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ZEROING THE TRANSFER MATRIX OF THE ROESSER MODEL OF 2–D LINEAR SYSTEMS

TADEUSZ KACZOREK [a]

[a] Faculty of Electrical Engineering
Bialystok University of Technology
Wiejska 45D, 15-351 Bialystok, Poland
e-mail: t.kaczorek@pb.edu.pl

Controllability, observability and the transfer matrix of the discrete 2-D Roesser model are analyzed. It is shown that the controllability of the Roesser model is invariant under state feedbacks and the observability under output feedbacks. Sufficient conditions are established for the zeroing of the transfer matrix of the Roesser model.

Keywords: controllability, observability, 2-D Roesser model, state feedback, output feedback, zeroing of the transfer matrix.

1. Introduction

In 2D linear systems the inputs, outputs and state variables are functions of two independent variables (Fornasini and Marchesini, 1978; Kaczorek, 1993; 1985; Kaczorek and Rogowski, 2015). Various types of models of 2-D linear systems have been proposed (Kaczorek, 1985). In the paper by Roesser (1975) a 2-D discrete-time Roesser model has been presented. The Roesser model is a particular case of the second Fornasini–Marchesini model (Fornasini and Marchesini, 1978; Kaczorek, 1985). A general model of 2-D linear discrete-time systems was set forth by Kurek (1985). Generalized (descriptor) 2-D linear systems were analyzed by Kaczorek (1993). The general response formula for CFD pseudo-fractional 2D continuous linear systems described by the Roesser model was given by Rogowski (2020). Stability of discrete-time fractional systems with delays was investigated by Ruszewski (2019) and that of descriptor fractional discrete-time system with two different fractional orders by Sajewski (2016).

In this paper, controllability, observability and the transfer matrix of the discrete-time 2-D Roesser model will be analyzed and sufficient conditions for zeroing the transfer matrix will be established.

The paper is organized as follows. In Section 2 basic definitions and theorems concerning controllability, observability and the transfer matrix of linear discrete-time systems are recalled. Necessary and sufficient conditions for the controllability and observability of the 2-D Roesser model are given in Section 3. The controllability and observability of the 2-D Roesser model with state and output feedbacks are investigated in Section 4. The decompositions of the 2-D Roesser model into controllable and uncontrollable parts and into observable and unobservable parts are analyzed in Section 5. Conditions for the zeroing of the transfer matrix of the 2-D Roesser model are established. Concluding remarks are given in Section 6.

The following notation will be used: $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^{n\times m}$ stands for the set of $n \times m$ real matrices and $\mathbb{R}^n = \mathbb{R}^{n\times 1}$, $\mathbb{R}_+^{n\times m}$ means the set of $n \times m$ real matrices with nonnegative entries and $\mathbb{R}_+^n = \mathbb{R}_+^{n\times 1}$, I_n signifies the $n \times n$ identity matrix $\mathbb{C}$.

2. Preliminaries

Consider the linear discrete-time system

$$x_{i+1} = Ax_i + Bu_i, \tag{1a}$$

$$y_i = Cx_i, \tag{1b}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, $y_i \in \mathbb{R}^p$ are the state, input and output vectors, respectively, and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$.

Definition 1. (*Kaczorek, 1989; 1993; 1985; Kaczorek and Rogowski, 2015; Kalman, 1960; Klamka, 1991*) The system (1) (the pair (A,B)) is called *controllable* if for any given initial state $x(0) \in \mathbb{R}^n$ and any given final state $x_f \in \mathbb{R}^n$ there exists an input $u(t)$ for $t \in [0,\ t_f]$ which steers the system from $x(0) \in \mathbb{R}^n$ to $x(t_f) = x_f$.

Theorem 1. (Kaczorek, 1989; 1993; 1985; Kaczorek and Rogowski; 2015; Kalman, 1960; Klamka 1991) *The system (1) (the pair (A,B)) is controllable if and only if one of the following equivalent conditions is satisfied:*

$$\operatorname{rank}[\ B \quad AB \quad \dots \quad A^{n-1}B\ ] = n, \tag{2a}$$

$$\operatorname{rank}[\ I_n z - A \quad B\ ] = n \quad \forall z \in \mathbb{C}. \tag{2b}$$

Definition 2. (*Kaczorek, 1989; 1993; 1985; Kaczorek and Rogowski; 2015; Kalman, 1960; Klamka 1991*) The system (1) (the pair (A,B)) is called observable if knowing the input $u(t)$ and output $y(t)$ of the system (1) for $t \in [0,\ t_f]$ it is possible to find its initial state $x(0)$.

Theorem 2. (Kaczorek, 1989; 1993;1985; Kaczorek and Rogowski; 2015; Kalman, 1960; Klamka 1991) *The system (1) is observable if and only if one of the following equivalent conditions is satisfied:*

$$\operatorname{rank}\begin{bmatrix} \mathrm{C} \\ \mathrm{CA} \\ \vdots \\ CA^{n-1} \end{bmatrix} = n, \tag{3a}$$

$$\operatorname{rank}\begin{bmatrix} I_n z - A \\ C \end{bmatrix} = n, \quad \forall z \in \mathbb{C} \tag{3b}$$

It is well known that if the pair $(A,\ \ B)$ is uncontrollable and the pair $(A,\ \ C)$ is unobservable then, according to the Kalman theorem (Kaczorek, 1993; Kalman, 1960; Klamka, 1991), the system (1) can be decomposed in the four independent parts

$$\begin{aligned} A = P^{-1}AP &= \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} & 0 & A_{24} \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{bmatrix}, \\ B = P^{-1}B &= \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix}, \\ C = CP &= \begin{bmatrix} 0 & C_2 & 0 & C_4 \end{bmatrix}, \end{aligned} \tag{4}$$

where $A_{i,j} \in \mathbb{R}^{n_i \times n_i}$, $B_j \in \mathbb{R}^{n_j \times m}$, $C_i \in \mathbb{R}^{p \times n_i}$, $n = n_1 + \dots + n_4$ and $P \in \mathbb{R}^{n \times n}$ is nonsingular similarity transformation matrix such that

- $(A_{11},\ B_1)$ and $(A_{22},\ B_2)$ are the controllable parts of the system,
- $(A_{33},\ 0)$ and $(A_{44},\ 0)$ are the uncontrollable parts of the system,
- $(A_{11},\ 0)$ and $(A_{22},\ 0)$ are the unobservable parts of the system,
- $(A_{22},\ C_2)$ and $(A_{44},\ C_4)$ are observable parts of the system.

Theorem 3. (Kaczorek, 1993; Kalman, 1960; Klamka, 1991) *The transfer matrix of the system (1) is equal to the transfer matrix of its controllable and observable parts,*

$$T(z) = C[I_n z - A]^{-1}B = C_2[I_{n_2} z - A_{22}]^{-1}B_2. \tag{5}$$

3. Roesser model and its controllability and observability

The Roesser model of a 2-D linear discrete-time system has the form

$$\begin{bmatrix} x^h_{i+1,j} \\ x^v_{i,j+1} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x^h_{ij} \\ x^v_{ij} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{ij}, \tag{6a}$$

$$y_{ij} = C \begin{bmatrix} x^h_{ij} \\ x^v_{ij} \end{bmatrix}, \tag{6b}$$

where $x^h_{ij} \in \mathbb{R}^{n_1}$ and $x^v_{ij} \in \mathbb{R}^{n_2}$ are the horizontal and vertical state vectors, respectively, $u_{ij} \in \mathbb{R}^m$ is the input vector and $y_{ij} \in \mathbb{R}^p$ is the output vector, and

$$\begin{aligned} &A_{11} \in \mathbb{R}^{n_1 \times n_1}, && A_{12} \in \mathbb{R}^{n_1 \times n_2}, \\ &A_{21} \in \mathbb{R}^{n_2 \times n_1}, && A_{22} \in \mathbb{R}^{n_2 \times n_2}, \\ &B_1 \in \mathbb{R}^{n_1 \times m}, && B_1 \in \mathbb{R}^{n_2 \times m}, \\ &C \in \mathbb{R}^{p \times (n_1+n_2)}. \end{aligned}$$

The boundary conditions for the model (6) in the rectangle $[(0,\ 0),\ \ (r_1,\ r_2)]$ have the form

$$\begin{aligned} x^h_{0j} &\in \mathbb{R}^{n_1}, \quad j = 0, 1, \dots, r_1, \\ x^v_{i0} &\in \mathbb{R}^{n_2}, \quad i = 0, 1, \dots, r_2. \end{aligned} \tag{6c}$$

The solution of (6a) for given boundary conditions (6c) has the form

$$\begin{aligned} \begin{bmatrix} x^h_{ij} \\ x^v_{ij} \end{bmatrix} &= \sum_{p=0}^{i} T_{i-p,j} \begin{bmatrix} 0 \\ x^v_{p0} \end{bmatrix} + \sum_{q=0}^{j} T_{i,j-q} \begin{bmatrix} x^h_{0q} \\ 0 \end{bmatrix} \\ &+ \sum_{p=0}^{i-1} \sum_{q=0}^{j} T_{i-p-1,j-q} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u_{pq} \\ &+ \sum_{p=0}^{i} \sum_{q=0}^{j-1} T_{i-p,j-q-1} \begin{bmatrix} 0 \\ B_2 \end{bmatrix} u_{pq}, \end{aligned} \tag{7a}$$

and the transition matrix of the model is defined as (Roesser, 1975)

$$T_{ij} = \begin{cases} I_n & \text{for} \quad i = j = 0, \\ A_{10}T_{i-1,j} + A_{01}T_{i,j-1} & \text{otherwise,} \end{cases}$$

$$A_{10} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix},$$

$$A_{01} = \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix}. \tag{7b}$$

The matrix transfer function of the Roesser model has the form (Kaczorek, 1993)

$$T(z_1, z_2) = C \begin{bmatrix} I_{n_1}z_1 - A_{11} & -A_{12} \\ -A_{21} & I_{n_2}z_2 - A_{22} \end{bmatrix}^{-1} \times \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}. \tag{8}$$

Definition 3. (*Roesser, 1975; Kaczorek, 1985; 1993*) The Roesser model (6) is called *controllable* in the rectangle $[r_1,\ r_2] = [0 \le i \le r_1,\ 0 \le j \le r_2]$ if for any boundary conditions $x[0,\ j]$, $j \in [0,\ r_2]$, $x[l,\ 0]$, $l \in [r_1,\ 0]$ and every vector $x_f \in \mathbb{R}^n$ there exists a sequence of inputs $u(l,j) \in \mathbb{R}^m$, $(0,\ 0) \le (l,\ j) < (r_1,\ r_2)$ such that $x(r_1,\ r_2) = x_f$.

Theorem 4. *The Roesser model (6) is controllable in the rectangle $[r_1 r_2]$, if and only if*

$$\operatorname{rank} C_{\mathrm{R}} = n, \tag{9a}$$

where

$$\begin{aligned} C_R &= C_R(r_1,\ r_2) \\ &= [M(0,\ 1),\ M(1,\ 0), \dots, \\ &\quad M(l,\ j), \dots,\ M(r_1,\ r_2)], \\ M(i,\ j) &= T(i-1,\ j)B + T(i,\ j-1)B, \\ &\quad i = 0, 1, \dots, r_1, \quad j = 0, 1, \dots, r_2 \end{aligned} \tag{9b}$$

$$\operatorname{rank} \begin{bmatrix} I_{n_1}z_1 - A_{11} & -A_{12} & B_1 \\ -A_{21} & I_{n_2}z_2 - A_{22} & B_2 \end{bmatrix} = n, \quad z_1, z_2 \in \mathbb{C}. \tag{9c}$$

The proof is given by Roesser (1975) and Kaczorek (1985; 1993).

Definition 4. The Roesser model (6) is called (locally) *observable* in the rectangle $[r_1,\ r_2]$ if there is no local initial state $x(0,\ 0) \neq 0$ such that for zero inputs $u(l,j) = 0$, $(0,\ 0) \le (l,\ j) < (r_1,\ r_2)$ and zero boundary conditions $x^h(0,\ j) = 0$, $j \in [1,\ r_2]$, $x^v(i,\ 0) = 0$, $l \in [1,\ r_1]$ the output is also zero $y(l,\ j) = 0$ for $(0,\ 0) \le (i,\ j) < (r_1,\ r_2)$.

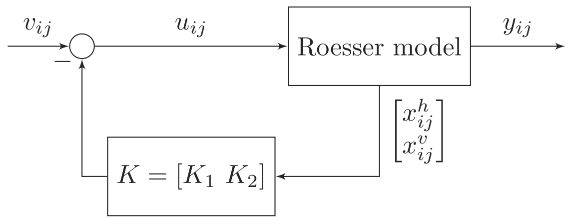

Fig. 1. Roesser model with state feedback.

Theorem 5. *The Roesser model (6) is observable in the rectangle $[r_1,\ r_2]$ if and only if*

$$\operatorname{rank} O_{\mathrm{R}} = n, \tag{10a}$$

where

$$O_R = \begin{bmatrix} C \\ CT_{10} \\ CT_{01} \\ \vdots \\ CT_{r_1 r_2} \end{bmatrix}, \tag{10b}$$

$$\operatorname{rank} \begin{bmatrix} I_{n_1}z_1 - A_{11} & -A_{12} \\ -A_{21} & I_{n_2}z_2 - A_{22} \\ C_1 & C_2 \end{bmatrix} = n, \quad \forall z_1, z_2 \in \mathbb{C}, \tag{10c}$$

where $C = [\ C_1 \quad C_2\ ] \in \mathbb{R}^{p\times n}$, $C_1 \in \mathbb{R}^{p\times n_1}$, $C_2 \in \mathbb{R}^{p\times n_2}$.

4. Controllability and observability of the Roesser model with feedbacks

Consider the Roesser model (6) (Fig. 1) with the state feedbacks

$$u_{ij} = v_{ij} - K_1 x^h_{ij} - K_2 x^v_{ij}, \tag{11}$$

where $v_{ij} \in \mathbb{R}^m$, $K_1 \in \mathbb{R}^{m\times n_1}$, $K_2 \in \mathbb{R}^{m\times n_2}$.

Substituting (11) into (6a), we obtain

$$\begin{aligned} \begin{bmatrix} x^h_{i+1,j} \\ x^v_{i,j+1} \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x^h_{ij} \\ x^v_{ij} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \left(v_{ij} - K \begin{bmatrix} x^h_{ij} \\ x^v_{ij} \end{bmatrix} \right) \\ &= A_c \begin{bmatrix} x^h_{ij} \\ x^v_{ij} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} v_{ij}, \end{aligned} \tag{12a}$$

where

$$A_c = \begin{bmatrix} A_{11} - B_1K_1 & A_{12} - B_1K_2 \\ A_{21} - B_2K_1 & A_{22} - B_2K_2 \end{bmatrix}. \tag{12b}$$

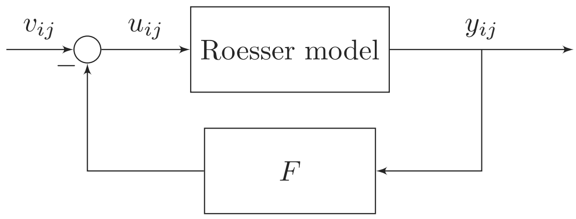

Fig. 2. Roesser model with output feedback.

Theorem 6. *The Roesser model with state feedbacks (12) is controllable if and only if the Roesser model (6a) is controllable.*

Proof. By Theorem 4 the Roesser model with feedback (12) is controllable if and only if

$$\operatorname{rank}\begin{bmatrix} I_{n_1}z_1 - A_{11} + B_1K_1 & -A_{12} + B_1K_2 & B_1 \\ -A_{21} + B_2K_1 & I_{n_2}z_2 - A_{22} + B_2K_2 & B_2 \end{bmatrix} = n, \quad z_1, z_2 \in \mathbb{C}. \tag{13}$$

Note that

$$\begin{bmatrix} I_{n_1}z_1 - A_{11} + B_1K_1 & -A_{12} + B_1K_2 & B_1 \\ -A_{21} + B_2K_1 & I_{n_2}z_2 - A_{22} + B_2K_2 & B_2 \end{bmatrix} = \begin{bmatrix} I_{n_1}z_1 - A_{11} & -A_{12} & B_1 \\ -A_{21} & I_{n_2}z_2 - A_{22} & B_2 \end{bmatrix} \times \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & I_{n_2} & 0 \\ K_1 & K_2 & I_m \end{bmatrix}. \tag{14}$$

From (14) it follows that the Roesser model with feedbacks (12) is controllable if and only if the Roesser model (6a) is controllable since the matrix

$$\begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & I_{n_2} & 0 \\ K_1 & K_2 & I_m \end{bmatrix} \tag{15}$$

is nonsingular for any K_1 and K_2. ■

Now let us consider the Roesser model (6) (Fig. 2) with the output feedbacks

$$u_{ij} = v_{ij} - Fy_{ij}, \tag{16}$$

where $F \in \mathbb{R}^{m\times p}$.

Substituting (6b) into (12), we obtain

$$u_{ij} = v_{ij} - FC\begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix}, \tag{17}$$

and the substituting (17) into (6a), we get

$$\begin{aligned}\begin{bmatrix} x_{i+1,j}^h \\ x_{i,j+1}^v \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}\left(v_{ij} - FC\begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix}\right) \\ &= \bar{A}_c\begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} v_{ij},\end{aligned} \tag{18a}$$

where

$$\bar{A}_c = \begin{bmatrix} A_{11} - B_1FC_1 & A_{12} - B_1FC_2 \\ A_{21} - B_2FC_1 & A_{22} - B_2FC_2 \end{bmatrix}, \quad C = [\ C_1 \quad C_2\ ], \quad C_1 \in \mathbb{R}^{p\times n_1}, \quad C_2 \in \mathbb{R}^{p\times n_2}. \tag{18b}$$

Theorem 7. *The Roesser model with output feedback (18) is observable if and only if the Roesser model (6a) is observable.*

Proof. The Roesser model with output feedback (18) is observable if and only if

$$\operatorname{rank}\begin{bmatrix} I_{n_1}z_1 - A_{11} + B_1FC_1 & -A_{12} + B_1FC_2 \\ -A_{21} + B_2FC_1 & I_{n_2}z_2 - A_{22} + B_2FC_2 \\ C_1 & C_2 \end{bmatrix} = n, z_1, \quad z_2 \in \mathbb{C}. \tag{19}$$

Note that

$$\begin{bmatrix} I_{n_1}z_1 - A_{11} + B_1FC_1 & -A_{12} + B_1FC_2 \\ -A_{21} + B_2FC_1 & I_{n_2}z_2 - A_{22} + B_2FC_2 \\ C_1 & C_2 \end{bmatrix} = \begin{bmatrix} I_{n_1} & 0 & B_1F \\ 0 & I_{n_2} & B_2F \\ 0 & 0 & I_p \end{bmatrix} \times \begin{bmatrix} I_{n_1}z_1 - A_{111} & -A_{12} \\ -A_{21} & I_{n_2}z_2 - A_{22} \\ C_1 & C_2 \end{bmatrix}. \tag{20}$$

From (20) it follows that the Roesser model with output feedback (18) is observable since the matrix

$$\begin{bmatrix} I_{n_1} & 0 & B_1F \\ 0 & I_{n_2} & B_2F \\ 0 & 0 & I_p \end{bmatrix} \tag{21}$$

is nonsingular for any matrix F. ■

5. Decomposition of the Roesser model

Let the Roesser model (6) be uncontrollable and unobservable. From the matrix (9) we choose the first independent columns $M_1, \dots, M_r$ and next $n-r$ additional independent columns $N_1, \dots, N_{n-r}$ such that the matrix

$$P = [M_1, \dots, M_r, N_1, \dots, N_{n-r}] \tag{22}$$

is nonsingular, $\det P \neq 0$.

We apply the similarity transformation to the uncontrollable pair $(A,\ B)$.

Lemma 1. *Applying the similarity transformation to the uncontrollable pair $(A,\ B)$, we obtain*

$$\begin{aligned} \bar{A} &= P^{-1}AP = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}, \\ \bar{B} &= P^{-1}B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \end{aligned} \tag{23}$$

$$A_1 \in \mathbb{R}^{r\times r}, \quad A_3 \in \mathbb{R}^{(n-r)\times(n-r)}, \quad B_1 \in \mathbb{R}^{r\times m},$$

where the pair $(A_1,\ B_1)$ is controllable and the pair $(A_3,\ 0)$ is uncontrollable.

Proof. Applying the similarity transformation to the pair $(A,\ B)$ and taking into account that $P^{-1}P = I_n$, we obtain

$$\begin{aligned} \bar{A} = P^{-1}AP &= [M_1, \dots, M_r, N_1, \dots, N_{n-r}]^{-1} \\ &\quad \times A[M_1, \dots, M_r, N_1, \dots, N_{n-r}] \\ &= [M_1, \dots, M_r, N_1, \dots, N_{n-r}]^{-1} \\ &\quad \times [AM_1, \dots, AM_r, AN_1, \dots, AN_{n-r}] \\ &= \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} \end{aligned} \tag{24a}$$

and

$$\begin{aligned} \bar{B} &= [M_1, \dots, M_r, N_1, \dots, N_{n-r}]^{-1}B \\ &= \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \end{aligned} \tag{24b}$$

where

$$\begin{aligned} A_2 &= \begin{bmatrix} p_1AN_1 & \dots & p_1AN_{n-r} \\ \dots & \dots & \dots \\ p_rAN_1 & \dots & p_rAN_{n-r} \end{bmatrix}, \\ A_2 &= \begin{bmatrix} p_{r+1}AN_1 & \dots & p_{r+1}AN_{n-r} \\ \dots & \dots & \dots \\ p_nAN_1 & \dots & p_nAN_{n-r} \end{bmatrix}. \end{aligned} \tag{24c}$$

and p_i, $i = 1, \dots, n$ is the i-th row of the matrix P^{-1}.

Note that the pair $(A_1,\ B_1)$ is controllable. ■

Lemma 2. *The transfer matrix (8) of the uncontrollable Roesser model (6) is equal to its controllable part*

$$\begin{aligned} T(z_1, z_2) &= \begin{bmatrix} C_1 & C_2 \end{bmatrix} \\ &\quad \begin{bmatrix} I_{n_1}z_1 - A_{11} & -A_{12} \\ -A_{21} & I_{n_2}z_2 - A_{22} \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\ &= C_1[I_rz_1 - A_1]^{-1}B_1, \end{aligned} \tag{25}$$

where the matrix P is defined by (22) and C_1 is the submatrix of the matrix $C = [\ C_1 \quad C_2\ ]$.

Proof. Taking into account (8), (23) and the definition of C, we obtain

$$\begin{aligned} &T(z_1, z_2) \\ &= C \begin{bmatrix} I_{n_1}z_1 - A_{11} & -A_{12} \\ -A_{21} & I_{n_2}z_2 - A_{22} \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\ &= CPP^{-1} \begin{bmatrix} I_{n_1}z_1 - A_{11} & -A_{12} \\ -A_{21} & I_{n_2}z_2 - A_{22} \end{bmatrix}^{-1} \\ &\quad \times PP^{-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\ &= \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} I_rz_1 - A_1 & -A_2 \\ 0 & I_{n-r}z_2 - A_3 \end{bmatrix}^{-1} \\ &\quad \times \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \\ &= C_1[I_rz_1 - A_1]^{-1}B_1. \end{aligned} \tag{26}$$

Therefore, the transfer matrix of the uncontrollable Roesser model is equal to the transfer matrix of only its controllable part. ■

Now consider the unobservable Roesser model (6). For, the matrix (10) we use the first r independent rows $\bar{M}_1, \dots, \bar{M}_r$ and next $n-r$ additional independent rows $\bar{N}_1, \dots, \bar{N}_{n-r}$ such that the matrix

$$\bar{P} = \begin{bmatrix} \bar{M}_1 \\ \vdots \\ \bar{M}_r \\ \bar{N}_1 \\ \vdots \\ \bar{N}_{n-r} \end{bmatrix} \tag{27}$$

is nonsingular, i.e., $\det \bar{P} \neq 0$.

We apply the similarity transformation to the unobservable pair $(A,\ C)$.

Lemma 3. *Applying the similarity transformation to the pair $(A,\ C)$, we obtain*

$$\begin{aligned} \hat{A} &= P^{-1}AP = \begin{bmatrix} \hat{A}_1 & 0 \\ \hat{A}_3 & \hat{A}_4 \end{bmatrix}, \\ \hat{C} &= CP = \begin{bmatrix} \hat{C}_1 & 0 \end{bmatrix}, \end{aligned} \tag{28}$$

where the pair $(\hat{A}_1,\ \hat{C}_1)$ is observable and the pair $(\hat{A}_4,\ 0)$ is unobservable.

The proof is similar (dual) to that of Lemma 1.

Lemma 4. *The transfer matrix (8) of the unobservable Roesser model (6) is equal to its observable part,*

$$\begin{aligned} &T(z_1,z_2)\\ &=\begin{bmatrix} \hat{C}_1 & 0 \end{bmatrix}\begin{bmatrix} I_{n_1}z_1-\hat{A}_1 & 0 \\ -\hat{A}_3 & I_{n_2}z_2-\hat{A}_4 \end{bmatrix}^{-1}\begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix}\\ &=\hat{C}_1[I_{n_1}z_1-\hat{A}_1]^{-1}\hat{B}_1. \end{aligned} \tag{29}$$

The proof is similar(dual) to that of Lemma 2.

Theorem 8. *The transfer matrix of the Roesser model (6) is zero if and only if the following conditions are satisfied:*

1. *the pair $(A,\ B)$ is uncontrollable,*
2. *the pair $(A,\ C)$ is unobservable,*
3. $CB = 0$.

Proof. By Lemma 2 the transfer matrix of the uncontrollable Roesser model is equal to its controllable part and, by Lemma 4, to its observable part. If Conditions 1 and 2 are satisfied then in the transfer matrix (14) the matrix A is zero and the transfer matrix of the model is zero if and only if Condition 3 is satisfied. ■

Example 1. Consider the Roesser model with the matrices

$$\begin{aligned} A&=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} =\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 1 \\ \hline 1 & 0 & 0 \end{array}\right], \\ B&=\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}=\left[\begin{array}{c} 0 \\ 1 \\ \hline 1 \end{array}\right] \end{aligned} \tag{30}$$

and two cases of the matrix C:
Case 1. $C_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$;
Case 2. $C_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$. ♦

Note that the pair (30) is uncontrollable since

$$\begin{aligned} &\operatorname{rank}\begin{bmatrix} M(0,1) & M(1,0) & M(1,1) \end{bmatrix}\\ &=\operatorname{rank}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}=2<n=3. \end{aligned} \tag{31}$$

In Case 1 the Roesser model is also unobservable since

$$\operatorname{rank}\begin{bmatrix} C_1 \\ C_1T_{10} \\ C_1T_{01} \\ C_1T_{11} \end{bmatrix}=\operatorname{rank}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}=2<n=3 \tag{32}$$

but

$$C_1B=\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}=1.$$

In this case the cancelation occurs in the transfer matrix but it is nonzero.

In Case 2 the Roesser model is also unobservable since

$$\begin{aligned} &T(z_1,z_2)\\ &=C\begin{bmatrix} I_{n_1}z_1-A_{11} & -A_{12} \\ -A_{21} & I_{n_2}z_2-A_{22} \end{bmatrix}^{-1}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}\\ &=\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} z_1-1 & 0 & 0 \\ 0 & z_1-1 & -1 \\ -1 & 0 & z_2 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\\ &=\frac{1}{z_2-1}, \end{aligned} \tag{33}$$

but $CB = 0$ and the transfer matrix

$$\begin{aligned} &T(z_1,z_2)\\ &=C_2\begin{bmatrix} I_{n_1}z_1-A_{11} & -A_{12} \\ -A_{21} & I_{n_2}z_2-A_{22} \end{bmatrix}^{-1}B\\ &=\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} z_1-1 & 0 & 0 \\ 0 & z_1-1 & -1 \\ -1 & 0 & z_2 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\\ &=0 \end{aligned} \tag{34}$$

is zero. This confirms Theorem 8.

6. Concluding remarks

Controllability, observability and the transfer matrix of the discrete 2-D Roesser model have been analyzed. It was shown that the controllability of the Roesser model is invariant under state feedbacks (Theorem 6) and the observability under output feedbacks (Theorem 7). Sufficient conditions are established for zeroing the transfer matrix of the Roesser model (Theorem 8). The analysis was illustrated by a simple numerical example. The considerations can be easily extended to Fornasini–Marchesini models and to general 2-D linear models. An open problem is an extension of these considerations to fractional orders 2-D linear systems.

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Tadeusz Kaczorek received his MSc, PhD and DSc degrees in electrical engineering from the Warsaw University of Technology in 1956, 1962 and 1964, respectively. In 1971 he became an associate professor and in 1974 a full professor at the same university. Since 2003 he has been a professor at the Bialystok University of Technology. In 1986 he was elected a corresponding member and in 1996 a full member of the Polish Academy of Sciences. In 2004 he was elected an honorary member of the Hungarian Academy of Sciences. He has been granted honorary doctorates by 13 universities. His research interests cover systems theory, especially singular multidimensional systems, positive multidimensional systems, singular positive 1D and 2D systems, as well as positive fractional 1D and 2D systems. He initiated research in the field of singular 2D, positive 2D and positive fractional linear systems. He has published 29 books (9 in English) and over 1200 scientific papers. He has also supervised 70 PhD theses. He is a member of editorial boards of 10 international journals.

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