_{\theta_i}^{\theta_{i+1}} q(s) \Delta s \\
 &\geq \frac{1}{\theta_1^\alpha} \sum_{i=1}^n \int_{\theta_i}^{\theta_{i+1}} s^\alpha q(s) \Delta s \\
 &= \frac{1}{\theta_1^\alpha} \int_{\theta_1}^{\theta_{n+1}} s^\alpha q(s) \Delta s.
 \end{aligned}$$

Letting  $t \rightarrow \infty$  and in view of Theorem 3.1, we get that every solution of (1) is oscillatory. This completes the proof.  $\square$

Next, we present some new oscillation criteria for (1) by using an integral averaging condition of the Kamenev type.

**THEOREM 3.4.** *Let all conditions of Lemma 2.1 hold and  $b_k^* \geq 1$ . Furthermore, assume that*

$$(H_8) \quad \limsup_{k \rightarrow \infty} \frac{1}{t^m} \int_{t_0}^{\theta_{k+1}} (t-s)^m q(s) \Delta s = \infty,$$

*then every solution of (1) oscillates.*

Proof. Proceeding as in the proof of Theorem 3.1, we get

$$w^\Delta(t) \leq -q(t), \quad \text{for } t \neq \theta_k.$$

Multiplying  $(t-s)^m$  to both side of the preceding inequality and integrating from  $\theta_k$  to  $\theta_{k+1}$ , we get

$$\int_{\theta_k}^{\theta_{k+1}} (t-s)^m w^\Delta(s) \, ds \leq - \int_{\theta_k}^{\theta_{k+1}} (t-s)^m q(s) \Delta s.$$



Indeed,

$$\begin{aligned}
 \int_{\theta_k}^{\theta_{k+1}} (t-s)^m w^\Delta(s) \Delta s &= (t-s)^m u(s) \Big|_{\theta_k}^{\theta_{k+1}} - \int_{\theta_k}^{\theta_{k+1}} ((t-s)^m)^{\Delta_s} w(s) \Delta s \\
 &= \int_{\theta_k}^{\theta_{k+1}} m(t-s)^{m-1} w(s) \Delta s \\
 &\quad + (t-\theta_{k+1})^m w(\theta_{k+1}) - (t-\theta_k)^m w(\theta_k^+),
 \end{aligned}$$

because

$$((t-s)^m)^{\Delta_s} = -m(t-s)^{m-1}.$$

As a result,

$$\int_{\theta_k}^{\theta_{k+1}} (t-s)^m w^\Delta(s) \Delta s \geq -(t-\theta_k)^m w(\theta_k^+).$$

Therefore,

$$\begin{aligned}
 \int_{\theta_k}^{\theta_{k+1}} (t-s)^m q(s) \Delta s &\leq - \int_{\theta_k}^{\theta_{k+1}} (t-s)^m w^\Delta(s) \Delta s \\
 &\leq (t-\theta_k)^m w(\theta_k^+) \leq b_k^* (t-\theta_k)^m w(\theta_k),
 \end{aligned}$$

that is,

$$\frac{1}{t^m} \int_{\theta_k}^{\theta_{k+1}} (t-s)^m q(s) \Delta s \leq b_k^* \left( \frac{t-\theta_k}{t} \right)^m w(\theta_k).$$

and hence

$$\limsup_{k \rightarrow \infty} \frac{1}{t^m} \int_{\theta_k}^{\theta_{k+1}} (t-s)^m q(s) \Delta s < \infty,$$

is a contradiction to (A<sub>8</sub>) This completes the proof of the theorem.  $\square$

**Remark 2.** Finally, we remark that, using the same technique and the same argument as above, one can obtain new oscillation criteria for the advanced dynamic equation with impulse of the form

$$(E) \begin{cases} [r(t)|u^\Delta(t)|^{\gamma-1} u^\Delta(t)]^\Delta + f(t, u(t), u(t+\delta)) = 0, & t \in \mathbb{T}, t \neq \theta_k, \\ r(\theta_k^+) |u^\Delta(\theta_k^+)|^{\gamma-1} u^\Delta(\theta_k^+) = I_k(r(\theta_k) |u^\Delta(\theta_k)|^{\gamma-1} u^\Delta(\theta_k)), & k \in \mathbb{N}, \\ u(t) = \phi(t), & t_0 - \delta \leq t \leq t_0. \end{cases}$$

## 4. Examples

EXAMPLE. Consider the impulsive system ( $\mathbb{T} = \mathbb{R}$ )

$$\begin{cases} u''(t) + \frac{(t+2)^2}{\ln t} u(t-2) = 0, & t > 2, t \neq \theta_k, \\ u'(\theta_k^+) = \left(\frac{k+2}{k+1}\right) u'(\theta_k), & k \in \mathbb{N}, \end{cases} \quad (5)$$

where

$$\begin{aligned} \gamma &= 1, & r(t) &= 1, & \delta &= 2, \\ q(t) &= \frac{(t+2)^2}{\ln t} \geq 0, & b_k^* &= b_k = \frac{k+2}{k+1}, & \theta_k &= 3k, \\ \theta_{k+1} - \theta_k &= 3 > 2, & k &\in \mathbb{N}, & f(x) &= x. \end{aligned}$$

Then, from (H<sub>4</sub>)

$$\begin{aligned} \int_{\theta_j}^{\infty} \frac{1}{r^{\frac{1}{\gamma}}(s)} \prod_{\theta_j < \theta_k < s} b_k^{\gamma} ds &= \int_2^{\infty} \prod_{2 < \theta_k < s} \frac{k+2}{k+1} ds \\ &= \int_2^{\theta_1} \prod_{2 < \theta_k < s} \frac{k+2}{k+1} ds + \int_{\theta_1^+}^{\theta_2} \prod_{2 < \theta_k < s} \frac{k+2}{k+1} ds + \int_{\theta_2^+}^{\theta_3} \prod_{2 < \theta_k < s} \frac{k+2}{k+1} ds + \dots \\ &= \frac{3}{2}(\theta_1 - 2) + \frac{3}{2} \times \frac{4}{3}(\theta_2 - \theta_1) + \frac{3}{2} \times \frac{4}{3} \times \frac{5}{4}(\theta_3 - \theta_2) + \dots \\ &= \frac{3}{2} \times 1 + 2 \times 3 + \frac{5}{2} \times 3 + \frac{1}{5} \times 3 + \dots \\ &\geq 1 + 2 + 3 + \dots = \sum_{i=1}^{\infty} i = \infty \end{aligned}$$

and from (H<sub>6</sub>) we have

$$\begin{aligned} \int_2^{\infty} \prod_{2 < \theta_k < s} \frac{1}{b_k^*} q(s) ds \\ = \left[ \int_2^{\theta_1} \prod_{2 < \theta_k < s} + \int_{\theta_1^+}^{\theta_2} \prod_{2 < \theta_k < s} + \dots + \int_{\theta_{k-1}^+}^{\infty} \prod_{2 < \theta_k < s} \right] \left( \frac{k+1}{k+2} \right) \left( \frac{(s+2)^2}{\ln s} \right) ds \rightarrow \infty. \end{aligned}$$

By Theorem 3.1, (5) has an oscillatory solution. In the mean time,

$$u''(t) + \frac{(t+2)^2}{\ln t} u(t-2) = 0$$

has a nonoscillatory solution  $u(t) = \ln(t+2)$ .

EXAMPLE. Consider  $(\mathbb{T} = \mathbb{Z})$

$$\begin{cases} \Delta^2 u(t) + \left(\frac{t}{t+1}\right) u^2(t-1) = 0, & t > 1, \quad t \neq \theta_k, \\ \Delta u(\theta_k^+) = \left(\frac{1}{k-1}\right) \Delta u(\theta_k), & k \in \mathbb{N}, \quad k > k_0, \end{cases} \quad (6)$$

where

$$\begin{aligned} \gamma &= 1, & \delta &= 1, & r(t) &= 1, & q(t) &= \frac{t}{t+1} \geq 0, & b_k^* &= b_k = \frac{1}{k-1}, \\ \theta_k &= 2^k, & \theta_{k+1} - \theta_k &= 2^k > 1, & k &\in \mathbb{N}, & k > k_0 &= 1, & f(x) &= x^2. \end{aligned}$$

Clearly,  $(H_4)$  is satisfied and from  $(H_6)$  we obtain

$$\begin{aligned} & \int_2^\infty \prod_{2 < \theta_k < s} \frac{1}{b_k^*} q(s) \, ds \\ &= \left[ \int_2^{\theta_1} \prod_{2 < \theta_k < s} + \int_{\theta_1^+}^{\theta_2} \prod_{2 < \theta_k < s} + \cdots + \int_{\theta_{k-1}^+}^\infty \prod_{2 < \theta_k < s} \right] (k-1) \left( \frac{s}{s+1} \right) \, ds \rightarrow \infty. \end{aligned}$$

All conditions of Theorem 3.4 are satisfied for (6) and hence (6) has an oscillatory solution.

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Gokula Nanda Chhatria  
 Department of Mathematics  
 Sambalpur University  
 Sambalpur-768019  
 INDIA  
 E-mail: c.gokulananda@gmail.com