Semantics of MML Query - Ordering

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Summary. Semantics of order directives of MML Query is presented. The formalization is done according to [1].

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The notation and terminology used in this paper have been introduced in the following articles: [2], [7], [13], [9], [10], [8], [4], [5], [11], [17], [19], [18], [6], [15], [16], [14], and [12].

1. Preliminaries

In this paper $X$ denotes a set, $R, R_1, R_2$ denote binary relations, $x, y, z$ denote sets, and $n, m, k$ denote natural numbers.

Let us consider a binary relation $R$ on $X$. Now we state the propositions:

1. field $R \subseteq X$.
2. If $x, y \in R$, then $x, y \in X$.

Now we state the propositions:

3. Let us consider sets $X, Y$. Then $(\text{id}_X)^0 Y = X \cap Y$.
4. $\langle x, y \rangle \in R \upharpoonright^2 X$ if and only if $x, y \in X$ and $\langle x, y \rangle \in R$.
5. $\text{dom}(X \upharpoonright R) \subseteq \text{dom} R$.
6. Let us consider a total reflexive binary relation $R$ on $X$ and a subset $S$ of $X$. Then $R \upharpoonright^2 S$ is a total reflexive binary relation on $S$. The theorem is a consequence of (4). Proof: Set $Q = R \upharpoonright^2 S$. dom $Q = S$. $\square$
7. Let us consider transfinite sequences $f, g$. Then $\text{rng}(f \upharpoonright g) = \text{rng} f \cup \text{rng} g$.

Let us consider $R$. Let us note that $R$ is transitive if and only if the condition (Def. 1) is satisfied.
(Def. 1) If \( x, y \in R \) and \( y, z \in R \), then \( x, z \in R \).

One can verify that \( R \) is antisymmetric if and only if the condition (Def. 2) is satisfied.

(Def. 2) If \( x, y \in R \) and \( y, x \in R \), then \( x = y \).

Now we state the proposition:

(8) Let us consider a non empty set \( X \), a total connected binary relation \( R \) on \( X \), and elements \( x, y \) of \( X \). If \( x \neq y \), then \( x, y \in R \) or \( y, x \in R \).

2. COMPOSITION OF ORDERS

Let \( R_1, R_2 \) be binary relations. The functor \( R_1, R_2 \) yielding a binary relation is defined by the term

(Def. 3) \( R_1 \cup (R_2 \setminus R_1^\sim) \).

Now we state the propositions:

(9) \( x, y \in R_1, R_2 \) if and only if \( x, y \in R_1 \) or \( y, x \not\in R_1 \) and \( x, y \in R_2 \).

(10) \( \text{field}(R_1, R_2) = \text{field} R_1 \cup \text{field} R_2 \). The theorem is a consequence of (9).

(11) \( R_1, R_2 \subseteq R_1 \cup R_2 \). The theorem is a consequence of (9).

Let \( X \) be a set and \( R_1, R_2 \) be binary relations on \( X \). Note that the functor \( R_1, R_2 \) yields a binary relation on \( X \). Let \( R_1, R_2 \) be reflexive binary relations. One can verify that \( R_1, R_2 \) is reflexive.

Let \( R_1, R_2 \) be antisymmetric binary relations. Note that \( R_1, R_2 \) is antisymmetric.

Let \( X \) be a set and \( R \) be a binary relation on \( X \). We say that \( R \) is \( \beta \)-transitive if and only if

(Def. 4) Let us consider elements \( x, y \) of \( X \). If \( x, y \not\in R \), then for every element \( z \) of \( X \) such that \( x, z \in R \) holds \( y, z \in R \).

Observe that every binary relation on \( X \) which is connected total and transitive is also \( \beta \)-transitive.

Let us observe that there exists an order in \( X \) which is connected.

Let \( R_1 \) be a \( \beta \)-transitive transitive binary relation on \( X \) and \( R_2 \) be a transitive binary relation on \( X \). Observe that \( R_1, R_2 \) is transitive.

Let \( R_1 \) be a binary relation on \( X \) and \( R_2 \) be a total reflexive binary relation on \( X \). Let us note that \( R_1, R_2 \) is total and reflexive as a binary relation on \( X \).

Let \( R_2 \) be a total connected reflexive binary relation on \( X \). One can verify that \( R_1, R_2 \) is connected.

Now we state the propositions:

(12) \( (R, R_1), R_2 = R, (R_1, R_2) \). The theorem is a consequence of (9).

(13) Let us consider a connected reflexive total binary relation \( R \) on \( X \) and a binary relation \( R_2 \) on \( X \). Then \( R, R_2 = R \). The theorem is a consequence of (9) and (2).
3. number of Ordering

Let $X$ be a set and $f$ be a function from $X$ into $\mathbb{N}$. The functor number of $f$ yielding a binary relation on $X$ is defined by

\begin{equation}
\text{number of } f \text{ if and only if } x, y \in X \text{ and } f(x) < f(y).
\end{equation}

Let us note that number of $f$ is antisymmetric transitive and $\beta$-transitive.

Let $X$ be a finite set and $O$ be an operation of $X$. The functor value of $O$ yielding a function from $X$ into $\mathbb{N}$ is defined by

\begin{equation}
\text{value of } O \text{ if and only if } x, y \in X \text{ and } O(x) < O(y).
\end{equation}

Now we state the proposition:

\begin{enumerate}
\item Let us consider a finite set $X$, an operation $O$ of $X$, and elements $x, y$ of $X$. Then $x, y \in \text{number of value of } O$ if and only if $x(O) < y(O)$.
\end{enumerate}

Let us consider $X$. Let $O$ be an operation of $X$. The functor first $O$ yielding a binary relation on $X$ is defined by

\begin{equation}
\text{first } O \text{ if and only if } x, y \in X \text{ and } O(x) \neq \emptyset \text{ and } O(y) = \emptyset.
\end{equation}

Let us observe that first $O$ is antisymmetric transitive and $\beta$-transitive.

4. Ordering by Resources

Let $A$ be a finite sequence and $x$ be an element. The functor $A \leftarrow x$ yielding a set is defined by the term

\begin{equation}
\cap(A^{-1}(\{x\})).
\end{equation}

Let us consider $x$. Note that $A \leftarrow x$ is natural.

Let us consider a finite sequence $A$. Now we state the propositions:

\begin{enumerate}
\item If $x \not\in \text{rng } A$, then $A \leftarrow x = 0$.
\item If $x \in \text{rng } A$, then $A \leftarrow x \in \text{dom } A$ and $x = A(A \leftarrow x)$.
\item If $A \leftarrow x = 0$, then $x \not\in \text{rng } A$.
\end{enumerate}

Let us consider $X$. Let $A$ be a finite sequence and $f$ be a function. The functor resource($X, A, f$) yielding a binary relation on $X$ is defined by

\begin{equation}
x, y \in \text{it if and only if } x, y \in X \text{ and } A \leftarrow (f(x)) \neq 0 \text{ and } A \leftarrow (f(x)) < A \leftarrow (f(y)) \text{ or } A \leftarrow (f(y)) = 0.
\end{equation}

Let us observe that resource($X, A, f$) is antisymmetric transitive and $\beta$-transitive.
5. Ordering by Number of Iteration

Let us consider $X$. Let $R$ be a binary relation on $X$ and $n$ be a natural number. One can check that the functor $R^n$ yields a binary relation on $X$. Now we state the propositions:

(18) If $(R^n)^oX = \emptyset$ and $m \geq n$, then $(R^m)^oX = \emptyset$.

(19) If for every $n$, $(R^n)^oX \neq \emptyset$ and $X$ is finite, then there exists $x$ such that $x \in X$ and for every $n$, $(R^n)^o x \neq \emptyset$. The theorem is a consequence of (18).

**Proof:** Define $P[\text{element}, \text{element}] \equiv$ there exists $n$ such that $S = n$ and $(R^n)^s S_1 = \emptyset$. For every element $x$ such that $x \in X$ there exists an element $y$ such that $y \in N$ and $P[x, y]$. Consider $f$ being a function such that $\text{dom}_f = X$ and $\text{rng}_f \subseteq N$ and for every element $x$ such that $x \in X$ holds $P[x, f(x)]$. Consider $n$ such that $\text{rng}_f \subseteq Z_n$. $\{x\}$ where $x$ is an element of $X : x \in X \subseteq 2^X$. Reconsider $Y = \{x\}$ where $x$ is an element of $X : x \in X$ as a family of subsets of $X$. $X = \bigcup Y$. $\{(R^n)^o y \}$ where $y$ is a subset of $X : y \in Y \subseteq \{0\}$. □

(20) If $R$ is reversely well founded and irreflexive and $X$ is finite and $R$ is finite, then there exists $n$ such that $(R^n)^o X = \emptyset$. The theorem is a consequence of (19).

**Proof:** Define $Q[\text{element}] \equiv$ for every $n$, $(R^n)^o S_1 \neq \emptyset$. Consider $x_0$ being a set such that $x_0 \in X$ and $Q[x_0]$. Define $P[\text{element}, \text{element}, \text{element}] \equiv$ if $Q[S_2]$, then $S_3 \in R^o S_2$ and $Q[S_3]$. For every natural number $n$ and for every set $x$, there exists a set $y$ such that $P[n, x, y]$. Consider $f$ being a function such that $\text{dom}_f = N$ and $f(0) = x_0$ and for every natural number $n$, $P[n, f(n), f(n + 1)]$. Define $R[\text{natural number}] \equiv Q[f(S_1)]$. $\text{rng}_f \subseteq \text{field} R$. Consider $z$ being an element such that $z \in \text{rng}_f$ and for every element $x$ such that $x \in \text{rng}_f$ and $z \neq x$ holds $\langle z, x \rangle \notin R$. Consider $y$ being an element such that $y \in N$ and $z = f(y)$. □

Let us consider $X$. Let $O$ be an operation of $X$. Assume $O$ is reversely well founded, irreflexive, and finite. The functor $\text{iteration of } O$ yielding a binary relation on $X$ is defined by

(Def. 10) There exists a function $f$ from $X$ into $N$ such that

(i) it = number of $f$, and

(ii) for every element $x$ of $X$ such that $x \in X$ there exists $n$ such that $f(x) = n$ and $x(O^n) \neq \emptyset$ or $n = 0$ and $x(O^n) = \emptyset$ and $x(O^{n+1}) = \emptyset$.

Let us note that every binary relation which is empty is also irreflexive and reversely well founded.

Let us consider $X$. Let us note that there exists an operation of $X$ which is empty.

Let $O$ be a reversely well founded irreflexive finite operation of $X$. One can check that $\text{iteration of } O$ is antisymmetric transitive and $\beta$-transitive.
6. value of Ordering

Let $X$ be a finite set. Let us observe that every order in $X$ is well-founded. Note that every connected order in $X$ is well-ordering.

Let us consider $X$. Let $R$ be a connected order in $X$ and $S$ be a finite subset of $X$. The functor $\text{order}(S, R)$ yielding a finite 0-sequence of $X$ is defined by

(Def. 11) (i) $\text{rng } it = S$, and
(ii) $it$ is one-to-one, and
(iii) for every natural numbers $i, j$ such that $i, j \in \text{dom } it$ holds $i \leq j$ iff $it(i), it(j) \in R$.

Now we state the proposition:

(21) Let us consider finite subsets $S_1, S_2$ of $X$ and a connected order $R$ in $X$. Then $\text{order}(S_1 \cup S_2, R) = \text{order}(S_1, R) \cap \text{order}(S_2, R)$ if and only if for every $x$ and $y$ such that $x \in S_1$ and $y \in S_2$ holds $x \neq y$ and $x, y \in R$.

The theorem is a consequence of (7). PROOF: Set $o1 = \text{order}(S_1, R)$. Set $o2 = \text{order}(S_2, R)$. $\text{order}(S_1, R) \cap \text{order}(S_2, R)$ is one-to-one. □

Let $X$ be a finite set, $O$ be an operation of $X$, and $R$ be a connected order in $X$. The functor $\text{value of}(O, R)$ yielding a binary relation on $X$ is defined by

(Def. 12) Let us consider elements $x, y$ of $X$. Then $x, y \in it$ if and only if $x(O) \neq \emptyset$ and $y(O) = \emptyset$ or $y(O) \neq \emptyset$ and $(\text{order}(x(O), R))_0, (\text{order}(y(O), R))_0 \in R$ and $(\text{order}(x(O), R))_0 \neq (\text{order}(y(O), R))_0$.

Let $R_1$ be a connected order in $X$. One can check that $\text{value of}(O, R_1)$ is antisymmetric transitive and $\beta$-transitive.

References


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